Quantum Field Theory

Uwe-Jens Wiese
Institute for Theoretical Physics
University of Bern

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Chapter 1

Introduction

Quantum mechanics describes the behavior of particles at atomic and subatomic scales. It is the quantum version of Newton's classical mechanics and reduces to this theory when $\hbar$ is sent to zero. Of course, Planck's quantum is a fundamental constant of Nature that is indeed non-zero. This has consequences not only for mechanics but for all of physics. In particular, Maxwell's theory of classical electrodynamics also needs to be modified by incorporating the principles of quantum physics. It turned out that this is a rather nontrivial enterprise which kept people like Feynman, Schwinger, and Tomonaga busy for some time after world war II. Their success was awarded with the Nobel prize for quantum electrodynamics (QED) — the quantum field theory that describes the electromagnetic interactions of electrons and positrons resulting from the exchange of photons. The fundamental degrees of freedom in QED are fields — not particles. In particular, photons — the elementary particles of light — emerge as quantum states of Maxwell's electromagnetic field. Similarly, electrons and positrons emerge as quanta of the 4-component spinor field that Dirac first introduced. In quantum field theory, particles result as quantized field fluctuations.

The most important feature of field theory is locality. The field degrees of freedom located at a given point of space are coupled only to the field values at infinitesimally neighboring points. The principle of locality is also at the heart of relativity theory which is inconsistent with instantaneous nonlocal interaction over a finite distance. Unifying relativity and quantum physics therefore naturally leads to quantum field theory; it cannot be achieved within quantum mechanics alone. In relativity theory, particle interactions cannot be described by instantaneous potentials but must be mediated locally. This is possible if there are physical degrees of freedom everywhere is space. These degrees of freedom

are nothing but the field values. Quantum field theory has enormously advanced our understanding of physics. It was originally developed to describe elementary particles but it is equally useful in condensed matter physics. In fact, it is extremely useful for describing any local interaction of many degrees of freedom at microscopic scales.

Our present understanding of fundamental physics is based on the field concept. At each point of space there is a set of field degrees of freedom whose fluctuations lead to the propagation of a variety of elementary particles. Each particle has its own field: the photon, the electron and positron, the gluons, the quarks and anti-quarks, the W- and Z-bosons, the neutrinos and anti-neutrinos, etc. These various fields are distinguished by their transformation properties under a variety of symmetries: space-time rotations, parity P, charge conjugation C, gauge transformations, etc. In fact, the various quantum field theories are characterized by their symmetry properties. For example, QED is a gauge theory with the Abelian gauge group $U(1)_{em}$ which respects the symmetries C and P. Quantum chromodynamics (QCD) is the quantum field theory that describes the strong interactions between quarks and anti-quarks mediated by the exchange of gluons. Its structure is similar to the one of QED. QCD is a gauge theory with non-Abelian gauge group $SU(3)_c$ which also respects the symmetries C and P. Both QED and QCD are incorporated in the standard model of particle physics which summarizes all we know about the fundamental forces of electromagnetism, as well as the weak and strong interactions (but not gravity). The standard model is a gauge theory with the non-Abelian gauge group $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ which, however, strongly violates parity P and charge conjugation C. Even the combined symmetry CP is weakly broken. Unlike QED and QCD, the standard model is a so-called chiral gauge theory in which left- and right-handed particles carry different charges.

The quantization of field theories is a highly non-trivial issue. After all, quantum field theories are quantum systems with infinitely many degrees of freedom — a given number per space point. This enormity of degrees of freedom gives rise to ultraviolet divergences which must be regularized and renormalized. Learning field theory is a non-trivial and sometimes rather technical task, which will, however, throw you right in the middle of current research in particle and condensed matter physics. Quantum field theory is such a powerful tool that it is indispensable in these fields of physics. You simply need to master quantum field theory if you want to understand — let alone contribute to — these very exciting fields of current research. It will naturally take a while before you can call yourself a quantum field theorist. It requires some time and a lot of hard work, but the awards are plenty. At the end of your studies you will have a basis for under-

standing all that is currently known about the microscopic quantum world. The regularization of chiral gauge theories like the standard model is a topic of current research. For example, only a few years ago, Lüscher from CERN achieved a breakthrough by quantizing a chiral gauge theory beyond perturbation theory. Naturally, when learning quantum field theory, we will start with much simpler field theories than the full standard model, or even QED. In fact, we will start with a simple relativistic theory of a scalar field. Scalar fields play an important role in both particle and condensed matter physics. The yet to be discovered Higgs particle — a corner stone of the standard model — is described by a scalar field. Similarly, scalar fields are used to describe Cooper pairs in superconductors or the order parameter in superfluid helium. Later we will move on to fermionic fields as well as to gauge fields. Even at the end of the course, we will not have reached the quantization of non-Abelian gauge theories like QCD or the standard model. However, by then you will hopefully have a solid basis in quantum field theory, which will allow you to proceed to these more advanced topics whenever a research project requires it.

Chapter 2

From Mechanics to Quantum Field Theory

This chapter provides a brief summary of the mathematical structure of quantum field theory. Classical field theories are discussed as a generalization of point mechanics to systems with infinitely many degrees of freedom — a given number per space point. Similarly, quantum field theories are just quantum mechanical systems with infinitely many degrees of freedom. In the same way as point mechanics systems, classical field theories can be quantized with path integral methods. The quantization of field theories at finite temperature leads to path integrals in Euclidean time. This provides us with an analogy between quantum field theory and classical statistical mechanics. We also mention the lattice regularization which has recently provided a mathematically satisfactory formulation of the standard model beyond perturbation theory.

2.1 From Point Mechanics to Classical Field Theory

Point mechanics describes the dynamics of classical nonrelativistic point particles. The coordinates of the particles represent a finite number of degrees of freedom. In the simplest case — a single particle moving in one spatial dimension — we are dealing with a single degree of freedom: the x-coordinate of the particle. The dynamics of a particle of mass m moving in an external potential $V(x)$ is described by Newton's equation

$$m\partial_t^2 x = ma = F(x) = -\frac{dV(x)}{dx}. \tag{2.1.1}$$

Once the initial conditions are specified, this ordinary second order differential equation determines the particle's path $x(t)$, i.e. its position as a function of time. Newton's equation results from the variational principle to minimize the action

$$S[x] = \int dt\ L(x, \partial_t x), \tag{2.1.2}$$

over the space of all paths $x(t)$. The action is a functional (a function whose argument is itself a function) that results from the time integral of the Lagrange function

$$L(x, \partial_t x) = \frac{m}{2}(\partial_t x)^2 - V(x). \tag{2.1.3}$$

The Euler-Lagrange equation

$$\partial_t \frac{\delta L}{\delta(\partial_t x)} - \frac{\delta L}{\delta x} = 0, \tag{2.1.4}$$

is nothing but Newton's equation.

Classical field theories are a generalization of point mechanics to systems with infinitely many degrees of freedom — a given number for each space point $\vec{x}$. In this case, the degrees of freedom are the field values $\phi(\vec{x})$, where ϕ is some generic field. In case of a neutral scalar field, ϕ is simply a real number representing one degree of freedom per space point. A charged scalar field, on the other hand, is described by a complex number and hence represents two degrees of freedom per space point. The scalar Higgs field $\phi^a(\vec{x})$ (with $a \in \{1, 2\}$) in the standard model is a complex doublet, i.e. it has four real degrees of freedom per space point. An Abelian gauge field $A_i(\vec{x})$ (with a spatial direction index $i \in \{1, 2, 3\}$) — for example, the photon field in electrodynamics — is a neutral vector field with 3 real degrees of freedom per space point. One of these degrees of freedom is redundant due to the $U(1)_{em}$ gauge symmetry. Hence, an Abelian gauge field has two physical degrees of freedom per space point which correspond to the two polarization states of the massless photon. Note that the time-component $A_0(\vec{x})$ does not represent a physical degree of freedom. It is just a Lagrange multiplier field that enforces the Gauss law. A non-Abelian gauge field $A_i^a(\vec{x})$ is charged and has an additional index a. For example, the gluon field in chromodynamics with a color index $a \in \{1, 2, ..., 8\}$ represents $2 \times 8 = 16$ physical degrees of freedom per space point, again because of some redundancy due to the $SU(3)_c$ color gauge symmetry. The field that represents the W- and Z-bosons in the standard model has an index $a \in \{1, 2, 3\}$ and transforms under the gauge group $SU(2)_L$. Thus, it represents $2 \times 3 = 6$ physical degrees of freedom. However, in contrast to the photon, the W- and Z-bosons are massive due to the Higgs mechanism and have three (not just two) polarization states. The extra degree of freedom is provided by the Higgs field.

The analog of Newton's equation in field theory is the classical field equation of motion. For example, for a neutral scalar field this is the Klein-Gordon equation

$$\partial_\mu \partial^\mu \phi = -\frac{dV(\phi)}{d\phi}. \tag{2.1.5}$$

Again, after specifying appropriate initial conditions it determines the classical field configuration $\phi(x)$, i.e. the values of the field ϕ at all space-time points $x = (t, \vec{x})$. Hence, the role of time in point mechanics is played by space-time in field theory, and the role of the point particle coordinates is now played by the field values. As before, the classical equation of motion results from minimizing the action

$$S[\phi] = \int d^4x \; \mathcal{L}(\phi, \partial_\mu \phi). \tag{2.1.6}$$

The integral over time in eq.(2.1.2) is now replaced by an integral over space-time and the Lagrange function of point mechanics gets replaced by the Lagrange density function (or Lagrangian)

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi). \tag{2.1.7}$$

A simple interacting field theory is the ϕ^4 theory with the potential

$$V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{2.1.8}$$

Here m is the mass of the scalar field and λ is the coupling strength of its self-interaction. Note that the mass term corresponds to a harmonic oscillator potential in the point mechanics analog, while the interaction term corresponds to an anharmonic perturbation. As before, the Euler-Lagrange equation

$$\partial_\mu \frac{\delta L}{\delta(\partial_\mu \phi)} - \frac{\delta L}{\delta \phi} = 0, \tag{2.1.9}$$

is the classical equation of motion, in this case the Klein-Gordon equation. The analogies between point mechanics and field theory are summarized in table 2.1.

2.2 The Path Integral in Real Time

The quantization of field theories is most conveniently performed using the path integral approach. Here we first discuss the path integral in quantum mechanics

Point Mechanics	Field Theory
time t	space-time $x = (t, \vec{x})$
particle coordinate x	field value ϕ
particle path $x(t)$	field configuration $\phi(x)$
action $S[x] = \int dt\ L(x, \partial_t x)$	action $S[\phi] = \int d^4x\ \mathcal{L}(\phi, \partial_\mu \phi)$
Lagrange function $L(x, \partial_t x) = \frac{m}{2}(\partial_t x)^2 - V(x)$	Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)$
equation of motion $\partial_t \frac{\delta L}{\delta(\partial_t x)} - \frac{\delta L}{\delta x} = 0$	field equation $\partial_\mu \frac{\delta L}{\delta(\partial_\mu \phi)} - \frac{\delta L}{\delta \phi} = 0$
Newton's equation $\partial_t^2 x = -\frac{dV(x)}{dx}$	Klein-Gordon equation $\partial_\mu \partial^\mu \phi = -\frac{dV(\phi)}{d\phi}$
kinetic energy $\frac{m}{2}(\partial_t x)^2$	kinetic energy $\frac{1}{2}\partial_\mu \phi \partial^\mu \phi$
harmonic oscillator potential $\frac{m}{2}\omega^2 x^2$	mass term $\frac{m^2}{2}\phi^2$
anharmonic perturbation $\frac{\lambda}{4!}x^4$	self-interaction term $\frac{\lambda}{4!}\phi^4$

Table 2.1: *The dictionary that translates point mechanics into the language of field theory.*

— quantized point mechanics — using the real time formalism. A mathematically more satisfactory formulation uses an analytic continuation to so-called Euclidean time. This will be discussed in the next section.

The real time evolution of a quantum system described by a Hamilton operator H is given by the time-dependent Schrödinger equation

$$i\hbar \partial_t |\Psi(t)\rangle = H|\Psi(t)\rangle. \tag{2.2.1}$$

For a time-independent Hamilton operator the time evolution operator is given by

$$U(t', t) = \exp(-\frac{i}{\hbar} H(t' - t)), \tag{2.2.2}$$

such that

$$|\Psi(t')\rangle = U(t', t)|\Psi(t)\rangle. \tag{2.2.3}$$

Let us consider the transition amplitude $\langle x'|U(t', t)|x\rangle$ of a nonrelativistic point particle that starts at position x at time t and arrives at position x' at time t'. Using

$$\langle x|\Psi(t)\rangle = \Psi(x, t) \tag{2.2.4}$$

we obtain

$$\Psi(x', t') = \int dx\ \langle x'|U(t', t)|x\rangle \Psi(x, t), \tag{2.2.5}$$

i.e. $\langle x'|U(t',t)|x\rangle$ acts as a propagator for the wave function. The propagator is of physical interest because it contains information about the energy spectrum. When we consider propagation from an initial position x back to the same position we find

$$\begin{aligned}\langle x|U(t',t)|x\rangle &= \langle x|\exp(-\frac{i}{\hbar}H(t'-t))|x\rangle \\ &= \sum_n |\langle x|n\rangle|^2 \exp(-\frac{i}{\hbar}E_n(t'-t)). \end{aligned} \tag{2.2.6}$$

We have inserted a complete set, $\sum_n |n\rangle\langle n| = \mathbb{1}$, of energy eigenstates $|n\rangle$ with

$$H|n\rangle = E_n|n\rangle. \tag{2.2.7}$$

Hence, according to eq.(2.2.6), the Fourier transform of the propagator yields the energy spectrum as well as the energy eigenstates $\langle x|n\rangle$.

Inserting a complete set of position eigenstates we arrive at

$$\begin{aligned}\langle x'|U(t',t)|x\rangle &= \langle x'|\exp(-\frac{i}{\hbar}H(t'-t_1+t_1-t))|x\rangle \\ &= \int dx_1 \langle x'|\exp(-\frac{i}{\hbar}H(t'-t_1))|x_1\rangle \\ &\times \langle x_1|\exp(-\frac{i}{\hbar}H(t_1-t))|x\rangle \\ &= \int dx_1 \langle x'|U(t',t_1)|x_1\rangle\langle x_1|U(t_1,t)|x\rangle. \end{aligned} \tag{2.2.8}$$

It is obvious that we can repeat this process an arbitrary number of times. This is exactly what we do in the formulation of the path integral. Let us divide the time interval $[t,t']$ into N elementary time steps of size ε such that

$$t'-t = N\varepsilon. \tag{2.2.9}$$

Inserting a complete set of position eigenstates at the intermediate times $t_i, i \in \{1,2,...,N-1\}$ we obtain

$$\begin{aligned}\langle x'|U(t',t)|x\rangle &= \int dx_1 \int dx_2 ... \int dx_{N-1} \langle x'|U(t',t_{N-1})|x_{N-1}\rangle ... \\ &\times \langle x_2|U(t_2,t_1)|x_1\rangle\langle x_1|U(t_1,t)|x\rangle. \end{aligned} \tag{2.2.10}$$

In the next step we concentrate on one of the factors and we consider a single nonrelativistic point particle moving in an external potential $V(x)$ such that

$$H = \frac{p^2}{2m} + V(x). \tag{2.2.11}$$

Using the Baker-Campbell-Haussdorff formula and neglecting terms of order ε^2 we find

$$\begin{aligned}\langle x_{i+1}|U(t_{i+1},t_i)|x_i\rangle &= \langle x_{i+1}|\exp(-\frac{i\varepsilon p^2}{2m\hbar})\exp(-\frac{i\varepsilon}{\hbar}V(x))|x_i\rangle \\ &= \frac{1}{2\pi}\int dp\langle x_{i+1}|\exp(-\frac{i\varepsilon p^2}{2m\hbar})|p\rangle\langle p|\exp(-\frac{i\varepsilon}{\hbar}V(x))|x_i\rangle \\ &= \frac{1}{2\pi}\int dp\exp(-\frac{i\varepsilon p^2}{2m\hbar})\exp(-\frac{i}{\hbar}p(x_{i+1}-x_i)) \\ &\times \exp(-\frac{i\varepsilon}{\hbar}V(x_i)). \qquad (2.2.12)\end{aligned}$$

The integral over p is ill-defined because the integrand is a very rapidly oscillating function. To make the expression well-defined we replace the time step ε by $\varepsilon - ia$, i.e. we go out into the complex time plane. After doing the integral we take the limit $a \to 0$. Still one should keep in mind that the definition of the path integral required an analytic continuation in time. One finds

$$\langle x_{i+1}|U(t_{i+1},t_i)|x_i\rangle = \sqrt{\frac{m}{2\pi i\hbar\varepsilon}}\exp(\frac{i}{\hbar}\varepsilon[\frac{m}{2}(\frac{x_{i+1}-x_i}{\varepsilon})^2 - V(x_i)]). \qquad (2.2.13)$$

Inserting this back into the expression for the propagator we obtain

$$\langle x'|U(t',t)|x\rangle = \int \mathcal{D}x\exp(\frac{i}{\hbar}S[x]). \qquad (2.2.14)$$

The action has been identified in the time continuum limit as

$$\begin{aligned}S[x] &= \int dt\ [\frac{m}{2}(\partial_t x)^2 - V(x)] \\ &= \lim_{\varepsilon\to 0}\sum_i \varepsilon[\frac{m}{2}(\frac{x_{i+1}-x_i}{\varepsilon})^2 - V(x_i)]. \qquad (2.2.15)\end{aligned}$$

The integration measure is defined as

$$\int \mathcal{D}x = \lim_{\varepsilon\to 0}\sqrt{\frac{m}{2\pi i\hbar\varepsilon}}^N \int dx_1 \int dx_2 ... \int dx_{N-1}. \qquad (2.2.16)$$

This means that we integrate over the possible particle positions for each intermediate time t_i. In this way we integrate over all possible paths of the particle starting at x and ending at x'. Each path is weighted with an oscillating phase factor $\exp(\frac{i}{\hbar}S[x])$ determined by the action. The classical path of minimum action has the smallest oscillations, and hence the largest contribution to the path integral. In the classical limit $\hbar \to 0$ only that contribution survives.

2.3 The Path Integral in Euclidean Time

As we have seen, it requires a small excursion into the complex time plane to make the path integral mathematically well-defined. Now we will make a big step into that plane and actually consider purely imaginary so-called Euclidean time. The physical motivation for this, however, comes from quantum statistical mechanics. Let us consider the quantum statistical partition function

$$Z = \mathrm{Tr}\exp(-\beta H), \tag{2.3.1}$$

where $\beta = 1/T$ is the inverse temperature. It is mathematically equivalent to the time interval we discussed in the real time path integral. In particular, the operator $\exp(-\beta H)$ turns into the time evolution operator $U(t', t)$ if we identify

$$\beta = \frac{i}{\hbar}(t' - t). \tag{2.3.2}$$

In this sense the system at finite temperature corresponds to a system propagating in purely imaginary (Euclidean) time. By dividing the Euclidean time interval into N time steps, i.e. by writing $\beta = Na/\hbar$, and again by inserting complete sets of position eigenstates we now arrive at the Euclidean time path integral

$$Z = \int \mathcal{D}x \exp(-\frac{1}{\hbar}S_E[x]). \tag{2.3.3}$$

The action now takes the Euclidean form

$$\begin{aligned} S_E[x] &= \int dt\ [\frac{m}{2}(\partial_t x)^2 + V(x)] \\ &= \lim_{a\to 0}\sum_i a[\frac{m}{2}(\frac{x_{i+1} - x_i}{a})^2 + V(x_i)]. \end{aligned} \tag{2.3.4}$$

In contrast to the real time case the measure now involves N integrations

$$\int \mathcal{D}x = \lim_{a\to 0}\sqrt{\frac{m}{2\pi\hbar a}}^N \int dx_1 \int dx_2 ... \int dx_N. \tag{2.3.5}$$

The extra integration over $x_N = x'$ is due to the trace in eq.(2.3.1). Note that there is no extra integration over $x_0 = x$ because the trace implies periodic boundary conditions in the Euclidean time direction, i.e. $x_0 = x_N$.

The Euclidean path integral allows us to evaluate thermal expectation values. For example, let us consider an operator $\mathcal{O}(x)$ that is diagonal in the position

state basis. We can insert this operator in the path integral and thus compute its expectation value

$$\langle \mathcal{O}(x) \rangle = \frac{1}{Z} \mathrm{Tr}[\mathcal{O}(x) \exp(-\beta H)] = \frac{1}{Z} \int \mathcal{D}x \; \mathcal{O}(x(0)) \exp(-\frac{1}{\hbar} S_E[x]). \quad (2.3.6)$$

Since the theory is translation invariant in Euclidean time one can place the operator anywhere in time, e.g. at $t = 0$ as done here. When we perform the low temperature limit, $\beta \to \infty$, the thermal fluctuations are switched off and only the quantum ground state $|0\rangle$ (the vacuum) contributes to the partition function, i.e. $Z \sim \exp(-\beta E_0)$. In this limit the path integral is formulated in an infinite Euclidean time interval, and describes the vacuum expectation value

$$\langle \mathcal{O}(x) \rangle = \langle 0 | \mathcal{O}(x) | 0 \rangle = \lim_{\beta \to \infty} \frac{1}{Z} \int \mathcal{D}x \; \mathcal{O}(x(0)) \exp(-\frac{1}{\hbar} S_E[x]). \quad (2.3.7)$$

It is also interesting to consider 2-point functions of operators at different instances in Euclidean time

$$\begin{aligned} \langle \mathcal{O}(x(0)) \mathcal{O}(x(t)) \rangle &= \frac{1}{Z} \mathrm{Tr}[\mathcal{O}(x) \exp(-Ht) \mathcal{O}(x) \exp(Ht) \exp(-\beta H)] \\ &= \frac{1}{Z} \int \mathcal{D}x \; \mathcal{O}(x(0)) \mathcal{O}(x(t)) \exp(-\frac{1}{\hbar} S_E[x]). \end{aligned} \quad (2.3.8)$$

Again, we consider the limit $\beta \to \infty$, but we also separate the operators in time, i.e. we also let $t \to \infty$. Then the leading contribution is $|\langle 0 | \mathcal{O}(x) | 0 \rangle|^2$. Subtracting this, and thus forming the connected 2-point function, one obtains

$$\lim_{\beta, t \to \infty} \langle \mathcal{O}(x(0)) \mathcal{O}(x(t)) \rangle - |\langle \mathcal{O}(x) \rangle|^2 = |\langle 0 | \mathcal{O}(x) | 1 \rangle|^2 \exp(-(E_1 - E_0)t). \quad (2.3.9)$$

Here $|1\rangle$ is the first excited state of the quantum system with an energy E_1. The connected 2-point function decays exponentially at large Euclidean time separations. The decay is governed by the energy gap $E_1 - E_0$. In a quantum field theory E_1 corresponds to the energy of the lightest particle. Its mass is determined by the energy gap $E_1 - E_0$ above the vacuum. Hence, in Euclidean field theory particle masses are determined from the exponential decay of connected 2-point correlation functions.

2.4 Spin Models in Classical Statistical Mechanics

So far we have considered quantum systems both at zero and at finite temperature. We have represented their partition functions as Euclidean path integrals

over configurations on a time lattice of length β. We will now make a completely new start and study classical discrete systems at finite temperature. We will see that their mathematical description is very similar to the path integral formulation of quantum systems. Still, the physical interpretation of the formalism is drastically different in the two cases. In the next section we will set up a dictionary that allows us to translate quantum physics language into the language of classical statistical mechanics.

For simplicity, let us concentrate on simple classical spin models. Here the word spin does not mean that we deal with quantized angular momenta. All we do is work with classical variables that can point in specific directions. The simplest spin model is the Ising model with classical spin variables $s_x = \pm 1$. (Again, these do not represent the quantum states up and down of a quantum mechanical angular momentum 1/2.) More complicated spin models with an $O(N)$ spin rotational symmetry are the XY model ($N = 2$) and the Heisenberg model ($N = 3$). The spins in the XY model are 2-component unit-vectors, while the spins in the Heisenberg model have three components. In all these models the spins live on the sites of a d-dimensional spatial lattice. The lattice is meant to be a crystal lattice (so typically $d = 3$) and the lattice spacing has a physical meaning. This is in contrast to the Euclidean time lattice that we have introduced to make the path integral mathematically well-defined, and that we finally send to zero in order to reach the Euclidean time continuum limit. The Ising model is characterized by its classical Hamilton function (not a quantum Hamilton operator) which simply specifies the energy of any configuration of spins. The Ising Hamilton function is a sum of nearest neighbor contributions

$$\mathcal{H}[s] = J \sum_{\langle xy \rangle} s_x s_y - \mu B \sum_x s_x, \tag{2.4.1}$$

with a ferromagnetic coupling constant $J < 0$ that favors parallel spins, plus a coupling to an external magnetic field B. The classical partition function of this system is given by

$$Z = \int \mathcal{D}s \ \exp(-\mathcal{H}[s]/T) = \prod_x \sum_{s_x = \pm 1} \exp(-\mathcal{H}[s]/T). \tag{2.4.2}$$

The sum over all spin configurations corresponds to an independent summation over all possible orientations of individual spins. Thermal averages are computed by inserting appropriate operators. For example, the magnetization is given by

$$\langle s_x \rangle = \frac{1}{Z} \prod_x \sum_{s_x = \pm 1} s_x \exp(-\mathcal{H}[s]/T). \tag{2.4.3}$$

Similarly, the spin correlation function is defined by

$$\langle s_x s_y \rangle = \frac{1}{Z} \prod_x \sum_{s_x=\pm 1} s_x s_y \exp(-\mathcal{H}[s]/T). \tag{2.4.4}$$

At large distances the connected spin correlation function typically decays exponentially

$$\langle s_x s_y \rangle - \langle s \rangle^2 \sim \exp(-|x-y|/\xi), \tag{2.4.5}$$

where ξ is the so-called correlation length. At general temperatures the correlation length is typically just a few lattice spacings. When one models real materials, the Ising model would generally be a great oversimplification, because real magnets, for example, not only have nearest neighbor couplings. Still, the details of the Hamilton function at the scale of the lattice spacing are not always important. There is a critical temperature T_c at which ξ diverges and universal behavior arises. At this temperature a second order phase transition occurs. Then the details of the model at the scale of the lattice spacing are irrelevant for the long range physics that takes place at the scale of ξ. In fact, at their critical temperatures some real materials behave just like the simple Ising model. This is why the Ising model is so interesting. It is just a very simple member of a large universality class of different models, which all share the same critical behavior. This does not mean that they have the same values of their critical temperatures. However, their magnetization goes to zero at the critical temperature with the same power of $T_c - T$, i.e. their critical exponents are identical.

2.5 Quantum Mechanics versus Statistical Mechanics

We notice a close analogy between the Euclidean path integral for a quantum mechanical system and a classical statistical mechanics system like the Ising model. The path integral for the quantum system is defined on a 1-dimensional Euclidean time lattice, just like an Ising model can be defined on a d-dimensional spatial lattice. In the path integral we integrate over all paths, i.e. over all configurations $x(t)$, while in the Ising model we sum over all spin configurations s_x. Paths are weighted by their Euclidean action $S_E[x]$ while spin configurations are weighted with their Boltzmann factors depending on the classical Hamilton function $\mathcal{H}[s]$. The prefactor of the action is $1/\hbar$, and the prefactor of the Hamilton function is $1/T$. Indeed $\hbar$ determines the strength of quantum fluctuations, while the temperature T determines the strength of thermal fluctuations. The kinetic energy $\frac{1}{2}((x_{i+1} - x_i)/a)^2$ in the path integral is analogous to the nearest neighbor spin coupling $s_x s_{x+1}$, and the potential term $V(x_i)$ is analogous to the coupling $\mu B s_x$

Quantum mechanics	Classical statistical mechanics
Euclidean time lattice	d-dimensional spatial lattice
elementary time step a	crystal lattice spacing
particle position x	classical spin variable s
particle path $x(t)$	spin configuration s_x
path integral $\int \mathcal{D}x$	sum over configurations $\prod_x \sum_{s_x}$
Euclidean action $S_E[x]$	classical Hamilton function $\mathcal{H}[s]$
Planck's constant $\hbar$	temperature T
quantum fluctuations	thermal fluctuations
kinetic energy $\frac{1}{2}(\frac{x_{i+1}-x_i}{a})^2$	neighbor coupling $s_x s_{x+1}$
potential energy $V(x_i)$	external field energy $\mu B s_x$
weight of a path $\exp(-\frac{1}{\hbar}S_E[x])$	Boltzmann factor $\exp(-\mathcal{H}[s]/T)$
vacuum expectation value $\langle\mathcal{O}(x)\rangle$	magnetization $\langle s_x\rangle$
2-point function $\langle\mathcal{O}(x(0))\mathcal{O}(x(t))\rangle$	correlation function $\langle s_x s_y\rangle$
energy gap $E_1 - E_0$	inverse correlation length $1/\xi$
continuum limit $a \to 0$	critical behavior $\xi \to \infty$

Table 2.2: *The dictionary that translates quantum mechanics into the language of classical statistical mechanics.*

to an external magnetic field. The magnetization $\langle s_x\rangle$ corresponds to the vacuum expectation value of an operator $\langle\mathcal{O}(x)\rangle$ and the spin-spin correlation function $\langle s_x s_y\rangle$ corresponds to the 2-point correlation function $\langle\mathcal{O}(x(0))\mathcal{O}(x(t))\rangle$. The inverse correlation length $1/\xi$ is analogous to the energy gap $E_1 - E_0$ (and hence to a particle mass in a Euclidean quantum field theory). Finally, the Euclidean time continuum limit $a \to 0$ corresponds to a second order phase transition where $\xi \to \infty$. The lattice spacing in the path integral is an artifact of our mathematical description which we send to zero while the physics remains constant. In classical statistical mechanics, on the other hand, the lattice spacing is physical and hence fixed, while the correlation length ξ goes to infinity at a second order phase transition. All this is summarized in the dictionary of table 2.2.

2.6 The Transfer Matrix

The analogy between quantum mechanics and classical statistical mechanics suggests that there is an analog of the quantum Hamilton operator in the context of classical statistical mechanics. This operator is the so-called transfer matrix.

The Hamilton operator induces infinitesimal translations in time. In classical statistical mechanics, on the other hand, the analog of continuous time is a 1-dimensional spatial lattice. Hence, the transfer matrix cannot induce infinitesimal space translations. Instead it induces translations by the smallest possible distance — namely by one lattice spacing. For a quantum mechanical system the transfer matrix transports us by one lattice spacing in Euclidean time, and it is given by

$$T = \exp(-\frac{a}{\hbar}H). \tag{2.6.1}$$

Now we want to construct the transfer matrix for the 1-dimensional Ising model without an external magnetic field. The corresponding partition function is given by

$$Z = \prod_x \sum_{s_x=\pm 1} \exp(\beta J \sum_x s_x s_{x+1}). \tag{2.6.2}$$

The transfer matrix obeys

$$Z = \mathrm{Tr} T^N, \tag{2.6.3}$$

where N is the number of lattice points, and its matrix elements are given by the Boltzmann factor corresponding to a nearest neighbor pair by

$$\langle s_{x+1}|T|s_x\rangle = \exp(\beta J s_x s_{x+1}). \tag{2.6.4}$$

This is a 2×2 matrix. The eigenvalues of the transfer matrix can be written as $\exp(-E_0)$ and $\exp(-E_1)$. The energy gap then determines the inverse correlation length as

$$1/\xi = E_1 - E_0. \tag{2.6.5}$$

It is instructive to compute ξ as a function of β to locate the critical point of the 1-d Ising model.

Here we will do the corresponding calculation for the 1-d xy-model. In the xy-model the spins are unit vectors $(\cos\varphi_x, \sin\varphi_x)$ in the xy-plane that are attached to the points x of a d-dimensional lattice. Here we consider $d = 1$, i.e. we study a chain of xy-spins. The standard Hamilton function of the xy-model is given by

$$\mathcal{H}[\varphi] = J \sum_{\langle xy\rangle} (1 - \cos(\varphi_{x+1} - \varphi_x)). \tag{2.6.6}$$

In complete analogy to the Ising model the transfer matrix is now given by

$$\langle \varphi_{x+1}|T|\varphi_x\rangle = \exp(-\beta J(1 - \cos(\varphi_{x+1} - \varphi_x)), \tag{2.6.7}$$

which is a matrix with an uncountable number of rows and columns, because there is a continuum of values for φ_x and φ_{x+1}. Still, we can ask about the eigenvalues of this matrix. For this purpose we consider the Fourier representation

$$\langle\varphi_{x+1}|T|\varphi_x\rangle = \sum_{m\in\mathbb{Z}} \langle\varphi_{x+1}|m\rangle \exp(-\beta J) I_m(\beta J) \langle m|\varphi_x\rangle, \tag{2.6.8}$$

where

$$\langle\varphi_x|m\rangle = \exp(im\varphi_x), \tag{2.6.9}$$

are the eigenvectors of the transfer matrix. The eigenvalues are given in terms of modified Bessel functions

$$\exp(-E_m) = \exp(-\beta J) I_m(\beta J). \tag{2.6.10}$$

The energy gap between the ground state and an excited state is given by

$$E_m - E_0 = \log \frac{I_0(\beta J)}{I_m(\beta J)}, \tag{2.6.11}$$

which is nonzero for finite β. In the zero temperature limit $\beta \to \infty$ we have

$$\frac{I_0(\beta J)}{I_m(\beta J)} \sim 1 + \frac{m^2}{2\beta J}, \tag{2.6.12}$$

such that

$$\xi = 1/(E_1 - E_0) \sim 2\beta J \to \infty. \tag{2.6.13}$$

Hence, there is a critical point at zero temperature. In the language of quantum mechanics this implies the continuum limit of a Euclidean lattice theory corresponding to a quantum mechanical problem. In the continuum limit the energies corresponding to the eigenvalues of the transfer matrix take the form

$$E_m - E_0 \sim \frac{m^2}{2\beta J}. \tag{2.6.14}$$

These energies are in lattice units (the lattice spacing was put to 1). Hence, to extract physics we need to consider energy ratios and we find

$$\frac{E_m - E_0}{E_1 - E_0} \sim m^2. \tag{2.6.15}$$

These are the appropriate energy ratios of a quantum rotor — a particle that moves on a circle. Indeed the xy-spins describe an angle, which can be interpreted as the position of the quantum particle. Also the eigenvectors of the transfer matrix are just the energy eigenfunctions of a quantum rotor. Hence, we just

solved the Schrödinger equation with a discrete Euclidean time step using the transfer matrix instead of the Hamilton operator. The fact that energy ratios approach physically meaningful constants in the continuum limit is known as scaling. Of course, the discretization introduces an error as long as we are not in the continuum limit. For example, at finite β the energy ratio is

$$\frac{E_m}{E_1} = \frac{\log(I_0(\beta J)/I_m(\beta J)}{\log(I_0(\beta J)/I_1(\beta J)}, \tag{2.6.16}$$

which is different from the continuum answer m^2. This cut-off effect due to a finite lattice spacing is known as a scaling violation.

2.7 Lattice Field Theory

So far we have restricted ourselves to quantum mechanical problems and to classical statistical mechanics. The former were defined by a path integral on a 1-d Euclidean time lattice, while the latter involved spin models on a d-dimensional spatial lattice. When we quantize field theories on the lattice, we formulate the theory on a d-dimensional space-time lattice, i.e. usually the lattice is 4-dimensional. Just as we integrate over all configurations (all paths) $x(t)$ of a quantum particle, we now integrate over all configurations $\phi(x)$ of a quantum field defined at any Euclidean space-time point $x = (\vec{x}, x_4)$. Again the weight factor in the path integral is given by the action. Let us illustrate this for a free neutral scalar field $\phi(x) \in R$. Its Euclidean action is given by

$$S_E[\phi] = \int d^4x \, [\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{m^2}{2}\phi^2]. \tag{2.7.1}$$

Interactions can be included, for example, by adding a $\frac{\lambda}{4!}\phi^4$ term to the action. The Feynman path integral for this system is formally written as

$$Z = \int \mathcal{D}\phi \exp(-S_E[\phi]). \tag{2.7.2}$$

(Note that we have put $\hbar = c = 1$.) The integral is over all field configurations, which is a divergent expression if no regularization is imposed. One can make the expression mathematically well-defined by using dimensional regularization of Feynman diagrams. This approach is, however, limited to perturbation theory. The lattice allows us to formulate field theory beyond perturbation theory, which is very essential for strongly interacting theories like QCD, but also for the standard model in general. For example, due to the heavy mass of the top quark,

the Yukawa coupling between the Higgs and top quark field is rather strong. The above free scalar field theory, of course, does not really require a nonperturbative treatment. We use it only to illustrate the lattice quantization method in a simple setting. On the lattice the continuum field $\phi(x)$ is replaced by a lattice field Φ_x, which is restricted to the points x of a d-dimensional space-time lattice. From now on we will work in lattice units, i.e. we put $a = 1$. The above continuum action can be approximated by discretizing the continuum derivatives such that

$$S_E[\Phi] = \sum_{x,\mu} \frac{1}{2}(\Phi_{x+\hat{\mu}} - \Phi_x)^2 + \sum_x \frac{m^2}{2}\Phi_x^2. \tag{2.7.3}$$

Here $\hat{\mu}$ is the unit vector in the μ-direction. The integral over all field configurations now becomes a multiple integral over all values of the field at all lattice points

$$Z = \prod_x \int_{-\infty}^{\infty} d\Phi_x \exp(-S_E[\Phi]). \tag{2.7.4}$$

For a free field theory the partition function is just a Gaussian integral. In fact, one can write the lattice action as

$$S_E[\Phi] = \frac{1}{2}\sum_{x,y} \Phi_x \mathcal{M}_{xy} \Phi_y, \tag{2.7.5}$$

where the matrix $\mathcal{M}$ describes the couplings between lattice points. Diagonalizing this matrix by a unitary transformation $\mathcal{U}$ one has

$$\mathcal{M} = \mathcal{U}^\dagger \mathcal{D} \mathcal{U}. \tag{2.7.6}$$

Introducing

$$\Phi'_x = \mathcal{U}_{xy}\Phi_y \tag{2.7.7}$$

one obtains

$$Z = \prod_x \int d\Phi'_x \exp(-\frac{1}{2}\sum_x \Phi'_x \mathcal{D}_{xx} \Phi'_x) = (2\pi)^{N/2} \mathrm{det}\mathcal{D}^{-1/2}, \tag{2.7.8}$$

where N is the number of lattice points.

To extract the energy values of the corresponding quantum Hamilton operator we need to study the 2-point function of the lattice field

$$\langle \Phi_x \Phi_y \rangle = \frac{1}{Z} \int \mathcal{D}\Phi \ \Phi_x \Phi_y \exp(-S_E[\Phi]). \tag{2.7.9}$$

This is most conveniently done by introducing a source field in the partition function, such that

$$Z[J] = \int \mathcal{D}\Phi \exp(-S_E[\Phi] + \sum_x J_x \Phi_x). \tag{2.7.10}$$

Then the connected 2-point function is given by

$$\langle \Phi_x \Phi_y \rangle - \langle \Phi \rangle^2 = \frac{\partial^2 \log Z[J]}{\partial J_x \partial J_y}|_{J=0}. \tag{2.7.11}$$

The Boltzmann factor characterizing the problem with the external sources is given by the exponent

$$\frac{1}{2}\Phi \mathcal{M} \Phi - J\Phi = \frac{1}{2}\Phi' \mathcal{M} \Phi' - \frac{1}{2} J \mathcal{M}^{-1} J. \tag{2.7.12}$$

Here we have introduced

$$\Phi' = \Phi - \mathcal{M}^{-1} J. \tag{2.7.13}$$

Integrating over Φ' in the path integral we obtain

$$Z[J] = (2\pi)^{N/2} \det \mathcal{D}^{-1/2} \exp(\frac{1}{2} J \mathcal{M}^{-1} J), \tag{2.7.14}$$

and hence

$$\langle \Phi_x \Phi_y \rangle = \frac{1}{2} \mathcal{M}^{-1}_{xy}. \tag{2.7.15}$$

It is instructive to invert the matrix $\mathcal{M}$ by going to Fourier space, i.e. by writing

$$\Phi_x = \frac{1}{(2\pi)^d} \int_B d^d p \ \Phi(p) \exp(ipx). \tag{2.7.16}$$

The momentum space of the lattice is given by the Brillouin zone $B =]-\pi, \pi]^d$. For the 2-point function in momentum space one then finds

$$\langle \Phi(-p)\Phi(p) \rangle = [\sum_\mu (2\sin(p_\mu/2))^2 + m^2]^{-1}. \tag{2.7.17}$$

This is the lattice version of the continuum propagator

$$\langle \Phi(-p)\Phi((p) \rangle = (p^2 + m^2)^{-1}. \tag{2.7.18}$$

From the lattice propagator we can deduce the energy spectrum of the lattice theory. For this purpose we construct a lattice field with definite spatial momentum $\vec{p}$ located in a specific time slice

$$\Phi(\vec{p})_t = \sum_x \Phi_{\vec{x},t} \exp(-i\vec{p} \cdot \vec{x}), \tag{2.7.19}$$

and we consider its 2-point function

$$\langle \Phi(-\vec{p})_0 \Phi(\vec{p})_t \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} dp_d \langle \Phi(-p)\Phi(p) \rangle \exp(ip_d t). \tag{2.7.20}$$

Inserting the lattice propagator of eq.(2.7.17) one can perform the integral. One encounters a pole in the propagator when $p_d = iE$ with

$$(2\sinh(E/2))^2 = \sum_i (2\sin(p_i/2))^2 + m^2. \tag{2.7.21}$$

The 2-point function then takes the form

$$\langle \Phi(-\vec{p})_0 \Phi(\vec{p})_t \rangle = C \exp(-Et), \tag{2.7.22}$$

i.e. it decays exponentially with slope E. This allows us to identify E as the energy of the lattice scalar particle with spatial momentum $\vec{p}$. In general, E differs from the correct continuum dispersion relation

$$E^2 = \vec{p}^2 + m^2. \tag{2.7.23}$$

Only in the continuum limit, i.e. when E, $\vec{p}$ and m are small in lattice units, the lattice dispersion relation agrees with the one of the continuum theory.

Chapter 3

Classical Scalar Field Theory

Scalar fields play an important role in various areas of physics. For example, the Higgs field of the standard model of particle physics is a scalar field that gives rise to the spontaneous breakdown of the $SU(2)_L \otimes U(1)_Y$ gauge group to the $U(1)_{em}$ gauge group of electromagnetism. This is how particles obtain their masses in the standard model. The lightest strongly interacting particle is the pion that arises in QCD. The pion is a so-called pseudo-Goldstone boson associated with the spontaneous breakdown of the approximate global chiral symmetry of QCD. In chiral perturbation theory, a technique developed by Gasser and Leutwyler, the pion is described by a scalar field. Furthermore, scalar fields are used to describe Cooper pairs of electrons in the condensed matter physics of superconductors. In that case, the scalar field dynamics leads to the spontaneous breaking of the $U(1)_{em}$ gauge symmetry itself. Besides being physically relevant, scalar fields are simpler to handle theoretically than fermion fields or gauge fields. This is the main reason why we begin our investigation of quantum field theory using scalar fields. It should be mentioned that there are reasons to believe that truly elementary scalar fields may not even exist. For example, the scalar field describing Cooper pairs is composed of electron fields. Similarly, at a more fundamental level the pion field of chiral perturbation theory is composed of quark-, antiquark-, and gluon fields. It is likely that the Higgs field of the standard model is also not truly fundamental. Since the Higgs particle has not yet been observed, this is remains an open question.

3.1 Scalar Fields

Scalar fields transform trivially under space-time transformations. In particular, they are invariant under Lorentz transformations. As a consequence, unlike vector fields, scalar fields do not carry any Lorentz indices. Still, scalar fields may transform non-trivially under certain internal symmetries. Such symmetries give rise to conserved charges which could be coupled to gauge fields. The simplest scalar field is neutral and has no additional indices. It is simply described by a single real number per space-time point x. A charged scalar field, such as the one representing a Cooper pair of electrons, is described by a complex number per space-time point. The scalar Higgs field of the standard model is a complex doublet. It is described by two complex (or alternatively four real) numbers per space-time point. Finally, the pion field of chiral perturbation theory has three internal degrees of freedom. It is described by a special unitary 2×2 matrix $U(x) \in SU(2)$ (i.e. with determinant 1) per space-time point x.

Let us consider an N-component scalar field $\phi^i(x)$ with $i \in \{1, 2, ..., N\}$. A neutral scalar field corresponds to $N = 1$, while a charged scalar field corresponds to $N = 2$. In case of the standard model Higgs field one has $N = 4$. The Lagrangian of the corresponding scalar field theory is given by

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2} \sum_{i=1}^{N} \partial_\mu \phi^i \partial^\mu \phi^i - V(\phi). \tag{3.1.1}$$

Here

$$V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4, \ \phi(x)^2 = \sum_{i=1}^{N} \phi^i(x)^2, \tag{3.1.2}$$

is a scalar potential that contains the so-called bare mass m of the field as well as the bare coupling constant λ of its self-interaction. Both m and λ get renormalized in the quantized theory. At this point we consider the theory at the classical level. The classical equation of motion for scalar field theory is given by

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^i} - \frac{\delta \mathcal{L}}{\delta \phi^i} = \partial_\mu \partial^\mu \phi^i + \frac{dV(\phi)}{d\phi^i} = \partial_\mu \partial^\mu \phi^i + m^2 \phi^i + \frac{\lambda}{3!}\phi^2 \phi^i = 0. \tag{3.1.3}$$

The classical vacuum configuration, i.e. the configuration of lowest energy, is simply given by $\phi^i(x) = 0$. Due to the uncertainty principle, the vacuum of scalar quantum field theory (i.e. its ground state) cannot just be given by $\phi^i(x) = 0$. This is in complete analogy to an anharmonic oscillator with potential $V(x) = \frac{1}{2}m\omega^2 x^2 + \frac{\lambda}{4!}x^4$. While the energy of the classical oscillator is minimized for $x = 0$, the quantum ground state contains quantum fluctuations around the

classical vacuum. For example, for the harmonic oscillator (with $\lambda = 0$) these fluctuations are described by a Gaussian wave function. Let us also look for non-zero solutions of the equation of motion from above. Due to the nonlinearity of the equations this is a non-trivial task. The situation simplifies significantly when we ignore the self-interaction of the scalar field and put $\lambda = 0$. The classical equation of motion of the resulting free field theory

$$\partial_\mu \partial^\mu \phi^i + m^2 \phi^i = \partial_t^2 \phi^i - \Delta \phi^i + m^2 \phi^i = 0, \tag{3.1.4}$$

is known as the Klein-Gordon equation. It admits simple plane-wave solutions

$$\phi^i(x) = \phi_0^i \exp(i(\vec{k} \cdot \vec{x} - \omega t)), \tag{3.1.5}$$

which obey the relativistic dispersion relation

$$\omega^2 = k^2 + m^2. \tag{3.1.6}$$

Here $E = \hbar\omega$ and $\vec{p} = \hbar\vec{k}$ can be identified as the energy and the momentum of a relativistic free particle with energy-momentum relation

$$E^2 = p^2 c^2 + (mc^2)^2. \tag{3.1.7}$$

Since we have put $\hbar = c = 1$ this is completely consistent with the previous equation. In particular, the parameter m in the Lagrangian can be identified as the mass of the resulting free particle. It should be pointed out that the Klein-Gordon equation admits solutions for both positive and negative energies. Upon quantization, the latter give rise to anti-particles.

3.2 Noether's Theorem

The Lagrangian of scalar field theory has an $O(N)$ symmetry, i.e. it is invariant against rotations

$$\phi(x)' = O\phi(x). \tag{3.2.1}$$

Here O is an $N \times N$ orthogonal rotation matrix, i.e. $O^T O = O O^T = 1$. In components the previous equations take the form

$$\phi^i(x)' = \sum_{j=1}^{N} O_{ij} \phi^j(x), \ \sum_{j=1}^{N} O_{ij}^T O_{jk} = \sum_{j=1}^{N} O_{ji} O_{jk} = \delta_{ik}. \tag{3.2.2}$$

Symmetries are always important because they give rise to conserved quantum numbers. It should be noted that the $O(N)$ symmetry of our scalar field theory

is global, i.e. the symmetry transformations O do not depend on space or time. This is in contrast to gauge theories in which symmetries are realized locally. Gauge theories have conserved charges. For example, electric charge is conserved both in classical and in quantum electrodynamics. In classical electrodynamics charge conservation is encoded in the continuity equation $\partial_\mu j^\mu = 0$ for the electromagnetic current $j_\mu(x) = (\rho(x), \vec{j}(x))$. Here $\rho(x)$ and $\vec{j}(x)$ are the charge and current densities, respectively.

The $O(N)$ symmetry of scalar field theory also gives rise to a conserved current. In order to derive this current we now consider local $O(N)$ transformations

$$\phi(x)' = O(x)\phi(x). \tag{3.2.3}$$

Interestingly, the potential contribution to the Lagrangian $V(\phi)$ is invariant even against these local $O(N)$ transformations. This follows simply from

$$\phi(x)'^2 = \phi(x)O(x)^T O(x)\phi(x) = \phi(x)^2. \tag{3.2.4}$$

Next we will compute the variation of the kinetic contribution to the Lagrangian with respect to infinitesimal local $O(N)$ transformations

$$O(x) = 1 + \epsilon(x). \tag{3.2.5}$$

The orthogonality of $O(x)$ implies

$$\begin{aligned} &O(x)^T O(x) = [1 + \epsilon(x)^T][1 + \epsilon(x)] \approx 1 + \epsilon(x) + \epsilon(x)^T = 1 \;\Rightarrow \\ &\epsilon(x)^T = -\epsilon(x). \end{aligned} \tag{3.2.6}$$

The infinitesimally transformed scalar field takes the form

$$\phi(x)' = [1 + \epsilon(x)]\phi(x), \tag{3.2.7}$$

and hence

$$\partial^\mu \phi(x)' = \partial^\mu \phi(x) + \partial^\mu \epsilon(x)\phi(x) + \epsilon(x)\partial^\mu \phi(x). \tag{3.2.8}$$

Consequently, one now obtains

$$\begin{aligned} \partial_\mu \phi' \partial^\mu \phi' &= \partial_\mu \phi \partial^\mu \phi + \partial_\mu \phi \partial^\mu \epsilon \phi + \partial_\mu \phi \epsilon \partial_\mu \phi \\ &+ \phi \partial_\mu \epsilon^T \partial^\mu \phi + \partial_\mu \phi \epsilon^T \partial^\mu \phi \\ &= \partial_\mu \phi \partial^\mu \phi + \partial_\mu \phi \partial^\mu \epsilon \phi - \phi \partial_\mu \epsilon \partial^\mu \phi, \end{aligned} \tag{3.2.9}$$

and the variation of the action under local infinitesimal $O(N)$ transformations takes the form

$$\begin{aligned} &S[\phi'] - S[\phi] = \int d^4x \; [\mathcal{L}(\phi', \partial_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi)] = \\ &- \int d^4x \; \sum_{i,j=1}^{N} \partial_\mu \epsilon_{ij} j^{\mu ij} = \int d^4x \; \sum_{i,j=1}^{N} \epsilon_{ij} \partial_\mu j^{\mu ij}, \end{aligned} \tag{3.2.10}$$

with the $O(N)$ Noether current given by

$$j^{\mu ij} = \phi^i \partial^\mu \phi^j - \phi^j \partial^\mu \phi^i. \tag{3.2.11}$$

Indeed, the current is conserved, i.e. $\partial_\mu j^{\mu ij} = 0$, as a consequence of the classical equations of motion. This follows directly from

$$\begin{aligned} \partial_\mu j^{\mu ij} &= \partial_\mu [\phi^i \partial^\mu \phi^j - \phi^j \partial^\mu \phi^i] = \phi^i \partial_\mu \partial^\mu \phi^j - \phi^j \partial_\mu \partial^\mu \phi^i \\ &= -\phi^i \frac{dV(\phi)}{d\phi^j} + \phi^j \frac{dV(\phi)}{d\phi^i} \\ &= \phi^i [m^2 \phi^j + \frac{\lambda}{3!} \phi^2 \phi^j] - \phi^j [m^2 \phi^i + \frac{\lambda}{3!} \phi^2 \phi^i] = 0. \end{aligned} \tag{3.2.12}$$

Note that the derivation of current conservation is similar to the one for the probability current in ordinary quantum mechanics.

Chapter 4

Canonical Quantization of a Scalar Field

Canonical quantization of field theory is a rather tedious approach. Therefore we will concentrate on the path integral for the rest of this course. However, canonical quantization has the advantage that it is rather similar to Schrödinger's approach to quantum mechanics which we are very familiar with. Here we consider the canonical quantization of a free 1-component scalar field, which is fairly easy to carry out. The complications of canonical quantization versus the path integral show up only when interactions are included.

4.1 From the Lagrange to the Hamilton Density

Let us consider a 1-component real scalar field $\phi(x)$ with the Lagrange density

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2. \tag{4.1.1}$$

The canonically conjugate momentum to the field $\phi(x)$ is given by

$$\Pi(x) = \frac{\delta \mathcal{L}}{\delta \partial_0 \phi(x)} = \partial^0 \phi(x), \tag{4.1.2}$$

which is just the time-derivative of $\phi(x)$. The classical Hamilton density is given by

$$\mathcal{H}(\phi, \Pi) = \Pi \partial^0 \phi - \mathcal{L} = \frac{1}{2}\Pi^2 + \frac{1}{2}\partial_i \phi \partial_i \phi + \frac{m^2}{2}\phi^2. \tag{4.1.3}$$

Here the index i runs over the spatial directions only. The classical Hamilton function is the spatial integral of the Hamilton density

$$H[\phi,\Pi] = \int d^3x\ \mathcal{H}(\phi,\Pi) = \int d^3x\ (\frac{1}{2}\Pi^2 + \frac{1}{2}\partial_i\phi\partial_i\phi + \frac{m^2}{2}\phi^2). \tag{4.1.4}$$

The Hamilton function is a functional of the classical field $\phi(\vec{x})$ and its canonically conjugate momentum field $\Pi(\vec{x})$. Upon quantization the Hamilton function will turn into the Hamilton operator of the corresponding quantum field theory.

4.2 Commutation Relations for Scalar Field Operators

In the canonical quantization of field theory the field values and their conjugate momenta become operators acting in a Hilbert space. As we discussed before, the field value $\phi(\vec{x})$ is analogous to the particle coordinate $\vec{x}$ in quantum mechanics. Similarly, $\Pi(\vec{x})$ is analogous to the momentum $\vec{p}$ of the particle. In quantum mechanics position and momentum do not commute

$$[x_i, p_j] = i\hbar\delta_{ij},\ [x_i, x_j] = [p_i, p_j] = 0. \tag{4.2.1}$$

Similarly, (now putting $\hbar = 1$) one postulates the following commutation relations for the field operators $\phi(\vec{x})$ and $\Pi(\vec{y})$

$$[\phi(\vec{x}), \Pi(\vec{y})] = i\delta(\vec{x}-\vec{y}),\ [\phi(\vec{x}), \phi(\vec{y})] = [\Pi(\vec{x}), \Pi(\vec{y})] = 0. \tag{4.2.2}$$

It is important to note that these commutation relations are completely local. In particular, fields at different points in space commute with each other. In quantum mechanics the momentum operator is represented as the derivative with respect to the position

$$p_i = \frac{\hbar}{i}\frac{\partial}{\partial x_i}. \tag{4.2.3}$$

Similarly, the field operator $\Pi(\vec{x})$ can be written as

$$\Pi(\vec{x}) = -i\frac{\partial}{\partial\phi(\vec{x})}, \tag{4.2.4}$$

i.e. as a derivative with respect to the field value.

4.3 Hamilton Operator in Scalar Quantum Field Theory

After turning the classical fields into operators it is straightforward to turn the classical Hamilton function $H[\phi, \Pi]$ into the quantum Hamilton operator

$$H = \int d^3x \ \frac{1}{2}(\Pi^2 + \partial_i\phi\partial_i\phi + m^2\phi^2). \tag{4.3.1}$$

At this level it should be obvious that quantum field theory is really just quantum mechanics with infinitely many degrees of freedom (in this case one for each point $\vec{x}$ in space).

As usual, solving this quantum theory amounts to diagonalizing the Hamiltonian. For this purpose it is convenient to go to momentum space. Hence, we introduce Fourier transformed fields

$$\tilde{\phi}(\vec{p}) = \int d^3x \ \phi(\vec{x}) \exp(-i\vec{p}\cdot\vec{x}), \ \tilde{\Pi}(\vec{p}) = \int d^3x \ \Pi(\vec{x}) \exp(-i\vec{p}\cdot\vec{x}). \tag{4.3.2}$$

Note that, unlike $\phi(\vec{x})$ and $\Pi(\vec{x})$, $\tilde{\phi}(\vec{p})$ and $\tilde{\Pi}(\vec{p})$ are not Hermitean but obey

$$\tilde{\phi}(\vec{p})^\dagger = \tilde{\phi}(-\vec{p}), \ \tilde{\Pi}(\vec{p})^\dagger = \tilde{\Pi}(-\vec{p}). \tag{4.3.3}$$

Using the commutations relations for $\phi(\vec{x})$ and $\Pi(\vec{y})$ one derives the commutation relations between $\tilde{\phi}(\vec{p})$ and $\tilde{\Pi}(\vec{q})$ as

$$[\tilde{\phi}(\vec{p}), \tilde{\Pi}(\vec{q})] = i(2\pi)^3\delta(\vec{p}+\vec{q}), \ [\tilde{\phi}(\vec{p}), \tilde{\phi}(\vec{q})] = [\tilde{\Pi}(\vec{p}), \tilde{\Pi}(\vec{q})] = 0. \tag{4.3.4}$$

Similarly, one can now write the Hamilton operator as

$$H = \frac{1}{(2\pi)^3}\int d^3p \ \frac{1}{2}(\tilde{\Pi}^\dagger\tilde{\Pi} + (\vec{p}^2 + m^2)\tilde{\phi}^\dagger\tilde{\phi}). \tag{4.3.5}$$

This Hamiltonian is reminiscent of the one for the harmonic oscillator with $\omega = \sqrt{\vec{p}^2 + m^2}$ playing the role of the frequency. This suggests to introduce creation and annihilation operators $a(\vec{p})^\dagger$ and $a(\vec{p})$ as

$$a(\vec{p}) = \frac{1}{\sqrt{2}}[\sqrt{\omega}\tilde{\phi}(\vec{p}) + \frac{i}{\sqrt{\omega}}\tilde{\Pi}(\vec{p})], \ a(\vec{p})^\dagger = \frac{1}{\sqrt{2}}[\sqrt{\omega}\tilde{\phi}(\vec{p})^\dagger - \frac{i}{\sqrt{\omega}}\tilde{\Pi}(\vec{p})^\dagger], \tag{4.3.6}$$

which obey the commutation relations

$$\begin{aligned} &[a(\vec{p}), a(\vec{q})^\dagger] = \frac{i}{2}[\tilde{\Pi}(\vec{p}), \tilde{\phi}(-\vec{q})] - \frac{i}{2}[\tilde{\phi}(\vec{p}), \tilde{\Pi}(-\vec{q})] = (2\pi)^3\delta(\vec{p}-\vec{q}), \\ &[a(\vec{p}), a(\vec{q})] = [a(\vec{p})^\dagger, a(\vec{q})^\dagger] = 0. \end{aligned} \tag{4.3.7}$$

In terms of these operators, the Hamiltonian takes the form

$$H = \frac{1}{(2\pi)^3} \int d^3p \ \sqrt{\vec{p}^2 + m^2} (a(\vec{p})^\dagger a(\vec{p}) + \frac{1}{2} V). \tag{4.3.8}$$

The volume factor arises from

$$\delta(\vec{p}) = \frac{1}{(2\pi)^3} \int d^3x \exp(-i\vec{p} \cdot \vec{x}) \ \Rightarrow \ (2\pi)^3 \delta(\vec{0}) = \int d^3x \ 1 = V. \tag{4.3.9}$$

4.4 Vacuum and Particle States

In analogy to a single harmonic oscillator, the vacuum state $|0\rangle$ of the scalar field theory is determined by

$$a(\vec{p})|0\rangle = 0, \tag{4.4.1}$$

for all $\vec{p}$. The vacuum is indeed an eigenstate of the Hamiltonian from above with the energy

$$E = \frac{1}{(2\pi)^3} \frac{1}{2} V \int d^3p \ \sqrt{\vec{p}^2 + m^2}. \tag{4.4.2}$$

The volume factor represents a harmless infrared divergence. It is natural in a field theory that the energy of the vacuum is proportional to the spatial volume. However, even the energy density

$$\frac{E}{V} = \frac{1}{(2\pi)^3} \frac{1}{2} \int d^3p \ \sqrt{\vec{p}^2 + m^2} = \frac{1}{4\pi^2} \int_0^\infty dp \ p^2 \sqrt{\vec{p}^2 + m^2}, \tag{4.4.3}$$

is still divergent in the ultraviolet. This is a typical short-distance (i.e. high-momentum) divergence of field theory. The theory must be regularized in order to make the vacuum energy density finite. This can be achieved, for example, by introducing a momentum cut-off Λ. The regularized vacuum energy density

$$\rho = \frac{E}{V} = \frac{1}{4\pi^2} \int_0^\Lambda dp \ p^2 \sqrt{\vec{p}^2 + m^2} \sim \Lambda^4, \tag{4.4.4}$$

of course, again diverges in the limit $\Lambda \to \infty$. The vacuum energy of field theory gives rise to one of the greatest mysteries in physics — the cosmological constant problem. When one couples classical gravity, i.e. Einstein's general relativity, to quantum field theory, the vacuum energy manifests itself as a cosmological constant. Recent observations have shown that the cosmological constant in Nature is extremely small but still positive. This leads to an accelerated expansion of the Universe, instead of the deceleration expected for a matter-dominated Universe.

Understanding the origin of the vacuum energy density is one of the most challenging questions in physics today. A naive consideration from field theory would perhaps identify the cut-off Λ with the Planck scale $M_P = 1/\sqrt{G} \approx 10^{18}$GeV, at which quantum effects of gravity become important. Here G is Newton's constant. Hence, a naive field theoretical estimate of the cosmological constant is about

$$\rho_{theory} \approx M_P^4. \tag{4.4.5}$$

The observed cosmological constant, on the other hand, is

$$\rho_{observation} \approx (10^{-3}\text{eV})^4, \tag{4.4.6}$$

such that

$$\frac{\rho_{theory}}{\rho_{observation}} \approx 10^{120}. \tag{4.4.7}$$

This is presumably the greatest discrepancy between theory and observation ever encountered in all of physics. To explain this discrepancy is the essence of the cosmological constant problem.

From a pure particle physics point of view, i.e. ignoring the gravitational effect of the vacuum energy, the divergence of ρ is rather harmless. In particular, the energies of particles (which are excitations above the vacuum) are differences between the energy of an excited state and the ground state (the vacuum). In these energy differences the divergent factor drops out. In particular, the single particle states of the theory are given by

$$|\vec{p}\rangle = a(\vec{p})^\dagger |0\rangle, \tag{4.4.8}$$

with an energy

$$E(\vec{p}) = \omega = \sqrt{\vec{p}^2 + m^2}. \tag{4.4.9}$$

This is indeed the energy of a free particle with rest mass m and momentum $\vec{p}$. More precisely, $E(\vec{p})$ is the finite energy difference between the divergent energy of the excited state and the ground state. Multi-particle states can be obtained by acting with more than one particle creation operator on the vacuum state. For example, the 2-particle states are obtained as

$$|\vec{p}_1, \vec{p}_2\rangle = a(\vec{p}_1)^\dagger a(\vec{p}_2)^\dagger |0\rangle, \tag{4.4.10}$$

Since $[a(\vec{p}_1)^\dagger, a(\vec{p}_2)^\dagger] = 0$ one finds

$$|\vec{p}_2, \vec{p}_1\rangle = |\vec{p}_1, \vec{p}_2\rangle, \tag{4.4.11}$$

i.e. the 2-particle state is symmetric under particle permutation. The same is true for multi-particle states. This shows that the scalar particles of our field theory are indeed indistinguishable identical bosons.

4.5 The Momentum Operator

Let us also consider the momentum operator of our theory. Energy and momentum are components of the so-called energy-momentum tensor

$$\theta_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}. \tag{4.5.1}$$

The Hamilton density from before is given by

$$\mathcal{H} = \theta_{00} = \partial_0\phi\partial_0\phi - g_{00}\mathcal{L} = \Pi^2 - \mathcal{L}. \tag{4.5.2}$$

Similarly, the momentum density is given by

$$\mathcal{P} = \theta_{0i} = \partial_0\phi\partial_i\phi - g_{0i}\mathcal{L} = \Pi\partial_i\phi. \tag{4.5.3}$$

Accordingly, in quantum field theory, a Hermitean momentum operator is constructed as

$$\begin{aligned} P_i &= \int d^3x\ \frac{1}{2}(\Pi\partial_i\phi + \partial_i\phi\Pi) \\ &= \frac{1}{(2\pi)^3}\int d^3p\ ip_i\frac{1}{2}(\tilde{\Pi}(\vec{p})\tilde{\phi}(-\vec{p}) + \tilde{\phi}(-\vec{p})\tilde{\Pi}(\vec{p})) \\ &= \frac{1}{(2\pi)^3}\int d^3p\ p_i a(\vec{p})^\dagger a(\vec{p}). \end{aligned} \tag{4.5.4}$$

The vacuum $|0\rangle$ is an eigenstate of the momentum operator with eigenvalue $\vec{0}$. Hence, as one would expect, the vacuum has zero momentum and is, consequently, translation invariant. The single particle states are again eigenstates,

$$\vec{P}|\vec{p}\rangle = \vec{p}|\vec{p}\rangle, \tag{4.5.5}$$

which shows that $\vec{p}$ is indeed the momentum of the particle.

Chapter 5

Path Integral for Scalar Field Theory

As we have seen, the quantum mechanical path integral is particularly well defined in Euclidean time. Even the path integral in Minkowski real-time requires an infinitesimal excursion into the complex time plane. In addition, the Euclidean path integral is intimately related to quantum statistical mechanics. Hence, we now consider quantum field theory in the framework of the Euclidean path integral. This has the additional advantage that this formulation also works beyond perturbation theory. In particular, lattice field theory is also formulated in Euclidean time. Just like the paths of a particle contributing to the quantum mechanical path integral in Euclidean time, Euclidean fields themselves are not directly physical objects. Instead they just serve as integration variables which allow us to derive physical quantities such as particle masses and coupling constants. There is a rigorous connection between the Euclidean time path integral and the real world. In particular, it yields the same physical results as the path integral formulated in Minkowski space-time. However, one should not confuse the Euclidean field configurations with time-evolutions of physical fields in real time.

5.1 From Minkowski to Euclidean Space-Time

Let us start from the Lagrangian of a scalar field in Minkowski space-time

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi) = \frac{1}{2}(\partial_t \phi \partial_t \phi - \partial_i \phi \partial_i \phi) - V(\phi), \qquad (5.1.1)$$

which gives rise to the action

$$S[\phi] = \int dt d^3x \ \mathcal{L}(\phi, \partial_\mu \phi). \tag{5.1.2}$$

The path integral in Minkowski space-time is given by the formal expression

$$Z = \int \mathcal{D}\phi \ \exp(iS[\phi]). \tag{5.1.3}$$

We now analytically continue the time coordinate to purely imaginary values

$$x_4 = it. \tag{5.1.4}$$

The Lagrangian then takes the form

$$\mathcal{L}(\phi, \partial_\mu \phi) = -[\frac{1}{2}(\partial_4 \phi \partial_4 \phi + \partial_i \phi \partial_i \phi) + V(\phi)] = -[\frac{1}{2}\partial_\mu \phi \partial_\mu \phi + V(\phi)]. \tag{5.1.5}$$

In Euclidean space-time the distinction between co- and contra-variant indices is no longer necessary because the metric is simply given by $g_{\mu\nu} = \delta_{\mu\nu}$. Due to the time-integration the action picks up an additional factor i and now takes the form

$$S[\phi] = i \int d^4x \ [\frac{1}{2}\partial_\mu \phi \partial_\mu \phi + V(\phi)], \tag{5.1.6}$$

where $d^4x = dx_4 d^3x$. The path integral in Euclidean time now takes the form

$$Z = \int \mathcal{D}\phi \ \exp(-S_E[\phi]), \tag{5.1.7}$$

with the Euclidean action given by

$$S_E[\phi] = \int_0^\beta dx_4 \int d^3x \ [\frac{1}{2}\partial_\mu \phi \partial_\mu \phi + V(\phi)]. \tag{5.1.8}$$

We have introduced a finite extent $\beta = 1/T$ of the periodic Euclidean time dimension, which puts the field theory at the finite temperature T. The path integral Z is nothing but the corresponding partition function of quantum statistical mechanics. It should be noted that, as it stands, the path integral Z is a highly divergent formal expression which needs to be regularized and properly renormalized. We have already seen how this can be done nonperturbatively by using the lattice regularization. While it is unavoidable in nonperturbative calculations, for perturbative calculations the lattice regularization is not the most convenient choice. It is much simpler to use dimensional regularization, i.e. an analytic continuation in the dimension of space-time. This is what we will concentrate on in

what follows. From now on we will keep the space-time dimension d as a continuous complex parameter. Then the formal expression for the Euclidean action reads

$$S_E[\phi] = \int d^d x \ [\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + V(\phi)], \tag{5.1.9}$$

and the corresponding partition function — including an external field $j(x)$ — takes the form

$$Z[j] = \int \mathcal{D}\phi \ \exp(-S_E[\phi] + \int d^d x \ j\phi). \tag{5.1.10}$$

Of course, one should not forget that this approach is limited to perturbation theory, and does not define quantum field theory nonperturbatively.

5.2 Euclidean Propagator and Contraction Rule

In Euclidean field theory, physical information is extracted from n-point correlation functions, which correspond to vacuum expectation values of time-ordered products of field operators

$$\langle 0|T\phi(x_1)\phi(x_2)...\phi(x_n)|0\rangle = \frac{1}{Z}\int \mathcal{D}\phi \ \phi(x_1)\phi(x_2)...\phi(x_n)\exp(-S_E[\phi]). \tag{5.2.1}$$

For example, the 2-point function can be obtained as

$$\begin{aligned}\langle 0|T\phi(x_1)\phi(x_2)|0\rangle &= \frac{1}{Z}\int \mathcal{D}\phi \ \phi(x_1)\phi(x_2)\exp(-S_E[\phi]) \\ &= \frac{1}{Z}\frac{d^2 Z[j]}{dj(x_1)dj(x_2)}|_{j=0}.\end{aligned} \tag{5.2.2}$$

In complete analogy to the lattice calculation discussed before, for a free scalar field one obtains

$$Z[j] = Z\exp[\frac{1}{2}\int d^d x d^d y \ j(x)G(x-y)j(y)], \tag{5.2.3}$$

where $G(x)$ is the Euclidean propagator. In momentum space it takes the form

$$G(p) = \int d^d x \ G(x)\exp(-ipx) = \frac{1}{p^2+m^2}. \tag{5.2.4}$$

For a free field the n-point functions are simply related to the 2-point function.

For example, the 4-point function takes the form

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = \frac{1}{Z}\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\exp(-S_E[\phi]) = \frac{1}{Z}\frac{d^4Z[j]}{dj(x_1)dj(x_2)dj(x_3)dj(x_4)}|_{j=0}. \tag{5.2.5}$$

Using eq.(5.2.3) one finds

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = G(x_1-x_2)G(x_3-x_4) + G(x_1-x_3)G(x_2-x_4) + G(x_1-x_4)G(x_2-x_3). \tag{5.2.6}$$

In a Feynman diagram the propagators are represented as lines connecting the external points x_i. This is an example of a general contraction rule for n-point functions with even n (the n-point functions for odd n simply vanish)

$$\langle 0|T\phi(x_1)\phi(x_2)...\phi(x_n)|0\rangle = \sum_{contractions} G(x_{i_1}-x_{i_2})G(x_{i_3}-x_{i_4})...G(x_{i_{n-1}}-x_{i_n}). \tag{5.2.7}$$

The sum extends over all partitions of the indices $1, 2, ..., n$ into pairs (i_1, i_2), (i_3, i_4), ..., (i_{n-1}, i_n). There are $(n-1)!! = 1 \cdot 3 \cdot 5 \cdot ... \cdot (n-1)$ such pairings. For $n = 4$ there are indeed $(4-1)!! = 1 \cdot 3 = 3$ contractions.

5.3 Perturbative Expansion of the Path Integral

Let us divide the Euclidean action into a free and an interacting part

$$S_E[\phi] = S_f[\phi] + S_i[\phi] \tag{5.3.1}$$

with

$$S_f[\phi] = \int d^dx\ \frac{1}{2}[\partial_\mu\phi\partial_\mu\phi + m^2\phi^2], \tag{5.3.2}$$

and

$$S_i[\phi] = \int d^dx\ \frac{1}{4!}\phi^4. \tag{5.3.3}$$

The interaction term makes it impossible to compute the full path integral analytically because it is no longer Gaussian. In perturbation theory one assumes that the coupling constant λ is small and one expands

$$\exp(-S_E[\phi]) = \exp(-S_f[\phi])[1 + S_i[\phi] + \frac{1}{2}S_i[\phi]^2 + ...]. \tag{5.3.4}$$

Inserting this expansion into the path integral, the expression for the n-point function takes the form

$$\langle 0|T\phi(x_1)\phi(x_2)...\phi(x_n)|0\rangle = \frac{1}{Z}\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)...\phi(x_n)\times$$
$$\sum_{k=0}^{\infty}\frac{\lambda^k}{(4!)^k k!}\int d^dv_1 d^dv_2...d^dv_k\ \phi(v_1)^4\phi(v_2)^4...\phi(v_k)^4\exp(-S_f[\phi]). \quad (5.3.5)$$

This expression is similar to the one for the n-point function in the free theory. However, in addition to the external points x_1, x_2, ..., x_n, we now also have k internal points v_1, v_2, ..., v_k whose positions are independently integrated over all of space-time and which are known as interaction vertices. These are the points at which the field ϕ experiences its self-interaction.

As before, we can apply the contraction rule. However, we now have four fields $\phi(v_i)$ at each interaction vertex. Correspondingly, there are now contractions that connect a vertex back to itself via a propagator $G(v_i - v_i) = G(0)$. In a Feynman diagram such propagators appear as internal lines, while the propagators connected to external points x_i are denoted as external lines. In four dimensions the propagator at zero distance $G(0)$ is an ultraviolet divergent quantity. This divergence is regularized by analytically continuing the dimension of space-time.

5.4 Dimensional Regularization

Let us consider the propagator in d-dimensional space-time

$$G(x) = \frac{1}{(2\pi)^d}\int d^dp\ \frac{\exp(ipx)}{p^2+m^2}. \quad (5.4.1)$$

Using

$$\frac{1}{p^2+m^2} = \int_0^\infty \exp(-t(p^2+m^2)), \quad (5.4.2)$$

we can write

$$\begin{aligned} G(x) &= \frac{1}{(2\pi)^d}\int_0^\infty \exp(-tm^2)\int d^dp\ \exp(ipx - tp^2) \\ &= \frac{1}{(2\pi)^d}\int_0^\infty \exp(-tm^2 - \frac{x^2}{4t})\int d^dq\ \exp(-tq^2) \\ &= \frac{1}{(4\pi)^{d/2}}\int_0^\infty t^{-d/2}\exp(-tm^2 - \frac{x^2}{4t}) \\ &= \frac{1}{(2\pi)^{d/2}} m^{d-2}(m|x|)^{1-d/2}K_{1-d/2}(m|x|). \end{aligned} \tag{5.4.3}$$

Here $K_\nu(z)$ is a Bessel function. For $|x| \neq 0$ the propagator is finite for any value of d. However, for $|x| = 0$ we have

$$G(0) = \frac{1}{(4\pi)^{d/2}}\int_0^\infty t^{-d/2}\exp(-tm^2) = \frac{1}{(4\pi)^{d/2}} m^{d-2}\Gamma(1-\frac{d}{2}), \tag{5.4.4}$$

which diverges for $d = 2, 4, 6, ...$ but is regular for other values of the space-time dimension. It should be noted that the above integral over t converges only for $d < 2$. However, the result of the integral can still be analytically continued to general d, except to even integer dimensions. Near $d = 4$ the Γ-function takes the form

$$\Gamma(1-\frac{d}{2}) = \frac{2}{d-4} - \Gamma'(1) - 1 + \mathcal{O}(d-4), \tag{5.4.5}$$

which reveals the ultraviolet singularity as a pole in the space-time dimension at $d = 4$. Of course, the analytic continuation in the space-time dimension is just a mathematical trick that makes the propagator well-defined in the ultraviolet limit. Unlike the lattice regularization, dimensional regularization is difficult to interpret physically. Still, it is a rather elegant way to make the formal expressions of continuum field theory mathematically well-defined — at least in perturbation theory. Furthermore, it yields the same results in the perturbative continuum limit as e.g. the lattice regularization or other regularization schemes, but is much easier to handle. It is remarkable and reassuring that the physics is ultimately regularization-independent.

5.5 The 2-Point Function to Order λ

We will now use perturbation theory to evaluate the 2-point function to order λ. One obtains

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \frac{1}{Z}\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)[1 - \frac{\lambda}{4!}\int d^dv_1\ \phi(v)^4]\exp(-S_f[\phi]). \tag{5.5.1}$$

Also the partition function itself must be expanded accordingly, i.e.

$$Z = \int \mathcal{D}\phi \; [1 - \frac{\lambda}{4!} \int d^d v \; \phi(v)^4] \exp(-S_f[\phi]), \tag{5.5.2}$$

which implies

$$\begin{aligned} \langle 0|T\phi(x_1)\phi(x_2)|0\rangle &= \frac{1}{Z_f} \int \mathcal{D}\phi \; \phi(x_1)\phi(x_2) \exp(-S_f[\phi]) \\ &- \frac{1}{Z_f} \int \mathcal{D}\phi \; \phi(x_1)\phi(x_2) \frac{\lambda}{4!} \int d^d v \; \phi(v)^4 \exp(-S_f[\phi]) \\ &+ \frac{1}{Z_f} \int \mathcal{D}\phi \; \frac{\lambda}{4!} \int d^d v \; \phi(v)^4 \exp(-S_f[\phi]) \\ &\times \frac{1}{Z_f} \int \mathcal{D}\phi \; \phi(x_1)\phi(x_2) \exp(-S_f[\phi]). \end{aligned} \tag{5.5.3}$$

Here

$$Z_f = \int \mathcal{D}\phi \; \exp(-S_f[\phi]), \tag{5.5.4}$$

is the partition function of the free theory. The first term in eq.(5.5.3) is the free propagator, and the other terms are corrections of order λ. Let us use the contraction rule to evaluate these terms. In the second term $\phi(x_1)$ can be contracted either with $\phi(x_2)$ or with one of the factors in $\phi(v)^4$, for which there are four possibilities. Hence, we obtain

$$\begin{aligned} &\frac{1}{Z_f} \int \mathcal{D}\phi \; \phi(x_1)\phi(x_2) \int d^d v \; \phi(v)^4 \exp(-S_f[\phi]) = \\ &G(x_1 - x_2) \frac{1}{Z_f} \int \mathcal{D}\phi \; \int d^d v \; \phi(v)^4 \exp(-S_f[\phi]) + \\ &\int d^d v \; G(x_1 - v) \frac{1}{Z_f} \int \mathcal{D}\phi \; \phi(x_2)\phi(v)^3 \exp(-S_f[\phi]). \end{aligned} \tag{5.5.5}$$

In the last term there are three ways to contract $\phi(x_2)$ with one of the factors in $\phi(v)^3$ such that

$$\frac{1}{Z_f} \int \mathcal{D}\phi \; \phi(x_2)\phi(v)^3 \exp(-S_f[\phi]) = 3G(v - x_2)G(0). \tag{5.5.6}$$

Similarly, by putting $x_2 = v$ in this expression, we obtain

$$\frac{1}{Z_f} \int \mathcal{D}\phi \; \phi(v)^4 \exp(-S_f[\phi]) = 3G(0)^2. \tag{5.5.7}$$

Inserting these relations into eq.(5.5.5) one finds

$$\frac{1}{Z_f}\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)\int d^d v\ \phi(v)^4 \exp(-S_f[\phi]) =$$
$$3G(x_1-x_2)G(0)^2\int d^d v\ 1 + 3G(0)\int d^d v\ G(x_1-v)G(v-x_2). \quad (5.5.8)$$

Similarly, the last term in eq.(5.5.3) gives rise to

$$\frac{1}{Z_f}\int \mathcal{D}\phi\ \int d^d v\ \phi(v)^4 \exp(-S_f[\phi]) \times$$
$$\frac{1}{Z_f}\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)\exp(-S_f[\phi]) = 3G(x_1-x_2)G(0)^2\int d^d v\ 1. \quad (5.5.9)$$

It should be noted that, like another term obtained before, this term is infrared divergent, i.e. it goes to infinity with the space-time volume $\int d^d v\ 1$. However, the two infrared divergent terms cancel. Both terms arise from so-called vacuum bubble diagrams. In those diagrams there are pieces completely disconnected from any external point x_i. In that case, the integral over the corresponding vertex v_k is unsuppressed and diverges with the space-time volume. One can prove that the contributions of vacuum bubbles to general n-point functions always cancel. We will not present the formal proof for this, but we will use this result and always drop vacuum bubble diagrams.

Putting everything together, one thus obtains

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = G(x_1-x_2) - \frac{\lambda}{2}G(0)\int d^d v\ G(x_1-v)G(v-x_2). \quad (5.5.10)$$

The integral over v corresponds to a convolution in coordinate space, which after a Fourier transform turns into a simple product in momentum space. Hence, in momentum space the previous equation takes the form

$$G'(p) = \frac{1}{p^2+m^2} - \frac{\lambda}{2}G(0)\frac{1}{(p^2+m^2)^2} + \mathcal{O}(\lambda^2) = \frac{1}{p^2+m'^2} + \mathcal{O}(\lambda^2). \quad (5.5.11)$$

Here $G'(p)$ is the full propagator to order λ and m' is the mass also corrected to that order. One reads off

$$m'^2 = m^2 + \frac{\lambda}{2}G(0) = m^2 + \frac{\lambda}{2}\frac{1}{(4\pi)^{d/2}}m^{d-2}\Gamma(1-\frac{d}{2}). \quad (5.5.12)$$

5.6 Mass Renormalization

We have seen that the mass m of the free scalar particle changes to m' when the ϕ^4 interaction is taken into account at leading order in λ. Although eq.(5.5.12) yields a finite value for m' for almost all values of d, m' still diverges when one approaches the physical space-time dimension $d = 4$. This is a manifestation of the ultraviolet divergences of field theory which, unlike infrared divergences, do not simply cancel. In particular, as it stands, the expression for m' still diverges in the physical limit. To cure this problem, in addition to regularizing the ultraviolet divergence (in this case, by an analytic continuation in the dimension of space-time), we must now renormalize the bare mass parameter m. As we have seen, in the presence of the ϕ^4 interaction, to order λ, the physical mass is now m' and no longer m. Hence, m' is physical and should thus be finite, while m is a so-called bare (i.e. unrenormalized) parameter that appears in the Lagrangian, but has no direct physical meaning. Renormalization of the mass means that we let the unphysical bare parameter m depend on the cut-off parameter (i.e. in this case on $d - 4$) such that the physical renormalized mass m' remains fixed. In particular, we now put

$$\begin{aligned} m^2 &= m'^2 - \frac{\lambda}{2}G(0) = m'^2 - \frac{\lambda}{2}\frac{1}{(4\pi)^{d/2}}m'^{d-2}\Gamma(1-\frac{d}{2}) \\ &= m'^2 - \frac{\lambda}{2}\frac{1}{(4\pi)^{d/2}}m'^{d-2}[\frac{2}{d-4} - \Gamma'(1) - 1 + \mathcal{O}(d-4)], \end{aligned} \quad (5.6.1)$$

which means that m depends on d. In particular, now m^2 diverges (it has a pole at $d = 4$) while the physical mass m' remains finite.

At the end we have traded an unphysical bare parameter m for a physical mass m'. However, the theory itself does not predict the physical value of the mass. Just as we could choose any value of m in the free theory, we can now choose any value of m' for the interacting theory.

5.7 Connected and Disconnected Diagrams

It is useful to classify Feynman diagrams according to their topology. An n-point function $\langle 0|T\phi(x_1)\phi(x_2)...\phi(x_n)|0\rangle$ is represented by diagrams with n external points x_i, a number k of internal vertices at points v_i, and a number of propagators represented by lines connecting the points. There is one line emanating from each external point and there are four lines running into each

vertex. Vertices that are not connected directly or indirectly to any external points belong to vacuum bubbles. Feynman diagrams that contain vacuum bubbles cancel and thus need not be considered. In the following we limit ourselves to diagrams without vacuum bubbles. Such diagrams decompose into various connected pieces. A connected piece is characterized by the set of external points that it contains. Those are connected to each other by propagators either directly or indirectly via some vertices. All vertices belong to some connected piece, otherwise there would be vacuum bubbles. A Feynman diagram that contains more than one connected piece is called disconnected. The contribution of such a diagram to the n-point function factorizes into contributions from each individual connected piece. Hence, the problem of computing a general n-point function reduces to the evaluation of connected diagrams. For example, the 2-point function $\langle 0|T\phi(x_1)\phi(x_2)|0\rangle$ automatically receives contributions from connected diagrams only. The 4-point function $\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle$, on the other hand, gets contributions from various connected pieces corresponding to the partitions $[(x_1,x_2),(x_3,x_4)]$, $[(x_1,x_3),(x_2,x_4)]$, $[(x_1,x_4),(x_2,x_3)]$, as well as from contributions in which all four external points are connected with each other.

A useful category of connected diagrams are the 1-particle irreducible ones. Those remain connected when any single propagator line in the diagram is cut. Diagrams that fall apart into two disconnected pieces under the cutting operation are 1-particle reducible. The full 2-point function gets contributions from chains of 1-particle irreducible subgraphs connected by single propagators. These terms build a geometric series. The first term of the series is the free propagator $(p^2+m^2)^{-1}$. The second term contains the remaining 1-particle irreducible diagrams and takes the form $-(p^2+m^2)^{-1}\Sigma(p^2)(p^2+m^2)^{-1}$, where $\Sigma(p^2)$ is the so-called self-energy. The third term is given by $(p^2+m^2)^{-1}\Sigma(p^2)(p^2+m^2)^{-1}\Sigma(p^2)(p^2+m^2)^{-1}$, etc., such that the full propagator is given by

$$\begin{aligned} G'(p) &= \frac{1}{p^2+m^2} - \frac{1}{p^2+m^2}\Sigma(p^2)\frac{1}{p^2+m^2} \\ &+ \frac{1}{p^2+m^2}\Sigma(p^2)\frac{1}{p^2+m^2}\Sigma(p^2)\frac{1}{p^2+m^2} - ... \\ &= \frac{1}{p^2+m^2}\frac{1}{1+\Sigma(p^2)/(p^2+m^2)} = \frac{1}{p^2+m^2+\Sigma(p^2)}. \end{aligned} \tag{5.7.1}$$

One can now read off the physical mass m' of the particle from the value $p^2 = -m'^2$ for which the full propagator has a pole, i.e.

$$m'^2 = m^2 + \Sigma(-m'^2). \tag{5.7.2}$$

The perturbative calculation from before implies

$$\Sigma(p^2) = \frac{\lambda}{2}\frac{1}{(4\pi)^{d/2}} m^{d-2}\Gamma(1-\frac{d}{2}) + \mathcal{O}(\lambda^2). \tag{5.7.3}$$

The full propagator can be expanded around the pole at $p^2 = -m'^2$ and one obtains

$$G'(p) = \frac{Z}{p^2 + m'^2}, \tag{5.7.4}$$

where Z is the residue of the pole. Expanding

$$\Sigma(p^2) = \Sigma(-m'^2) + (p^2 + m'^2)\Sigma'(-m'^2) + ... \tag{5.7.5}$$

one identifies

$$Z = \frac{1}{1 - \Sigma'(-m'^2)}. \tag{5.7.6}$$

The factor $\sqrt{Z}$ is known as the wave function renormalization constant.

5.8 Feynman Rules for ϕ^4 Theory

It is convenient to consider the Fourier transform of the n-point function

$$\begin{aligned}&\int d^dx_1 d^dx_2 ... d^dx_n \ \langle 0|T\phi(x_1)\phi(x_2)...\phi(x_n)|0\rangle \\ &\times \exp[i(p_1x_1 + p_2x_2 + ... + p_nx_n)] = \\ &(2\pi)^d\delta(p_1 + p_2 + ... + p_n)\Gamma(p_1, p_2, ..., p_n).\end{aligned} \tag{5.8.1}$$

The δ-function results from translation invariance. Dropping Feynman diagrams containing vacuum bubbles or disconnected pieces, in perturbation theory one then obtains

$$\begin{aligned}&(2\pi)^d\delta(p_1 + p_2 + ... + p_n)\Gamma(p_1, p_2, ..., p_n) = \\ &\frac{1}{Z_f}\int \mathcal{D}\phi \ \tilde{\phi}(p_1)\tilde{\phi}(p_2)...\tilde{\phi}(p_n)\exp(-S_f[\phi]) \times \\ &\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{(4!)^k k!}\int d^dv_1 d^dv_2 ... d^dv_k \ \phi(v_1)^4\phi(v_2)^4...\phi(v_k)^4.\end{aligned} \tag{5.8.2}$$

Here $\tilde{\phi}(p)$ is the Fourier transform of the field $\phi(x)$. Since we are interested in connected diagrams only, we must contract a field $\tilde{\phi}(p)$ carrying an external

momentum p with a field $\phi(v)$ at an internal vertex v. The resulting contraction takes the form

$$\frac{1}{Z_f}\int \mathcal{D}\phi\ \tilde{\phi}(p)\phi(v)\exp(-S_f[\phi]) = \frac{\exp(ipv)}{p^2+m^2}. \tag{5.8.3}$$

In addition, the remaining fields $\phi(v_i)$ must be contracted among themselves, which implies

$$\frac{1}{Z_f}\int \mathcal{D}\phi\ \phi(v_i)\phi(v_j)\exp(-S_f[\phi]) = \frac{1}{(2\pi)^d}\int d^dq \frac{\exp(-iq(v_i-v_j))}{p^2+m^2}. \tag{5.8.4}$$

The momentum q associated with the internal line connecting the vertices v_i and v_j must be integrated over. After performing the contractions, the positions of the vertices v_i appear only in exponential factors. Hence, when one integrates over v_i one generates a δ-function for the four momenta flowing into that vertex.

These observations lead to the following Feynman rules for the evaluation of connected n-point functions in a ϕ^4 theory:

• Consider all pairwise contractions in the product of internal and external fields $\tilde{\phi}(p_1)\tilde{\phi}(p_2)...\tilde{\phi}(p_n)\phi(v_1)^4\phi(v_2)^4...\phi(v_k)^4$ that lead to a connected Feynman diagram. Each diagram has a multiplicity factor that counts the number of pairings leading to the same topology of the diagram. Also one must take into account the factor $1/k!$.

• Write down a propagator $(p_i^2+m^2)^{-1}$ for each oriented external line.

• Associate a momentum q_i with each oriented internal line and write down a factor $(q_i^2+m^2)^{-1}$.

• Write down a momentum conserving δ-function $(2\pi)^d\delta(q_1+q_2+q_3+q_4)$ for each vertex v_i taking into account the orientation of the internal or external lines. To each vertex is also associated a factor $-\lambda/4!$.

• Finally, integrate the resulting expression over all internal momenta, i.e. write $\int d^dq_1...d^dq_I/(2\pi)^{Id}$.

The number I of internal lines can be determined as follows. There are four lines emanating from each vertex, n of which are external. Each internal line connects two vertices and thus

$$I = \frac{1}{2}(4k-n). \tag{5.8.5}$$

There are I integrations and k δ-functions. However, one of the δ-functions reflects translation invariance and will remain in the final result. Hence, only $k-1$

δ-functions can be used to perform some integrations trivially. The remaining

$$l = I - (k-1) = k + 1 - \frac{n}{2} \tag{5.8.6}$$

integrations determine the number of loops l in the diagram. The number of loops increases with the order k of the expansion and also depends on the number of external momenta n of the n-point function. For example, at order λ (i.e. for $k = 1$) a 2-point function (with $n = 2$) receives a 1-loop contribution, while at the same order a 4-point function has no contributions from loop diagrams. Diagrams without loops (i.e. with $l = 0$) are called tree diagrams. The main difficulty in evaluating Feynman diagrams is to perform the loop-integrations over internal momenta. While tree diagrams are easy to evaluate, multi-loop diagrams are much harder to deal with.

5.9 The 4-Point Function to Order λ^2

Let us begin with something simple, the tree diagram for the 4-point function. There is just one vertex and all four external momenta flow into that vertex. There are four possible contractions for the external line with momentum p_1, three remaining possible contractions for the external line with momentum p_2, two remaining possible contractions for the external line with momentum p_3, and finally only one possible contraction for the external line with momentum p_4. Hence, the multiplicity factor is $4 \times 3 \times 2 = 4!$. Following the Feynman rules we obtain

$$\begin{aligned}
&(2\pi)^d \delta(p_1 + p_2 + p_3 + p_4)\Gamma(p_1, p_2, p_3, p_4) = \\
&4! \frac{1}{(p_1^2 + m^2)(p_2^2 + m^2)(p_3^2 + m^2)(p_4^2 + m^2)} \times \\
&\frac{(-\lambda)}{4!}(2\pi)^d \delta(p_1 + p_2 + p_3 + p_4),
\end{aligned} \tag{5.9.1}$$

such that

$$\Gamma(p_1, p_2, p_3, p_4) = -\frac{\lambda}{(p_1^2 + m^2)(p_2^2 + m^2)(p_3^2 + m^2)(p_4^2 + m^2)}. \tag{5.9.2}$$

In the next order $k = 2$ we need to study the 1-loop diagrams. Now there are two vertices. If all external lines would flow into the same vertex, the diagram would be disconnected. Hence, either three or two external lines may run into the same vertex. First, we consider the case where the external line with the

momentum p_1 runs into one vertex and the other external lines (with momenta p_2, p_3, and p_4) run into the other vertex. There are 8 possible contractions for the first external line with momentum p_1. Then there are $4 \times 3 \times 2$ possible contractions of the three other external lines at the other vertex. Finally, there are 3 contractions for an internal line with momentum q_1 connecting the two vertices and one remaining internal line with momentum q_2 leading from the first vertex back to itself. Hence, the total multiplicity factor is $8 \times 4 \times 3 \times 2 \times 3 = (4!)^2$. Following the Feynman rules one obtains the contribution

$$\frac{1}{(2\pi)^{2d}} \int d^d q_1 d^d q_2 \; (4!)^2 \frac{1}{2!} \frac{1}{(p_1^2+m^2)(p_2^2+m^2)(p_3^2+m^2)(p_4^2+m^2)} \times$$
$$\frac{1}{(q_1^2+m^2)(q_2^2+m^2)} \frac{(-\lambda)^2}{(4!)^2} (2\pi)^{2d} \delta(p_1 - q_2) \delta(q_2 + p_2 + p_3 + p_4) =$$
$$\frac{\lambda^2 G(0)/2}{(p_1^2+m^2)^2(p_2^2+m^2)(p_3^2+m^2)(p_4^2+m^2)} (2\pi)^d \delta(p_1 + p_2 + p_3 + p_4). \quad (5.9.3)$$

The integration over q_2 was performed trivially using a δ-function while the integration over the loop-momentum q_1 gives the divergent (but dimensionally regularized) factor $G(0)$. Similar expressions exist for diagrams where the loop is attached to the external lines with momenta p_2, p_3, and p_4. These four 1-loop diagrams just give rise to the renormalization of the mass of the incoming particles. Together with the tree diagram they lead to

$$\Gamma(p_1, p_2, p_3, p_4) = -\frac{\lambda}{(p_1^2+m'^2)(p_2^2+m'^2)(p_3^2+m'^2)(p_4^2+m'^2)}. \quad (5.9.4)$$

However, this is not the full answer to order λ^2. We also need to take into account the 1-loop diagrams where two external lines run into the same vertex. Then there are again 8 possible contractions for the external line with momentum p_1. If the external line with momentum p_2 runs into the same vertex there are 3 remaining possible contractions. The external line with momentum p_3 has 4 possible contractions at the other vertex, and the external line with momentum p_4 has 3 remaining contractions also at that other vertex. Finally there are 2 possible contractions for internal lines with momenta q_1 and q_2 connecting the two vertices. Hence, the total multiplicity factor for this diagram is again $8 \times 3 \times 4 \times 3 \times 2 = (4!)^2$. The Feynman rules lead to the expression

$$\frac{1}{(2\pi)^{2d}} \int d^d q_1 d^d q_2 \; (4!)^2 \frac{1}{2!} \frac{1}{(p_1^2+m^2)(p_2^2+m^2)(p_3^2+m^2)(p_4^2+m^2)} \times$$
$$\frac{1}{(q_1^2+m^2)(q_2^2+m^2)} \frac{(-\lambda)^2}{(4!)^2} (2\pi)^{2d} \delta(p_1 + p_2 - q_1 - q_2) \delta(q_1 + q_2 + p_3 + p_4) =$$
$$\frac{\lambda^2 J(s)/2}{(p_1^2+m^2)^2(p_2^2+m^2)(p_3^2+m^2)(p_4^2+m^2)} (2\pi)^d \delta(p_1 + p_2 + p_3 + p_4), \quad (5.9.5)$$

where

$$J(p^2) = \frac{1}{(2\pi)^d}\int d^dq\ \frac{1}{(q^2+m^2)((p-q)^2+m^2)}. \tag{5.9.6}$$

We have introduced the so-called Mandelstam variable $s = (p_1+p_2)^2$. There are two other diagrams with similar topology which depend on the Mandelstam variables $t = (p_1+p_3)^2$ and $u = (p_1+p_4)^2$. The total contribution to the 4-point function at the order λ^2 takes the form

$$\Gamma(p_1,p_2,p_3,p_4) = -\frac{\lambda - \lambda^2[J(s)+J(t)+J(u)]/2}{(p_1^2+m'^2)(p_2^2+m'^2)(p_3^2+m'^2)(p_4^2+m'^2)}. \tag{5.9.7}$$

5.10 Dimensional Regularization of $J(p^2)$

Just like $G(0)$, $J(p^2)$ is divergent in four space-time dimensions and must be regularized. As before we choose dimensional regularization. In order to evaluate the corresponding integral we use a trick due to Feynman and Schwinger

$$\frac{1}{AB} = \int_0^1 d\tau\ \frac{1}{[(1-\tau)A+\tau B]^2}, \tag{5.10.1}$$

and we write

$$\begin{aligned} J(p^2) &= \frac{1}{(2\pi)^d}\int d^dq\ \frac{1}{(q^2+m^2)((p-q)^2+m^2)} \\ &= \frac{1}{(2\pi)^d}\int_0^1 d\tau \int d^dq\ \frac{1}{[(1-\tau)(q^2+m^2)+\tau((p-q)^2+m^2)]^2}, \\ &= \int_0^1 d\tau\ \frac{1}{(2\pi)^d}\int d^d\tilde{q}\ \frac{1}{(\tilde{q}^2+\tilde{m}^2)^2}. \end{aligned} \tag{5.10.2}$$

In the last step we have shifted the integration variable from q to $\tilde{q} = q - \tau p$ and we have introduced $\tilde{m}^2 = m^2 - \tau(1-\tau)p^2$. Also using

$$\frac{1}{(\tilde{q}^2+\tilde{m}^2)^2} = \int_0^\infty dt\ t\exp(-t(q'^2+m'^2)), \tag{5.10.3}$$

and performing the d-dimensional Gaussian integral

$$\frac{1}{(2\pi)^d}\int d^d\tilde{q}\ \exp(-t\tilde{q}^2) = (4\pi t)^{-d/2}, \tag{5.10.4}$$

one obtains

$$\begin{aligned} J(p^2) &= \int_0^1 d\tau \int_0^\infty dt\ (4\pi t)^{-d/2} t\exp(-t\tilde{m}^2) \\ &= \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\int_0^1 d\tau\ \tilde{m}^{d-4} = -\frac{1}{8\pi^2(d-4)} + ... \end{aligned} \tag{5.10.5}$$

The expression has a pole at $d = 4$. The ellipses indicate the remaining finite piece. It should be noted that the divergent piece is independent of p^2.

5.11 Renormalization of the Coupling Constant

Since the divergence of the 4-point function is momentum-independent, it can be absorbed into a redefinition (a renormalization) of the coupling constant λ. The coupling depends on the momenta of the interacting particles. Hence, in order to define the physical coupling in a unique way one must specify a so-called renormalization condition. The specific form of the renormalization condition is entirely a matter of convention. It is conventional to define the physical coupling at the center of the so-called Mandelstam triangle

$$s = t = u = \frac{4}{3}m'^2, \tag{5.11.1}$$

where m' is the physical renormalized mass. We define the finite part of the function $J(p^2)$ as

$$\tilde{J}(p^2) = J(p^2) - J(\frac{4}{3}m'^2), \tag{5.11.2}$$

and we obtain

$$\Gamma(p_1, p_2, p_3, p_4) = -\frac{\lambda' - \lambda'^2[\tilde{J}(s) + \tilde{J}(t) + \tilde{J}(u)]/2}{(p_1^2 + m'^2)(p_2^2 + m'^2)(p_3^2 + m'^2)(p_4^2 + m'^2)}, \tag{5.11.3}$$

where the physical renormalized coupling constant is given by

$$\lambda' = \lambda - \frac{3}{2}J(\frac{4}{3}m'^2)\lambda^2 + \mathcal{O}(\lambda^3). \tag{5.11.4}$$

The cut-off is now removed by demanding that the physical coupling constant λ' remains fixed at the value that describes experiments, while the bare coupling λ diverges. It is remarkable that the renormalizations of m and λ are sufficient to make all higher n-point functions finite without further adjustments. In other words, after fixing the renormalized mass and coupling at their physical values m' and λ', all other results of the theory are completely fixed. This implies a large predictive power of the theory. Just like the classical ϕ^4 theory, the corresponding quantum field theory has only two free parameters. Quantum field theories with a finite number of such parameters are called renormalizable.

5.12 Renormalizable Scalar Field Theories

Let us consider a scalar field theory in d space-time dimensions with a general polynomial self-interaction potential

$$V(\phi) = \sum_p g_p \phi^p. \tag{5.12.1}$$

The corresponding Euclidean action takes the form

$$S[\phi] = \int d^d x \, [\frac{1}{2}\partial_\mu \phi \partial_\mu \phi + V(\phi)]. \tag{5.12.2}$$

Since the action enters the Boltzmann factor $\exp(-S[\phi])$ in the path integral as an exponent, it must be dimensionless. Consequently, the field ϕ has the dimension

$$d_\phi = \frac{d-2}{2}. \tag{5.12.3}$$

For example, in $d = 2$ a scalar field is dimensionless ($d_\phi = 0$), while in $d = 4$ it has the dimension of a mass ($d_\phi = 1$). Similarly, the coupling constant g_p has the dimension

$$d_{g_p} = d - p d_\phi = d(1 - \frac{p}{2}) + p. \tag{5.12.4}$$

For example, the ϕ^2 term has dimension $d_{g_2} = 2$ irrespective of the dimension d. Of course, the prefactor $g_2 = \frac{1}{2}m^2$ of this term is indeed given by the mass squared of the scalar particle. It will turn out that theories containing couplings with negative mass dimension (i.e. $d_{g_p} < 0$) are not renormalizable. Hence, renormalizability requires

$$d(1 - \frac{p}{2}) + p \geq 0 \;\Rightarrow\; p \leq \frac{2d}{d-2}. \tag{5.12.5}$$

As a result, in $d = 2$ space-time dimensions any polynomial potential ($p \leq \infty$) leads to a renormalizable scalar field theory. In $d = 3$, on the other hand, renormalizability requires $p \leq 6$, and in $d = 4$ dimensions $p \leq 4$. In $d = 6$ we have $p \leq 3$, i.e. a ϕ^3 interaction is formally still renormalizable. However, in the absense of a stabalizing ϕ^4 term, such a theory has an unstable vacuum and is thus inconsistent. Consequently, in more than $d = 4$ dimensions, renormalizable scalar field theories are necessarily trivial, i.e. they are free field theories because the potential is limited to at most quadratic terms, which lead to simple Gaussian integrals.

Let us now derive eq.(5.12.5) by analyzing the degree of divergence of Feynman diagrams. For this purpose we choose a more physical regularization than

dimensional regularization, which is nothing more (and nothing less) than a very neat mathematical device. The most natural physical regularization is the one using a space-time lattice. Here we choose something that is more easy to handle in the calculation of Feynman diagrams, namely a regularization using a momentum cut-off Λ. In this regularization the integrals over internal momenta q_i are limited to $|q| \leq \Lambda$.

Let us consider a connected Feynman diagram for an n-point function in a d-dimensional scalar field theory with general polynomial self-interactions. There are n external lines, k_p vertices of type $p > 2$ (with p lines emanating from them), and I internal lines. The total number of vertices is $k = \sum_p k_p$ and the total number of lines emanating from these vertices is $\sum_p k_p p$. Hence, the number of internal lines is given by

$$I = \frac{1}{2}(\sum_p k_p p - n). \tag{5.12.6}$$

As before, each vertex is associated with a momentum-conserving δ-function, and, as a manifestation of translation invariance, there is one remaining overall δ-function. Hence, again there are

$$l = I - k + 1 \tag{5.12.7}$$

non-trivial loop integrations. Each integration over an internal line momentum q_i contains a measure factor q_i^{d-1}, and each internal line propagator takes the form $(q_i^2 + m^2)^{-1}$. We are interested in an upper limit on the degree of ultraviolet divergence of the Feynman diagram. Hence, we neglect m^2 in the propagator (whoose ultraviolet contributions are dominated by $q_i^2 \approx \Lambda^2$) and we replace the propagator by q_i^{-2}. The so-called superficial degree of divergence is then given by

$$\delta = dl - 2I = \frac{d-2}{2}(\sum_p k_p p - n) - d\sum_p k_p + d. \tag{5.12.8}$$

The integral in the corresponding Feynman diagram is convergent if $\delta < 0$, i.e. if

$$\sum_p [k_p(\frac{d-2}{2}p - d)] < \frac{d-2}{2}n - d. \tag{5.12.9}$$

In order to prevent the superficial degree of divergence to become arbitrarily large for a large number k_p of vertices, i.e. in order for a theory to be renormalizable, one needs

$$\frac{d-2}{2}p - d \leq 0 \;\Rightarrow\; p \leq \frac{2d}{d-2}. \tag{5.12.10}$$

This is just the condition of eq.(5.12.5). Theories for which $p < 2d/(d-2)$ for all vertices are called super-renormalizable. For example, ϕ^4 theory (with $p = 4$) is

super-renormalizable in $d = 3$ space-time dimensions. In four dimensions ϕ^4 theory is still renormalizable because then $p = 2d/(d-2)$. In a renormalizable (but not super-renormalizable) theory Feynman diagrams are superficially divergent if

$$\frac{d-2}{2}n \leq d. \tag{5.12.11}$$

For example, in $d = 4$, 2- and 4-point functions are divergent, but all higher n-point functions are convergent, while in $d = 3$ dimensions 6-point functions are still divergent.

Chapter 6

Canonical Quantization of Electrodynamics

In order to get a good intuitive understanding of photons, we first consider the canonical quantization of the free electromagnetic field. Interactions with scalar fields will be discussed in the next chapter using the Euclidean path integral.

6.1 From the Lagrange to the Hamilton Density

Let us consider the Lagrange density of the free electromagnetic field

$$\mathcal{L}(A_\mu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{6.1.1}$$

where the field strength tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{6.1.2}$$

The field strength (and thus the action) is invariant against Abelian gauge transformation

$$A'_\mu = A_\mu + \partial_\mu \varphi. \tag{6.1.3}$$

In order to isolate the physical gauge invariant degrees of freedom of the gauge field we fix the gauge by choosing $A_0(x) = 0$. This leaves a remnant invariance against time-independent (but still space-dependent) gauge transformations. In particular, after a time-independent gauge transformation we still have

$$A'_0 = A_0 + \partial_0 \varphi = A_0 = 0. \tag{6.1.4}$$

It is convenient to fix the remnant gauge freedom by choosing the Coulomb gauge

$$\partial_i A_i = 0, \tag{6.1.5}$$

at an initial instance of time. Then the electric and magnetic field strengths are given by

$$E_i = F_{i0} = \partial_i A_0 - \partial_0 A_i, \;\; B_i = \frac{1}{2}\varepsilon_{ijk} F_{jk} = \varepsilon_{ijk}\partial_j A_k, \tag{6.1.6}$$

and the Lagrange density takes the form

$$\mathcal{L}(A_i) = \frac{1}{2}(E_i E_i - B_i B_i). \tag{6.1.7}$$

The canonically conjugate momentum to the vector potential $A_i(x)$ is given by

$$\Pi_i(x) = \frac{\delta \mathcal{L}}{\delta \partial_0 A_i(x)} = \partial^0 A^i(x) = E_i(x), \tag{6.1.8}$$

which is just the electric field. The classical Hamilton density is given by

$$\mathcal{H}(A_i, \Pi_i) = \Pi_i \partial^0 A^i - \mathcal{L} = \frac{1}{2}(\Pi_i \Pi_i + B_i B_i) = \frac{1}{2}(E_i E_i + B_i B_i). \tag{6.1.9}$$

The classical Hamilton function is the spatial integral of the Hamilton density

$$H[A_i, E_i] = \int d^3x \; \mathcal{H}(A_i, E_i) = \int d^3x \; \frac{1}{2}[E_i^2 + (\varepsilon_{ijk}\partial_j A_k)^2]. \tag{6.1.10}$$

The Hamilton function is a functional of the classical field $A_i(\vec{x})$ and its canonically conjugate momentum field $\Pi_i(\vec{x}) = E_i(\vec{x})$. Upon quantization the Hamilton function will turn into the Hamilton operator of the corresponding quantum field theory.

As in the canonical quantization of scalar fields we now postulate commutation relations between the gauge field variables $A_i(x)$ and their canonically conjugate momenta $\Pi_i(x) = E_i(x)$

$$[A_i(\vec{x}), E_j(\vec{y})] = i\delta_{ij}\delta(\vec{x} - \vec{y}), \;\; [A_i(\vec{x}), A_j(\vec{y})] = [E_i(\vec{x}), E_j(\vec{y})] = 0. \tag{6.1.11}$$

Again, these commutation relations are completely local, i.e. fields at different points in space commute with each other. The field operator $E_i(\vec{x})$ takes the form

$$E_i(\vec{x}) = -i\frac{\partial}{\partial A_i(\vec{x})}. \tag{6.1.12}$$

6.2 The Hamilton Operator for the Photon Field

We now turn the classical Hamilton function $H[A_i, E_i]$ into the Hamilton operator

$$H = \int d^3x \ \frac{1}{2}[E_i^2 + (\varepsilon_{ijk}\partial_j A_k)^2]. \tag{6.2.1}$$

In order to diagonalize the Hamiltonian it is convenient to go to momentum space. We introduce Fourier transformed fields

$$\tilde{A}_i(\vec{p}) = \int d^3x \ A_i(\vec{x}) \exp(-i\vec{p}\cdot\vec{x}), \ \tilde{E}_i(\vec{p}) = \int d^3x \ E_i(\vec{x}) \exp(-i\vec{p}\cdot\vec{x}), \tag{6.2.2}$$

which obey

$$\tilde{A}_i(\vec{p})^\dagger = \tilde{A}_i(-\vec{p}), \ \tilde{E}_i(\vec{p})^\dagger = \tilde{E}_i(-\vec{p}). \tag{6.2.3}$$

The corresponding commutations relations take the form

$$[\tilde{A}_i(\vec{p}), \tilde{E}_i(\vec{q})] = i(2\pi)^3\delta_{ij}\delta(\vec{p}+\vec{q}), \ [\tilde{A}_i(\vec{p}), \tilde{A}_j(\vec{q})] = [\tilde{E}_i(\vec{p}), \tilde{E}_i(\vec{q})] = 0, \tag{6.2.4}$$

and the Hamilton operator can be written as

$$H = \frac{1}{(2\pi)^3}\int d^3p \ \frac{1}{2}[\tilde{E}_i^\dagger\tilde{E}_i + (\varepsilon_{ijk}p_j\tilde{A}_k^\dagger)(\varepsilon_{ilm}p_l\tilde{A}_m)]. \tag{6.2.5}$$

It is important to note that one must also respect the Coulomb gauge constraint $\partial_i A_i = 0$ which in momentum space takes the form

$$p_i\tilde{A}_i = 0. \tag{6.2.6}$$

Hence, the direction of $\tilde{A}_i$ must be perpendicular to the direction of the momentum p_i. For example, when the momentum points in the z-direction, i.e. when $\vec{p} = (0, 0, p)$, there are two linearly independent modes $\tilde{A}_x$ and $\tilde{A}_y$, corresponding to the two polarization states of an electromagnetic wave or a photon. Using the gauge constraint we can write

$$(\varepsilon_{ijk}p_j\tilde{A}_k^\dagger)(\varepsilon_{ilm}p_l\tilde{A}_m) = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})p_jp_l\tilde{A}_k^\dagger\tilde{A}_m) = \vec{p}^2\tilde{A}_k^\dagger\tilde{A}_k. \tag{6.2.7}$$

Hence, keeping in mind that their are only two physical transverse degrees of freedom for the electromagnetic field, the Hamiltonian takes the form

$$H = \frac{1}{(2\pi)^3}\int d^3p \ \frac{1}{2}(\tilde{E}_i^\dagger\tilde{E}_i + \vec{p}^2\tilde{A}_i^\dagger\tilde{A}_i). \tag{6.2.8}$$

It should be noted that, for a given momentum $\vec{p}$, the sum over i is restricted to the two polarization directions perpendicular to $\vec{p}$.

In complete analogy to the canonical quantization of a scalar field, using $\omega = |\vec{p}|$, we now introduce creation and annihilation operators $a_i(\vec{p})^\dagger$ and $a_i(\vec{p})$ as

$$a_i(\vec{p}) = \frac{1}{\sqrt{2}}[\sqrt{\omega}\tilde{A}_i(\vec{p}) + \frac{i}{\sqrt{\omega}}\tilde{E}_i(\vec{p})],\ a_i(\vec{p})^\dagger = \frac{1}{\sqrt{2}}[\sqrt{\omega}\tilde{A}_i(\vec{p})^\dagger - \frac{i}{\sqrt{\omega}}\tilde{E}_i(\vec{p})^\dagger], \quad (6.2.9)$$

which obey the commutation relations

$$\begin{aligned}
&[a_i(\vec{p}), a_j(\vec{q})^\dagger] = \frac{i}{2}[\tilde{E}_i(\vec{p}), \tilde{A}_j(-\vec{q})] - \frac{i}{2}[\tilde{A}_i(\vec{p}), \tilde{E}_j(-\vec{q})] = (2\pi)^3\delta_{ij}\delta(\vec{p}-\vec{q}), \\
&[a_i(\vec{p}), a_j(\vec{q})] = [a_i(\vec{p})^\dagger, a_j(\vec{q})^\dagger] = 0.
\end{aligned} \quad (6.2.10)$$

In terms of these operators, the Hamiltonian takes the form

$$H = \frac{1}{(2\pi)^3}\int d^3p\ |\vec{p}|(a_i(\vec{p})^\dagger a_i(\vec{p}) + V). \quad (6.2.11)$$

Here V is the spatial volume. Again, the sum over i is restricted to the two polarization directions perpendicular to the momentum vector $\vec{p}$.

6.3 Vacuum and Photon States

In complete analogy to the case of a scalar field, the vacuum state of the electromagnetic field is determined by

$$a_i(\vec{p})|0\rangle = 0, \quad (6.3.1)$$

for all $\vec{p}$ and all polarizations i. The vacuum is an eigenstate of the Hamiltonian from above with the energy

$$E = \frac{1}{(2\pi)^3}V\int d^3p\ |\vec{p}|. \quad (6.3.2)$$

As usual, the volume factor represents a harmless infrared divergence which disappears when one considers the vacuum energy density

$$\rho = \frac{E}{V} = \frac{1}{(2\pi)^3}\int d^3p\ |\vec{p}| = \frac{1}{2\pi^2}\int_0^\infty dp\ p^3. \quad (6.3.3)$$

As for the scalar field, the vacuum energy is still divergent in the ultraviolet. When we use a momentum cut-off, the energy density diverges as $\rho \sim \Lambda^4$.

The energies of photons are differences between the energy of an excited state and the vacuum. In these energy differences the divergent factor drops out. The single photon states are given by

$$|\vec{p}, i\rangle = a_i(\vec{p})^\dagger |0\rangle, \tag{6.3.4}$$

with an energy

$$E(\vec{p}) = \omega = |\vec{p}|. \tag{6.3.5}$$

This is the energy of a free particle with vanishing rest mass and momentum $\vec{p}$. Multi-photon states are obtained by acting with more than one creation operator on the vacuum state. The 2-photon states are obtained as

$$|\vec{p}_1, i, \vec{p}_2, j\rangle = a_i(\vec{p}_1)^\dagger a_j(\vec{p}_2)^\dagger |0\rangle, \tag{6.3.6}$$

Since $[a_i(\vec{p}_1)^\dagger, a_j(\vec{p}_2)^\dagger] = 0$ one has

$$|\vec{p}_2, j, \vec{p}_1, i\rangle = |\vec{p}_1, i, \vec{p}_2, j\rangle, \tag{6.3.7}$$

which shows that photons are bosons.

6.4 Electromagnetic Momentum Operator

Let us also consider the momentum operator of our theory. The energy-momentum tensor of the electromagnetic field takes the form

$$\theta_{\mu\nu} = F_\mu^\rho F_{\nu\rho} - g_{\mu\nu}\mathcal{L}. \tag{6.4.1}$$

The Hamilton density from before is given by

$$\mathcal{H} = \theta_{00} = F_{0i}F_{0i} - g_{00}\mathcal{L} = E_i^2 - \mathcal{L} = \frac{1}{2}(E_iE_i + B_iB_i). \tag{6.4.2}$$

The momentum density is given by

$$\mathcal{P} = \theta_{0i} = F_{0j}F_{ij} - g_{0i}\mathcal{L} = \frac{1}{2}\varepsilon_{ijk}E_jB_k. \tag{6.4.3}$$

Accordingly, the Hermitean momentum operator of the photon field theory is given by

$$\vec{P} = \int d^3x \; \frac{1}{2}\vec{E} \times \vec{B}. \tag{6.4.4}$$

One can show that

$$\vec{P} = \frac{1}{(2\pi)^3}\int d^3p \; \vec{p} \; a_i(\vec{p})^\dagger a_i(\vec{p}). \tag{6.4.5}$$

As expected, the vacuum $|0\rangle$ is an eigenstate of the momentum operator with eigenvalue $\vec{0}$. The single photon states are again eigenstates,

$$\vec{P}|\vec{p}, i\rangle = \vec{p}|\vec{p}, i\rangle, \tag{6.4.6}$$

which shows that $\vec{p}$ is indeed the momentum of the photon.

Chapter 7

Path Integral for Scalar Electrodynamics

In this chapter we consider the perturbative quantization of scalar electrodynamics using the path integral in Euclidean space-time. This theory describes the interactions of a complex (i.e. 2-component) scalar field with itself and with the electromagnetic field. Scalar QED can be used to describe the physics of superconductors with the scalar field representing the Cooper pairs consisting of two electrons. In particle physics the more relevant theory is spinor QED — the theory describing the interactions between electrons, positrons, and photons. At first, in order to avoid extra complications due to the fermionic nature of electrons and positrons (which are described by the Dirac equation), we concentrate on scalar quantum electrodynamics. As in the case of ϕ^4 theory we consider the phase in which the symmetry is unbroken.

7.1 Gauge Fixing and Photon Propagator

The Euclidean action of the free electromagnetic field is given by

$$S_E[A] = \int d^d x \; \frac{1}{4} F_{\mu\nu} F_{\mu\nu}, \tag{7.1.1}$$

where the field strength is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{7.1.2}$$

Both the field strength and the action are invariant under gauge transformations

$$A'_\mu = A_\mu + \partial_\mu \varphi. \tag{7.1.3}$$

The path integral in Euclidean time now takes the form

$$Z = \int \mathcal{D}A \ \exp(-S_E[A]). \tag{7.1.4}$$

Again, this formal expression must be properly defined by imposing an appropriate regularization. In addition, in a perturbatively quantized gauge theory one must also fix the gauge. Otherwise there would be a divergence resulting from path integration over infinitely many gauge copies with the same Euclidean action. Without gauge fixing the photon propagator would take the form $(p^2\delta_{\mu\nu} - p_\mu p_\nu)^{-1}$. This is expression is a singular matrix (with an infinite eigenvalue) in the space of Lorentz indices because

$$(p^2\delta_{\mu\nu} - p_\mu p_\nu)p_\mu = 0. \tag{7.1.5}$$

In the perturbative path integral quantization of non-Abelian gauge theories gauge fixing is a non-trivial issue. Interestingly, when non-Abelian gauge theories are quantized nonperturbatively, using a space-time lattice, gauge fixing is not even necessary. Still, even on the lattice, the quantization of (non-compact) Abelian gauge theories does require gauge fixing. Fortunately, in Abelian gauge theories gauge fixing is much easier to implement than in non-Abelian gauge theories. In contrast to canonical quantization were we chose the Coulomb gauge, in the path integral quantization it is easier to work with a Lorentz-covariant gauge such as the Landau gauge

$$\partial_\mu A_\mu = 0. \tag{7.1.6}$$

The gauge fixing is incorporated as an additional term in the Euclidean action which then becomes

$$S_E[A] = \int d^d x \ [\frac{1}{4} F_{\mu\nu}F_{\mu\nu} + \frac{1}{2\alpha}(\partial_\mu A_\mu)^2]. \tag{7.1.7}$$

In the limit $\alpha \to 0$ only those gauge fields that satisfy the Landau gauge fixing condition have a finite action and thus contribute to the path integral. However, we'll keep the gauge fixing parameter α as an arbitrary free constant. As we'll see, physical results are independent of α.

In momentum space the Euclidean action takes the form

$$S_E[A] = \frac{1}{(2\pi)^d} \int d^d p \ \frac{1}{2} A_\mu [p^2\delta_{\mu\nu} - p_\mu p_\nu (1 - \frac{1}{\alpha})] A_\nu, \tag{7.1.8}$$

from which we read off the photon propagator $[p^2\delta_{\mu\nu} - p_\mu p_\nu(1-1/\alpha)]^{-1}$. This expression is no longer singular unless $\alpha \to \infty$. A particularly simple situation emerges in the so-called Feynman gauge, $\alpha = 1$, for which the photon propagator reduces to $\delta_{\mu\nu}p^{-2}$.

7.2 Feynman Rules for Scalar QED

Let us now couple the electromagnetic field to charged matter described by a complex scalar field $\phi = \phi^1 + i\phi^2$. The total Euclidean action of scalar QED takes the form

$$S_E[\phi, A] = \int d^dx \, [\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\alpha}(\partial_\mu A_\mu)^2 + \frac{1}{2}D_\mu\phi^* D^\mu\phi + V(\phi), \qquad (7.2.1)$$

where the covaraint derivative is given by

$$D_\mu\phi(x) = (\partial_\mu - ieA_\mu(x))\phi(x), \qquad (7.2.2)$$

and the potential for the scalar self-interaction takes the familiar form

$$V(\phi) = \frac{1}{2}m^2|\phi|^2 + \frac{\lambda}{4!}|\phi|^4. \qquad (7.2.3)$$

Since we now have both scalar fields and gauge fields we can consider mixed n-point functions consisting of n_ϕ scalar fields and n_γ photon fields. The corresponding n-point function is given by

$$\begin{aligned}&\langle 0|T\phi(x_1)...\phi(x_{n_\phi})A_{\mu_1}(x'_1)...A_{\mu_{n_\gamma}}(x'_{n_\gamma})|0\rangle = \\ &\frac{1}{Z}\int \mathcal{D}\phi\mathcal{D}A \; \phi(x_1)...\phi(x_{n_\phi})A_{\mu_1}(x'_1)...A_{\mu_{n_\gamma}}(x'_{n_\gamma})\exp(-S[\phi, A]).\end{aligned} \qquad (7.2.4)$$

Again, it is natural to consider the Fourier transform of the n-point function

$$\begin{aligned}&\int d^dx_1...d^dx_{n_\phi}d^dx'_1...d^dx'_{n_\gamma} \; \langle 0|T\phi(x_1)...\phi(x_{n_\phi})A_{\mu_1}(x'_1)...A_{\mu_{n_\gamma}}(x'_{n_\gamma})|0\rangle \\ &\times \exp[i(p_1x_1... + p_{n_\phi}x_{n_\phi})]\exp[i(p'_1x'_1... + p'_{n_\gamma}x'_{n_\gamma})] = \\ &(2\pi)^d\delta(p_1 + ... + p_{n_\phi} + p'_1 + ... + p'_{n_\gamma})\Gamma_{\mu_1...\mu_{n_\gamma}}(p_1, ..., p_{n_\phi}, p'_1, ..., p'_{n_\gamma}).\end{aligned} \qquad (7.2.5)$$

Again, the δ-function results from translation invariance.

From the terms in the action which are quadratic in the fields one reads off the propagators and from the non-quadratic terms one reads off the vertices. The covaraint derivative term

$$\frac{1}{2}D_\mu\phi^* D^\mu\phi = \frac{1}{2}(\partial_\mu + ieA_\mu)\phi^*(\partial_\mu - ieA_\mu)\phi, \tag{7.2.6}$$

gives rise to a 2-scalar-1-photon and a 2-scalar-2-photon vertex. One arrives at the following Feynman rules for scalar QED.

• Consider all pairwise contractions in the product of internal and external fields that lead to a connected Feynman diagram. Each diagram has a multiplicity factor that counts the number of pairings leading to the same topology of the diagram.

• Write down a propagator $(p_i^2 + m^2)^{-1}$ for each oriented external scalar field line, and a propagator $\delta_{\mu\nu}{p'_i}^{-2}$ (in Feynman gauge) for each external photon line.

• Associate a momentum q_i or q'_i with each oriented internal line and write down a factor $(q_i^2 + m^2)^{-1}$ for each scalar line and $\delta_{\mu\nu}{q'_i}^{-2}$ for each photon line.

• Write down a momentum conserving δ-function $(2\pi)^d\delta(q_1 + q_2 + q_3 + q_4)$ for each 4-scalar vertex taking into account the orientation of the internal or external lines. To each vertex of this kind is also associated a factor $-\lambda/4!$.

• Write down a momentum conserving δ-function $(2\pi)^d\delta(q_1 + q_2 + q'_1)$ for each 2-scalar-1-photon vertex. To each vertex of this kind is associated a factor $e(q_{1\mu} - q_{2\mu})$.

• Write down a momentum conserving δ-function $(2\pi)^d\delta(q_1 + q_2 + q'_1 + q'_2)$ for each 2-scalar-2-photon vertex. To each vertex of this kind is associated a factor $2e^2\delta_{\mu\nu}$.

• Finally, integrate the resulting expression over all internal momenta, i.e. write $\int d^dq_1...d^dq_{I_\phi}/(2\pi)^{I_\phi d}\int d^dq'_1...d^dq'_{I_\gamma}/(2\pi)^{I_\gamma d}$.

The numbers I_ϕ and I_γ of internal scalara and photon lines can be determined as follows. There are four scalar lines emanating from each of the k_{ϕ^4} scalar self-interaction vertices. There are two scalar and one photon line emerging from a 2-scalar-1-photon vertex of which there are $k_{\phi^2 A}$, and finally there two scalar and two photon lines emanating from a 2-scalar-2-photon vertex of which there are $k_{\phi^2 A^2}$. Hence, one obtains

$$I_\phi = \frac{1}{2}(4k_{\phi^4} + 2k_{\phi^2 A} + 2k_{\phi^2 A^2} - n_\phi),\ I_\gamma = \frac{1}{2}(k_{\phi^2 A} + 2k_{\phi^2 A^2} - n_\gamma). \tag{7.2.7}$$

There are $I = I_\phi + I_\gamma$ integrations and $k = k_{\phi^4} + k_{\phi^2 A} + k_{\phi^2 A^2}$ δ-functions. Again, one of the δ-functions reflects translation invariance and remains in the final result. Hence, only $k-1$ δ-functions can be used to perform some integrations trivially. The remaining

$$\begin{aligned} l = I - (k-1) &= \frac{1}{2}(4k_{\phi^4} + 2k_{\phi^2 A} + 2k_{\phi^2 A^2} - n_\phi) \\ &+ \frac{1}{2}(k_{\phi^2 A} + 2k_{\phi^2 A^2} - n_\gamma) - k_{\phi^4} - k_{\phi^2 A} - k_{\phi^2 A^2} + 1 \\ &= k_{\phi^4} + \frac{1}{2}k_{\phi^2 A} + k_{\phi^2 A^2} - n_\phi/2 - n_\gamma/2 + 1 \end{aligned} \tag{7.2.8}$$

integrations determine the number of loops l in the diagram.

Chapter 8

Lattice Field Theory

When we regularize individual Feynman diagrams, e.g. using dimensional regularization, we quantize a field theory in the framework of perturbation theory. Even if we could calculate to all orders of perturbation theory (which is, of course, practically impossible), the sum of all perturbative diagrams would not define the theory nonperturbatively. In order to quantize a field theory beyond perturbation theory, we must regularize the entire theory at once. A natural nonperturbative regularization arises when we replace the Euclidean space-time continuum by a 4-dimensional hypercubic lattice. The lattice spacing then serves as an ultraviolet cut-off.

8.1 Fermionic Path Integrals and Grassmann Algebras

We have defined the path integral by using the classical action. Theories with fermions have no immediate classical limit, and the definition of the path integral needs special care. The first step is to define a so-called Grassmann algebra, which works with anticommuting classical variables η_i with $i \in 1, 2, ..., N$. A Grassmann algebra is characterized by the anticommutation relations

$$\{\eta_i, \eta_j\} = \eta_i \eta_j + \eta_j \eta_i = 0. \tag{8.1.1}$$

An element of the Grassmann algebra is a polynomial in the generators

$$f(\eta) = f + \sum_i f_i \eta_i + \sum_{ij} f_{ij} \eta_i \eta_j + \sum_{ijk} f_{ijk} \eta_i \eta_j \eta_k + ... \tag{8.1.2}$$

The $f_{ij...l}$ are ordinary complex (or sometimes real) numbers, which are anti-symmetric in $i, j, ..., l$. One also defines formal differentiation and integration procedures. The differentiation rules are

$$\frac{\partial}{\partial\eta_i}\eta_i = 1,\ \frac{\partial}{\partial\eta_i}\eta_i\eta_j = \eta_j,\ \frac{\partial}{\partial\eta_i}\eta_j\eta_i = -\eta_j, \tag{8.1.3}$$

and integration is defined by

$$\int d\eta_i = 0,\ \int d\eta_i\ \eta_i = 1,\ \int d\eta_i d\eta_j\ \eta_i\eta_j = -1. \tag{8.1.4}$$

These integrals are formal expressions. One should not ask over which range of η_i we actually integrate.

The Grassmann algebra we use to define fermion fields is generated by Grassmann numbers Ψ_x and $\bar{\Psi}_x$, which are completely independent. The index x runs over all space-time points as well as over all spin, flavor or color indices. Let us consider the simplest (completely unrealistic) case of just two degrees of freedom Ψ and $\bar{\Psi}$, and let us perform the Gaussian integral

$$\int d\bar{\Psi}d\Psi\ \exp(-m\bar{\Psi}\Psi) = \int d\bar{\Psi}d\Psi\ (1 - m\bar{\Psi}\Psi) = m. \tag{8.1.5}$$

Note that the expansion of the exponential terminates because $\Psi^2 = \bar{\Psi}^2 = 0$. When we enlarge the Grassmann algebra to an arbitrary number of elements the above formula generalizes to

$$\prod_x \int d\bar{\Psi}_x d\Psi_x\ \exp(-\bar{\Psi}_x\mathcal{M}_{xy}\Psi_y) = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\ \exp(-\bar{\Psi}\mathcal{M}\Psi) = \det\mathcal{M}. \tag{8.1.6}$$

In the two variable case we have

$$\int d\bar{\Psi}d\Psi\ \bar{\Psi}\Psi\exp(-m\bar{\Psi}\Psi) = 1, \tag{8.1.7}$$

which generalizes to

$$\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\ \bar{\Psi}_x\Psi_y\exp(-\bar{\Psi}\mathcal{M}\Psi_y) = \mathcal{M}^{-1}_{ij}\det\mathcal{M}. \tag{8.1.8}$$

8.2 The Fermion Doubling Problem

In the continuum the Euclidean free fermion action is given by

$$S[\bar{\psi},\psi] = \int d^dx\ \bar{\psi}(x)(\gamma_\mu\partial_\mu + m)\psi(x). \tag{8.2.1}$$

In Euclidean space the Dirac matrices γ_μ are Hermitean. Again the partition function

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \ \exp(-S[\bar{\psi},\psi]) \tag{8.2.2}$$

is a formal expression that requires regularization. On the lattice the continuum fermion field $\bar{\psi}(x), \psi(x)$ is replaced by variables $\bar{\Psi}_x, \Psi_x$ that live on the lattice points x. The continuum derivative is discretized by finite differences, such that

$$S[\bar{\Psi}_x, \Psi_x] = \sum_{x,\mu} \frac{1}{2}(\bar{\Psi}_x \gamma_\mu \Psi_{x+\hat{\mu}} - \bar{\Psi}_{x+\hat{\mu}} \gamma_\mu \Psi_x) + \sum_x m \bar{\Psi}_x \Psi_x. \tag{8.2.3}$$

It is instructive to bring this expression to momentum space, because it allows us to read off the lattice fermion propagator

$$\langle \bar{\Psi}(-p)\Psi(p)\rangle = [i \sum_\mu \gamma_\mu \sin p_\mu + m]^{-1}. \tag{8.2.4}$$

In complete analogy to the scalar field case we can analyze the fermionic 2-point function for its exponential decay. The energies E of lattice fermions with spatial momentum $\vec{p}$ show up as poles in the propagator. Here the lattice dispersion relation takes the form

$$\sinh^2 E = \sum_i \sin^2 p_i + m^2. \tag{8.2.5}$$

Like in the scalar field case the continuum dispersion relation is recovered in the continuum limit for small E, $\vec{p}$ and m. However, there are other momenta where E becomes small for small m. These are at the corners of the Brillouin zone at which at least one component of the momentum vector takes the value $p_i = \pi$, because also then $\sin p_i$ goes to zero. This is in contrast to the scalar field case, and it is due to the fact that the fermionic action contains only a single derivative. As a consequence, the lattice dispersion relation leads to extra states in the spectrum that are absent in the continuum theory. The extra states do not disappear in the continuum limit, such that the naive lattice discretization of the fermionic action does not lead to the correct continuum theory. The extra states appearing in the lattice dispersion relation show up as extra physical particles. These so-called doubler fermions are a manifestation of a fundamental problem of lattice regularized fermionic theories. This so-called doubling problem leads to a multiplication of fermion species. The above lattice fermion propagator has 2^d poles instead of just 1 as in the continuum. Hence, the naively discretized fermion theory contains $2^d - 1$ extra fermion species. The origin of the doubling problem is deeply connected with chiral symmetry and can even be traced back to the Adler-Bell-Jackiw anomaly. The doubler fermions pose a severe problem in lattice field theory. Without removing them we cannot describe QCD (or the Standard model) which has definitely less than 16 quarks.

8.3 The Nielsen-Ninomiya No-Go Theorem

Before we try to eliminate the doubler fermions let us prove a general theorem due to Nielsen and Ninomiya: a chiral invariant free fermion lattice action, which is translation invariant, Hermitean, and local, necessarily has fermion doubling. The theorem is based on topology. In fact, it holds because the lattice momentum space (the Brillouin zone) is a torus. A general chiral and translation invariant lattice action leads to a propagator

$$\langle \bar{\Psi}(-p)\Psi(p)\rangle = [i\sum_{\mu}\gamma_\mu\rho_\mu(p)]^{-1}. \tag{8.3.1}$$

Of course, the mass term is now absent because it breaks chiral symmetry. The above action is Hermitean only if $\rho_\mu(p)$ is real. Locality of the action, i.e. exponential suppression of couplings at large distances, implies that $\rho_\mu(p)$ is regular in momentum space, i.e. there are no poles or discontinuities. Most important, translation invariance implies that $\rho_\mu(p)$ is periodic over the Brillouin zone. Poles of the propagator — and hence physical or doubler fermions — correspond to points p with $\rho_\mu(p) = 0$ for all μ. Hence, the theorem states that with the above properties, $\rho_\mu(p) = 0$ at more than just one point. Let us prove this first in $d = 1$. Then there is a single regular periodic function $\rho_1(p)$, which should at least have one zero to describe the physical fermion pole. Then the function is positive on one side of the zero and negative on the other side, such that it must go through zero again in order to satisfy periodicity. A double zero (with the function not changing sign) is not allowed, because this would lead to a wrong dispersion relation for the physical fermion. In higher dimensions the proof is analogous. In $d = 2$ there are two functions $\rho_1(p)$ and $\rho_2(p)$. The zeros of $\rho_1(p)$ lie on a closed curve in the 2-dimensional Brillouin zone. This curve may be closed via the periodic boundary conditions. The zeros of $\rho_2(p)$ lie on another closed curve that intersects the first one in the pole position of the physical fermion. Due to the periodic boundary conditions of the Brillouin zone, the curves than necessarily must also intersect somewhere else. Again, the curves cannot just touch each other, because this would lead to a wrong dispersion relation. In d dimensions the zeros of $\rho_\mu(p)$ (with $\mu = 1, 2, ..., d$) lie on d closed $d-1$-dimensional manifolds. Those cannot intersect in just one point. If they intersect once they necessarily intersect also somewhere else. This proves lattice fermion doubling for a chiral invariant lattice action. The theorem does not specify the number of doubler fermions. It is definitely possible to reduce the number of doublers from $2^d - 1$ to 1, but it is impossible to eliminate doublers completely. Of course, one may try to evade the theorem by violating one of its basic assumptions. In fact, people have worked on random lattices because it has no translation invariance,

and hence no periodic momentum space. However, there is not much one can do analytically on a random lattice, even for a free fermion, and it is unclear if the consequences of the no-go theorem are indeed evaded. When one violates Hermiticity one looses contact with Minkowski space and it is unclear what the Euclidean results mean. People have also worked with nonlocal actions. Then $\rho_\mu(p)$ is not a continuous function (it may have poles or discontinuities), and the theorem which relies on topology obviously does not apply. Still, it has turned out that most nonlocal actions have serious problems. For example, the resulting continuum theory may be nonlocal or not Lorentz invariant. In work on lattice gauge theory Wilson removed the fermion doublers in a very direct and radical way, simply by breaking chiral symmetry explicitly. Then the theorem is evaded because the propagator then contains terms without γ_μ. Of course, this leads to all kinds of complications. In particular, it is nontrivial that chiral symmetry will be recovered in the continuum limit. For vector-like theories like QCD this is under control. For chiral theories like the Standard model, however, it is still unclear if we can define them nonperturbatively due to the notorious fermion doubling problem.

8.4 Wilson Fermions

In his original work on lattice gauge theory Wilson has avoided the fermion doubling problem by breaking chiral symmetry explicitly. The so-called Wilson term gives the doubler fermions a mass at the order of the cut-off while the physical fermion remains massless. Hence, in the continuum limit chiral symmetry is restored in the physical sector. Wilson's modification of the naive fermion action takes the form of a discretized second derivative

$$\begin{aligned} S[\bar{\Psi},\Psi] &= \sum_{x,\mu}\frac{1}{2}(\bar{\Psi}_x\gamma_\mu\Psi_{x+\hat{\mu}} - \bar{\Psi}_{x+\hat{\mu}}\gamma_\mu\Psi_x) + \sum_x \bar{\Psi}_x\Psi_x \\ &+ \frac{1}{2}\sum_{x,\mu}(2\bar{\Psi}_x\Psi_x - \bar{\Psi}_x\Psi_{x+\hat{\mu}} - \bar{\Psi}_{x+\hat{\mu}}\Psi_x). \end{aligned} \tag{8.4.1}$$

Then the lattice propagator takes the form

$$\langle\bar{\Psi}(-p)\Psi(p)\rangle = [i\sum_\mu \gamma_\mu \sin p_\mu + m + \frac{1}{2}\sum_\mu (2\sin(p_\mu/2))^2]^{-1}. \tag{8.4.2}$$

It is instructive to work out the corresponding dispersion relation, and in particular to determine the fermion mass as a function of the bare mass m. The Wilson term acts as a momentum dependent mass term. For small momenta it

vanishes quadratically, and hence it does not effect the dispersion of the physical fermion at least in the continuum limit. For the doubler fermions, on the other hand, the Wilson term is nonzero, and effectively gives them a mass of the order of the cut-off. In the continuum limit the doubler fermions are hence eliminated from the spectrum of the theory.

8.5 Abelian Lattice Gauge Fields

Let us consider an Abelian gauge field a_μ in the continuum. A gauge transformation then amounts to

$$a'_\mu(y) = a_\mu(y) + \partial_\mu \varphi(y). \tag{8.5.1}$$

Since a gauge field is described by a Lorentz vector the corresponding lattice field is naturally associated with the links connecting neighboring lattice points. We denote the lattice gauge potential by $A_{\mu,x}$ where x denotes the center of the corresponding lattice link. The gauge transformation equation for Abelian lattice gauge field then takes the form

$$A'_{\mu,x} = A_{\mu,x} + \Phi_{x-\hat{\mu}/2} - \Phi_{x+\hat{\mu}/2}, \tag{8.5.2}$$

where Φ_x is the lattice gauge transformation associated with the lattice point x. The derivative of the continuum gauge transformation has been replaced by a finite difference, and we have again put the lattice spacing to 1. The field strength of the continuum gauge field is

$$f_{\mu\nu}(y) = \partial_\mu a_\nu(y) - \partial_\nu a_\mu(y), \tag{8.5.3}$$

while the lattice field strength is

$$F_{\mu\nu,x} = A_{\nu,x+\hat{\mu}/2} - A_{\nu,x-\hat{\mu}/2} - A_{\mu,x+\hat{\mu}/2} + A_{\mu,x-\hat{\mu}/2}. \tag{8.5.4}$$

Here x denotes the center of a lattice plaquette (an elementary lattice square) such that the lattice field strength is the oriented sum of gauge links around the plaquette. In complete analogy with the continuum action the standard lattice gauge action takes the form

$$S[A_\mu] = \sum_{\mu\nu,x} \frac{1}{4e^2} F^2_{\mu\nu,x}. \tag{8.5.5}$$

Again one can construct a perfect gauge action by blocking from the continuum. The natural way to block a continuum gauge field is

$$A_{\mu,x} = \int_{c_{x-\hat{\mu}/2}} d^d y \ (1 + y_\mu - x_\mu) a_\mu(y) + \int_{c_{x+\hat{\mu}/2}} d^d y \ (1 - y_\mu + x_\mu) a_\mu(y). \tag{8.5.6}$$

It is instructive to prove that with this blocking a continuum gauge transformation $\varphi(y)$ induces a lattice gauge transformation

$$\Phi_x = \int_{c_x} d^d y \ \varphi(y). \tag{8.5.7}$$

The derivation of the perfect gauge action is in complete analogy to the scalar and fermionic cases, and will not be presented here. As in the scalar field case, the perfect action turns out to be local.

8.6 The Notion of Lattice Differential Forms

Lattice fields have a natural place to live on the lattice. For example, gauge potentials live on the links, while gauge transformations live on the lattice points and field strength variables live on plaquettes. The notion of lattice differential forms is based on this fact, and it allows to perform complicated manipulations of Abelian lattice fields in a simple and transparent manner.

Let us consider a 4-dimensional hypercubic lattice. It decomposes naturally into k-dimensional oriented elementary cells c_k. These are the lattice points c_0, the links c_1, the plaquettes c_2, the cubes c_3 and the hypercubes c_4. The dual lattice is obtained by shifting the original lattice by half a lattice spacing in each direction. The dual lattice consists of $(4-k)$-dimensional dual cells ${}^*c_k$. The points ${}^*c_4$ of the dual lattice are dual to the hypercubes of the original lattice, the links ${}^*c_3$ are dual to the original cubes, the plaquettes ${}^*c_2$ are dual to the original plaquettes, etc. On the cells c_k one defines fields — the so-called k-forms $\Phi(c_k)$. For example, a gauge transformation is a 0-form because it lives on points, while a gauge potential is a 1-form because it lives on links. To each k-form there is associated a dual $(4-k)$-form simply by

$${}^*\Phi({}^*c_k) = \Phi(c_k). \tag{8.6.1}$$

The exterior differential d increases the rank of a form by one

$$d\Phi(c_{k+1}) = \sum_{c_k \in \partial c_{k+1}} \Phi(c_k), \tag{8.6.2}$$

i.e. $d\Phi$ is a $(k+1)$-form defined on the cells c_{k+1}. It is a sum of the k-forms Φ living on the oriented boundary of c_{k+1}. For example, the field strength is a 2-form constructed as

$$F = dA, \tag{8.6.3}$$

and a gauge transformation takes the form

$$A' = A + d\Phi. \tag{8.6.4}$$

Applying the exterior differential twice always gives $d^2 = 0$. For example, the Abelian field strength is gauge invariant because

$$F' = dA' = dA + d^2\Phi = dA = F. \tag{8.6.5}$$

The codifferential $\delta =^* d^*$ lowers the rank of a form by one and is given by

$$\delta\Phi(c_{k-1}) = \sum_{^*c_k \in \partial^* c_{k-1}} \Phi(c_k). \tag{8.6.6}$$

For example, the continuum Landau gauge condition $\partial_\mu a_\mu(y) = 0$ takes the form $\delta A = 0$ on the lattice. The Laplacian $\Delta = d\delta + \delta d$ leaves the rank of a form unchanged. Each form can be written as a Hodge decomposition

$$\Phi = d\Delta^{-1}\delta\Phi + \delta\Delta^{-1}d\Phi. \tag{8.6.7}$$

One defines a scalar product of two k-forms Φ and Ψ by

$$(\Phi, \Psi) = \sum_{c_k} \Phi(c_k)\Psi(c_k). \tag{8.6.8}$$

This induces a norm $||\Phi||^2 = (\Phi, \Phi)$. For example, the standard lattice gauge action takes the form

$$S[A] = \frac{1}{2e^2}||F||^2 = \frac{1}{2e^2}(dA, dA). \tag{8.6.9}$$

The rule for partial integration on the lattice simply reads

$$(d\Phi, \Psi) = (\Phi, \delta\Psi), \tag{8.6.10}$$

where Φ is a k-form, and Ψ is now a $(k+1)$-form. For a gauge field in the Landau gauge one can rewrite the gauge action as

$$S[A] = \frac{1}{2e^2}(A, \delta dA) = \frac{1}{2e^2}(A, \delta dA + d\delta A) = \frac{1}{2e^2}(A, \Delta A). \tag{8.6.11}$$

The notion of differential forms is very useful, in particular because it allows us to perform partial integrations on the lattice in a simple manner.

8.7 Wilson loops and the Lattice Coulomb Potential

Let us consider an Abelian lattice gauge theory (a theory of free photons) coupled to external electric currents. The currents may represent infinitely heavy charged particles. In the continuum the interaction of a gauge field with an external current is described by $\int d^d y\ j_\mu(y) a_\mu(y)$. Expressed in form language the corresponding lattice expression reads (J, A). Let us consider the lattice current of a heavy charged particle moving along the links of the lattice. The current is then 1 on the links that form the world line of the particle, and zero on all other links. The Wilson loop observable is associated with the propagation of a charge-anticharge pair that is created and later annihilated. The corresponding world line is a closed loop $\mathcal{C}$ and the so-called Wilson loop is given by

$$W_{\mathcal{C}} = \exp(i(J, A)). \tag{8.7.1}$$

The expectation value of a rectangular Wilson loop of spatial size R and temporal size T is related to the potential $V(R)$ of a static charge-anticharge pair

$$\lim_{T\to\infty} \langle W_{\mathcal{C}} \rangle \sim \exp(-V(R)T). \tag{8.7.2}$$

Let us calculate this quantity for an Abelian gauge theory. The lattice quantized path integral

$$Z = \prod_{c_1} \int_{-\infty}^{\infty} dA(c_1)\ \exp(-\frac{1}{2e^2}||F||^2) \tag{8.7.3}$$

is still undefined because the implicit integration over all gauges results in an infinite factor. The quantization of Abelian (noncompact) gauge fields on the lattice does indeed require gauge fixing. Here we choose to work in the Landau gauge, i.e. we introduce a δ-function constraint in the functional integral

$$Z = \prod_{c_1} \int_{-\infty}^{\infty} dA(c_1)\ \prod_{c_0} \delta(\delta A(c_0)) \exp(-\frac{1}{2e^2}||F||^2). \tag{8.7.4}$$

This expression is now well-defined. In the Landau gauge we can write the Wilson loop expectation value as

$$\begin{aligned} \langle W_{\mathcal{C}} \rangle &= \frac{1}{Z} \prod_{c_1} \int_{-\infty}^{\infty} dA(c_1)\ \prod_{c_0} \delta(\delta A(c_0)) \exp(-\frac{1}{2e^2}(A, \Delta A) + i(J, A)) \\ &\sim \exp(-\frac{e^2}{2}(J, \Delta^{-1} J)). \end{aligned} \tag{8.7.5}$$

Going to momentum space and performing the large T limit one can derive the lattice Coulomb potential from this expression. The lattice Coulomb potential is

slightly distorted at short distances due to discretization artifacts, but it agrees with the ordinary Coulomb potential at large separations. The Wilson-loop expectation value is an order parameter of lattice gauge theory. In the case of an Abelian gauge theory it indicates that we are in an ordinary Coulomb phase with the typical interactions between static charged particles.

8.8 Lattice QED

To construct the action of lattice QED we have to add interaction terms to the free fermion (electron) and free gauge field (photon) actions from before. In particular, the free Wilson fermion action

$$\begin{aligned} S[\bar{\Psi},\Psi] &= \sum_{x,\mu}\frac{1}{2}(\bar{\Psi}_x\gamma_\mu\Psi_{x+\hat{\mu}}-\bar{\Psi}_{x+\hat{\mu}}\gamma_\mu\Psi_x)+\sum_x m\bar{\Psi}_x\Psi_x \\ &+ \frac{1}{2}\sum_{x,\mu}(2\bar{\Psi}_x\Psi_x-\bar{\Psi}_x\Psi_{x+\hat{\mu}}-\bar{\Psi}_{x+\hat{\mu}}\Psi_x) \end{aligned} \tag{8.8.1}$$

is not gauge invariant. Under a $U(1)$ gauge transformation the fermion field transforms as

$$\Psi'_x=\exp(i\Phi_x)\Psi_x, \tag{8.8.2}$$

while the gauge field transforms as

$$A'_{\mu,x}=A_{\mu,x}+\Phi_{x+\hat{\mu}/2}-\Phi_{x-\hat{\mu}/2}. \tag{8.8.3}$$

To make the fermion action gauge invariant we now introduce the parallel transporter field

$$U_{\mu,x}=\exp(-iA_{\mu,x})\in U(1). \tag{8.8.4}$$

Under a gauge transformation it transforms as

$$U'_{\mu,x}=\exp(i\Phi_{x-\hat{\mu}/2})U_{\mu,x}\exp(-i\Phi_{x+\hat{\mu}/2}), \tag{8.8.5}$$

such that

$$\begin{aligned} S[\bar{\Psi},\Psi,U_\mu] &= \sum_{x,\mu}\frac{1}{2}(\bar{\Psi}_x\gamma_\mu U_{x+\hat{\mu}/2,\mu}\Psi_{x+\hat{\mu}}-\bar{\Psi}_{x+\hat{\mu}}\gamma_\mu U^+_{x+\hat{\mu}/2,\mu}\Psi_x) \\ &+ \sum_x m\bar{\Psi}_x\Psi_x \\ &+ \frac{1}{2}\sum_{x,\mu}(2\bar{\Psi}_x\Psi_x-\bar{\Psi}_x U_{x+\hat{\mu}/2,\mu}\Psi_{x+\hat{\mu}}-\bar{\Psi}_{x+\hat{\mu}}U^+_{x+\hat{\mu}/2,\mu}\Psi_x) \end{aligned} \tag{8.8.6}$$

is indeed gauge invariant and has the correct naive continuum limit. Note that $^+$ here denotes complex conjugation. The concept of a parallel transporter connecting neighboring lattice sites in a gauge covariant way extends naturally to the continuum. In fact, one can define a parallel transporter along any curve $\mathcal{C}$ connecting the points x_0 and x_1 as

$$U_{\mathcal{C}} = \exp(-i \int_{\mathcal{C}} dy_\mu a_\mu(y)). \tag{8.8.7}$$

Under a continuum gauge transformation

$$a'_\mu(y) = a_\mu(y) + \partial_\mu \varphi(y) \tag{8.8.8}$$

the parallel transporter transforms as

$$U'_{\mathcal{C}} = \exp(i\varphi(x_0)) U_{\mathcal{C}} \exp(-i\varphi(x_1)). \tag{8.8.9}$$

The Wilson loop defined above is nothing but a parallel transport around a closed curve. When one adds the free photon action to the above expression for the electron-photon interaction one obtains the action of lattice QED.

Lattice QED is of theoretical interest because it is unknown how QED behaves for large values of the electric charge. This can only be investigated in a nonperturbative formulation such as lattice gauge theory. Numerical simulations suggest that the renormalized electric charge never becomes large. In fact, some calculations indicate that QED is trivial, i.e. the renormalized electric charge vanishes in the continuum limit. To locate the continuum limit in such calculations it is essential to measure the electron mass, i.e. the inverse correlation length of the electron field. The naive correlation function $\langle \bar{\Psi}_x \Psi_y \rangle$, however, is not gauge invariant, and therefore does not contain information about the electron mass. The reason for the gauge dependence is that the physical electron consists not just of the bare electric charge carried by the fermion field, but also of the photon cloud surrounding the charge. The charged field of the physical electron can be constructed as follows

$$\Psi_{c,x} = \exp(-i(E_x, A))\Psi_x. \tag{8.8.10}$$

Here E_x is a 1-form describing the electric field of a static charge located at the point x, i.e.

$$\delta E_x = \delta_x, \tag{8.8.11}$$

where δ_x is a 0-form which is 1 at x and zero otherwise. The field E_x is localized in the specific time slice defined by x_d, and it is zero in all other time slices. This is very essential, because otherwise the definition of the electron field would

require information about the Euclidean past or future. This would destroy the causal structure of the theory, and would prevent us from constructing a transfer matrix, and hence from rotating our physical results back into Minkowski space-time. Under a gauge transformation one finds

$$\begin{aligned} \Psi'_{c,x} &= \exp(-i(E_x, A'))\Psi'_x = \exp(-i(E_x, A + d\Phi)) \exp(i\Phi_x)\Psi_x \\ &= \exp(-i(\delta E_x, \Phi)) \exp(i\Phi_x)\Psi_{c,x} = \Psi_{c,x}. \end{aligned} \tag{8.8.12}$$

Here $(\delta E_x, \Phi) = (\delta_x, \Phi) = \Phi_x$ has been used. Hence the physical charged field is gauge invariant. Its correlation function $\langle \bar{\Psi}_{c,x}\Psi_{c,y}\rangle$ does indeed decay exponentially with the electron mass.

8.9 Lattice QCD

The concept of the parallel transporter is even more important in non-Abelian gauge theories. Not even the pure gluon part of lattice QCD can be formulated in terms of vector potentials $G_{\mu,x}$. Instead, the whole action is formulated in terms of link parallel transporters

$$U_{x,\mu} = \exp(iG^a_{\mu,x}\lambda_a) = \exp(iG_{\mu,x}) \in SU(3). \tag{8.9.1}$$

Again, the parallel transporter has a meaning already in the continuum. In non-Abelian gauge theory it arises as a path ordered exponential

$$U_{\mathcal{C}} = \mathcal{P} \exp(i \int_{\mathcal{C}} dy_\mu G_\mu(y)) = \lim_{\varepsilon \to 0} \prod_{y \in \mathcal{C}} \exp(i\varepsilon G_\mu(y)). \tag{8.9.2}$$

It is instructive to prove that under a non-Abelian gauge transformation

$$G'_\mu(y) = g(y)(G_\mu(y) + \partial_\mu)g(y)^+, \tag{8.9.3}$$

the parallel transporter transforms as

$$U'_{\mathcal{C}} = g(x_0)U_{\mathcal{C}}g(x_1)^+, \tag{8.9.4}$$

in complete analogy to the Abelian case. Consequently, the quark-gluon interaction part of the lattice QCD action has exactly the same form as the electron-photon interaction part of the lattice QED action of eq.(8.8.6). Of course, now the parallel transporters are $SU(3)$ matrices and the fermion fields have a color index (which is suppressed in our notation).

The pure gluon part of the action, on the other hand, is quite different from the free photon action in QED. This is because the gauge transformation equation (8.9.3) cannot be discretized in a gauge covariant way, when one uses the vector potential G_μ. Instead Wilson has constructed the pure gluon part of the lattice QCD action again using parallel transporters. First, he builds a parallel transporter around an elementary plaquette

$$U_{\mu\nu,x} = U_{\mu,x-\hat{\nu}/2} U_{\nu,x+\hat{\mu}/2} U^+_{\mu,x+\hat{\nu}/2} U^+_{\nu,x-\hat{\mu}/2}. \tag{8.9.5}$$

Then the action is constructed as a sum over all plaquettes

$$S[U_\mu] = \sum_{\mu\nu,x} \frac{1}{2g_s^2} \text{Re Tr}(1 - U_{\mu\nu,x}). \tag{8.9.6}$$

It is a good exercise to prove that this expression reduces to the correct continuum action

$$S[G_\mu] = \int d^d y \ \frac{1}{2g_s^2} \text{Tr} G_{\mu\nu}(y) G_{\mu\nu}(y). \tag{8.9.7}$$

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To fully define the lattice QCD path integral we also must consider the measure. For the quarks we have the ordinary Grassmann variable integrations. For the gluons we must integrate over the configurations of parallel transporters U_μ. To discuss this, let us concentrate on a pure gluon theory. The lattice partition function of that theory is given by

$$Z = \prod_{x,\mu} \int_{SU(3)} dU_{\mu,x} \ \exp(-S[U_\mu]). \tag{8.9.8}$$

We integrate independently over all link variables using the so-called Haar measure $dU_{\mu,x}$ for each parallel transporter. The Haar measure is a left and right invariant measure, i.e.

$$\int_{SU(3)} dU \ f(g_L U) = \int_{SU(3)} dU \ f(U g_R) = \int_{SU(3)} dU \ f(U), \tag{8.9.9}$$

for any function f and for any $SU(3)$ matrices g_L and g_R. It is convenient to normalize the measure such that

$$\int_{SU(3)} dU \ 1 = 1. \tag{8.9.10}$$

One can show that the Haar measure is unique and that it has the following additional properties

$$\int_{SU(3)} dU \ U_{ij} = 0, \ \int_{SU(3)} dU \ U_{ij} U_{kl} = \frac{1}{3} \delta_{jk} \delta_{il}. \tag{8.9.11}$$

Here ij determines a specific matrix element, i.e. $i, j, k, l \in \{1, 2, 3\}$. For compact groups like $SU(3)$ the integration is over a finite domain. This is in contrast to QED in which we integrated the gauge potential A_μ over all real numbers. This made it necessary to fix the gauge in lattice QED. In lattice QCD, on the other hand, the path integral is finite from the beginning, and gauge fixing is not necessary. This is a big advantage of the formulation using parallel transporters.

The above pure gluon partition function contains a single parameter — the bare gauge coupling g_s. When we want to perform the continuum limit, we must search for values of g_s for which the correlation length of the lattice theory diverges. As always, we are looking for a second order phase transition. To analyze the phase structure of a gauge theory, one needs to study order parameters like the magnetization $\langle s_x \rangle$ in the Ising model. A simple local order parameter like $\langle U_{\mu,x} \rangle$ does not make sense in a gauge theory. This follows from Elitzur's theorem, which states that gauge non-invariant observables vanish in lattice gauge theory (formulated with parallel transporters). This is trivial to prove using eq.(8.9.11). A useful order parameter in a gauge theory must be gauge invariant and, in addition, nonlocal. A good order parameter in the pure gluon theory is the Wilson loop

$$\langle W_{\mathcal{C}} \rangle = \langle \mathrm{Tr} \prod_{\mu,x \in \mathcal{C}} U_{\mu,x} \rangle. \tag{8.9.12}$$

For a rectangular curve with side lengths R and T the Wilson loop describes the instantaneous creation and annihilation of a static quark-antiquark pair at distance R that lives for a time T. As in QED the Wilson loop is related to the static potential

$$\lim_{T \to \infty} \langle W_{\mathcal{C}} \rangle \sim \exp(-V(R)T). \tag{8.9.13}$$

In QCD we expect quarks and antiquarks to be confined to one another by a potential rising linearly at large separations R, i.e.

$$\lim_{R \to \infty} V(R) \sim \sigma R, \tag{8.9.14}$$

where σ is the string tension. In a confinement phase the Wilson loop hence shows an area law

$$\lim_{R,T \to \infty} \langle W_{\mathcal{C}} \rangle \sim \exp(-\sigma RT). \tag{8.9.15}$$

If confinement would be lost (by going through a phase transition) the Wilson loop would no longer show area law behavior. A Coulomb phase, for example, is characterized by a perimeter law.

8.10 Confinement in the Strong Coupling Limit

In lattice gauge theory it is straight forward to prove confinement for large values of the bare gauge coupling g_s. In the strong coupling region, however, the correlation length is small and we cannot take the continuum limit. In fact, due to asymptotic freedom we expect the continuum limit to be at $g_s \to 0$. It is an open question if confinement persists in the continuum limit. However, there is a lot of positive numerical evidence that this is indeed the case. Assuming that there is no phase transition between the strong and weak coupling regions, the derivation of confinement in the strong coupling regime would carry over to the continuum limit. In the strong coupling expansion we expand in powers of $1/g_s$ around $g_s = \infty$. To leading order the pure gluon action is then simply zero. The Wilson loop operator takes the form

$$W_{\mathcal{C}} = U_{1ij}U_{2jk}U_{3kl}...U_{Nmi}, \tag{8.10.1}$$

where $N = 2(R + T)$ is the number of links along the loop. Using the first formula of eq.(8.9.11) then immediately implies $W_{\mathcal{C}} = 0$. The second formula of eq.(8.9.11) suggests that to get a nonzero result one needs to have the product of two matrix elements for each link variable on the Wilson loop. This can be achieved by expanding the Boltzmann factor of the action to higher orders in $1/g_s$. The lowest nonzero contribution comes from tiling the Wilson loop with plaquettes that result from expanding

$$\exp(-\frac{1}{2g_s^2}\text{Re Tr}(1 - U_{\mu\nu,x})) = \exp(-\frac{3}{2g_s^2})(1 + \frac{1}{2g_s^2}\text{Re Tr}U_{\mu\nu,x}) \tag{8.10.2}$$

for the plaquettes in the interior of the Wilson loop. Taking the $U_{\mu\nu,x}$ term for all these plaquettes and using the second formula of eq.(8.9.11) gives

$$\langle W_{\mathcal{C}}\rangle = \frac{1}{(g_s^2)^{RT}}. \tag{8.10.3}$$

Indeed we find an area law and we read off the string tension as

$$\sigma = -\log(\frac{1}{g_s^2}). \tag{8.10.4}$$

Higher order corrections arise from deformations of the simple tiling from above. The leading correction comes from eliminating one of the RT plaquettes of the original tiling, and replacing it with five plaquettes at the surface of an elementary lattice cube, such that the resulting surface bounded by the Wilson loop has no holes. The cube can be attached above and below the plane of the Wilson loop,

and it can go out in the two possible orthogonal directions. This results in $4RT$ contributions, which all have four more plaquettes than the leading term, and are hence suppressed by $1/(g_s^2)^4$. Up to that order one finds

$$\langle W_{\mathcal{C}} \rangle = \frac{1}{(g_s^2)^{RT}}(1 + 4RT\frac{1}{(g_s^2)^4}). \tag{8.10.5}$$

The last two terms are the first term in an exponential, and hence we can read off the corrected string tension as

$$\sigma = -\log(\frac{1}{g_s^2}) + 4\frac{1}{(g_s^2)^4}. \tag{8.10.6}$$

The string tension has been computed to higher orders. Still, one is unable to go to high enough orders to reach the physically interesting scaling region in which we can perform the continuum limit. At present, only numerical simulation techniques are powerful enough to enter that regime. Consequently, the above result for the string tension cannot yet be compared with experimental results. It is instructive to derive the glueball mass in a similar manner. For this purpose one considers the correlation function of two elementary plaquette operators separated in Euclidean time. Using the strong coupling expansion one can identify the exponential decay, and extract the correlation length whose inverse is the massgap or glueball mass.

8.11 Confinement in Compact Abelian Gauge Theory

The above strong coupling calculations prove that lattice QCD confines at large values of the bare gauge coupling. However, the calculation does not really hint at the dynamical mechanism that confines color. It is a subject of ongoing research to understand the confinement mechanism in non-Abelian gauge theories. In abelian gauge theories, on the other hand, the issue is completely understood, even analytically. Of course, as we saw earlier, the Wilson loop in an Abelian gauge theory yields the Coulomb potential, which does not confine in $d = 4$. It is, however, possible to modify the formulation of Abelian gauge theories, such that they also confine at strong coupling. This is achieved simply by working with parallel transporters in the pure gauge action, as we were forced to do in non-Abelian theories. This leads to the compact formulation of Abelian lattice gauge theory, which is very similar to Wilson's lattice formulation of non-Abelian gauge theory. The use of (compact) parallel transporters instead of (noncompact) vector potentials has drastic consequences in Abelian gauge theory. As we will see, the compact theory has magnetic monopoles as additional degrees of freedom, which

turn out to be responsible for confinement. In fact, the monopoles condense in the confinement phase (like Cooper pairs condense in a superconductor), while they are ordinary charged particles in the Coulomb phase.

As we will see, the confinement mechanism in a compact Abelian gauge theory is the dual of the Meissner effect in a superconductor. An external magnetic field is constricted to flux strings within the superconductor. The flux strings cost energy proportional to their length, i.e. they have a nonzero string tension. If there would be single magnetic monopoles, in a superconductor their magnetic flux would form a string ending in an antimonopole, and hence monopoles and antimonopoles would be confined to one another. In QCD color electric charges (quarks and gluons) are confined. This could be described by a dual Meissner effect. The Meissner effect in a superconductor arises because electric charges (Cooper pairs) are condensed. A dual superconductor would require the condensation of magnetic charges (monopoles). This is exactly what happens in compact QED. It is still unclear if the same mechanism is responsible for confinement in QCD, although there is some positive numerical evidence.

To understand the role of the monopoles let us now consider 4-d compact $U(1)$ lattice gauge theory. This is lattice QED without electrons, but — as we will see — with magnetic charges. For simplicity we consider the theory in the so-called Villain formulation, for which the partition function takes the form

$$Z = \prod_{c_1} \int_{-\pi}^{\pi} d\theta(c_1) \prod_{c_2} \sum_{n(c_2)\in\mathbb{Z}} \exp(-\frac{1}{2e^2}\|d\theta + 2\pi n\|^2), \tag{8.11.1}$$

where the gauge field θ is a 1-form on the links c_1, n is a 2-form on the plaquettes c_2 and e is the bare electric charge. First we replace the link integration in the path integral by an integration over the plaquette angles $\theta_\Box = d\theta \mathrm{mod} 2\pi$. The Bianchi identity is implemented by a δ-function constraint

$$\begin{aligned} Z &= \prod_{c_2} \int_{-\pi}^{\pi} d\theta_\Box(c_2) \prod_{c_3} \sum_{k(c_3)\in\mathbb{Z}} \delta(\frac{1}{2\pi} d\theta_\Box - k) \\ &\quad \prod_{c_2} \sum_{n(c_2)\in\mathbb{Z}} \exp(-\frac{1}{2e^2}\|\theta_\Box + 2\pi n\|^2) \\ &= \prod_{c_3} \sum_{k(c_3)\in\mathbb{Z}} \prod_{c_2} \int_{-\pi}^{\pi} d\theta_\Box(c_2) \\ &\quad \prod_{c_2} \sum_{n(c_2)\in\mathbb{Z}} \exp(-\frac{1}{2e^2}\|\theta_\Box + 2\pi n\|^2) \exp(i(d\theta_\Box, k)). \end{aligned} \tag{8.11.2}$$

The constraint has been eliminated using the Poisson summation formula. Thereby we have introduced an integer valued 3-form k. Now we perform the integration over the variable $F = \theta_\Box + 2\pi n$, and we obtain

$$\begin{aligned} Z &= \prod_{c_3} \sum_{k(c_3)\in\mathbb{Z}} \prod_{c_2} \int_{-\infty}^{\infty} dF(c_2) \exp(-\frac{1}{2e^2}\|F\|^2) \exp(i(dF,k)) \\ &= \prod_{c_3} \sum_{k(c_3)\in\mathbb{Z}} \prod_{c_2} \int_{-\infty}^{\infty} dF(c_2) \exp(-\frac{1}{2e^2}\|F\|^2) \exp(i(F,{}^* d^* k)) \\ &= \prod_{{}^*c_3} \sum_{{}^*k({}^*c_3)\in\mathbb{Z}} \exp(-\frac{e^2}{2}\|d^* k\|^2). \end{aligned} \tag{8.11.3}$$

Eq.(8.11.3) describes a $\mathbb{Z}$ gauge theory with dual link variables ${}^*k$ and with a quadratic action.

The dual $\mathbb{Z}$ gauge theory is the $\kappa \to \infty$ limit of scalar QED with gauge group $\mathbb{R}$. The dual noncompact gauge field ${}^*A \in \mathbb{R}$ is a 1-form on the dual links ${}^*c_3$, and the complex scalar field $\Phi_x = \exp(i{}^*\chi_x)$ is defined in terms of the dual 0-form ${}^*\chi \in]-\pi, \pi]$. To make the action periodic in ${}^*\chi$ we also introduce a dual integer valued 1-form ${}^*k \in \mathbb{Z}$ such that the partition function reads

$$\begin{aligned} Z &= \prod_{{}^*c_3} \int_{-\infty}^{\infty} d^* A({}^*c_3) \prod_{{}^*c_4} \int_{-\pi}^{\pi} d^*\chi({}^*c_4) \prod_{{}^*c_3} \sum_{{}^*k({}^*c_3)\in\mathbb{Z}} \\ &\quad \exp(-\frac{1}{2g^2}\|d^* A\|^2 - \frac{\kappa}{2}\|d^*\chi + 2\pi {}^*k - {}^*A\|^2). \end{aligned} \tag{8.11.4}$$

The scalar field Φ_x carries the bare charge g. In the unitary gauge ${}^*\chi = 0$ the partition function is

$$\begin{aligned} Z &= \prod_{{}^*c_3} \int_{-\infty}^{\infty} d^* A({}^*c_3) \prod_{{}^*c_3} \sum_{{}^*k({}^*c_3)\in\mathbb{Z}} \\ &\quad \exp(-\frac{1}{2g^2}\|d^* A\|^2 - \frac{\kappa}{2}\|2\pi {}^*k - {}^*A\|^2). \end{aligned} \tag{8.11.5}$$

Now it is clear that for $\kappa \to \infty$ only configurations with ${}^*A = 2\pi {}^*k$ contribute to the functional integral and that we thus recover the dual $\mathbb{Z}$ gauge theory with

$$Z = \prod_{{}^*c_3} \sum_{{}^*k({}^*c_3)\in\mathbb{Z}} \exp(-\frac{e^2}{2}\|d^* k\|^2) \tag{8.11.6}$$

for $e = 2\pi/g$. This is exactly the Dirac quantization condition for the bare electric and magnetic charges. Since the scalar particle of the dual gauge theory carries

the charge g, it is natural to identify it with the magnetic monopole of the original model. We will see later that such an identification is indeed justified.

Lüscher has derived an inequality between the bare and renormalized charge in noncompact scalar QED in the Coulomb phase

$$g_r \leq g. \tag{8.11.7}$$

Here g_r is the renormalized magnetic charge. The Dirac quantization condition relates the renormalized electric and magnetic charges by

$$e_r = \frac{2\pi}{g_r}. \tag{8.11.8}$$

Duality in the $\kappa \to \infty$ limit together with the Dirac quantization condition thus turns the inequality eq.(8.11.7) into an inequality between the bare and the renormalized electric charge in compact $U(1)$ lattice QED

$$e_r = \frac{2\pi}{g_r} \geq \frac{2\pi}{g} = e. \tag{8.11.9}$$

Consequently, magnetic monopoles lead to antiscreening of electric charge, as opposed to the screening of electric charges in ordinary QED. Now it becomes clear why the presence of monopoles can lead to confinement of electric charges. Because monopoles enhance the renormalized electric charge, they can drive the theory to strong coupling, where Coulomb's law breaks down and confinement sets in. This is in fact what happens in compact lattice QED. At small bare electric charges the theory is in the Coulomb phase. Then the monopoles are stable magnetically charged particles, and external electric charges interact via Coulomb forces. For large bare electric charges the model is in the confined phase, in which the monopoles are condensed and magnetic charge is not a good quantum number. In the dual formulation the confined phase is equivalent to the Higgs phase of scalar QED, in which the scalar charged particle (the monopole) is condensed. The Coulomb phase of compact QED, on the other hand, corresponds to the Coulomb phase of noncompact scalar QED.

In the dual formulation of the theory the monopole is an ordinary charged scalar field. Therefore it is straightforward to construct a creation operator for it. As discussed before the bare charged field Φ_x cannot create the physical charged state, because it is not gauge invariant. Φ_x only creates the bare monopole, but not the dual photon cloud surrounding it. The gauge dependence of Φ_x is fixed by going to the Coulomb gauge. We perform a gauge transformation ${}^*\varphi$ in the time slice containing the point x

$$\begin{aligned} {}^*A' &= {}^*A + d^*\varphi, \\ {}^*\chi' &= ({}^*\chi + {}^*\varphi) \mathrm{mod} 2\pi. \end{aligned} \tag{8.11.10}$$

Coulomb gauge fixing generates the dual Coulomb field surrounding the monopole. The appropriate gauge transformation at the point x is given by

$$^*\varphi_x = -(^*B_x, ^* A), \tag{8.11.11}$$

with

$$\delta^* B_x = \delta_x, \tag{8.11.12}$$

where δ_x is the Kronecker delta function on the dual lattice, i.e.

$$\delta_x = \begin{cases} 1 & \text{for } y = x \\ 0 & \text{otherwise.} \end{cases} \tag{8.11.13}$$

$^*B_x$ is the magnetic field of a monopole located at the point x. It is nonzero only on the dual links located in the time slice containing the monopole creation point x. The explicit form of $^*B_x$ is given by

$$^*B_x = d_3 \Delta_3^{-1} \delta_x, \tag{8.11.14}$$

where d_3 is the 3-dimensional exterior differential and Δ_3 is the 3-dimensional Laplacian. In the Coulomb gauge the charged field is

$$\Phi_{c,x} = \exp(i^* \chi'_x) = \exp(-i(^*B_x, ^* A))\Phi_x. \tag{8.11.15}$$

This has exactly the same form as the charge creation operator discussed earlier, except that we are now using a dual formulation. If we again go to the limit $\kappa \to \infty$ and fix to unitary gauge $\Phi_x = 1$, $^*A = 2\pi ^*k$ we find

$$\Phi_{c,x} = \exp(-2\pi i(^*B_x, ^* k)). \tag{8.11.16}$$

This is the monopole field from which one can construct all monopole Green functions. Now we will investigate the vacuum expectation value of the monopole field

$$\langle \Phi_{c,x} \rangle = \frac{1}{Z} \prod_{^*c_3} \sum_{^*k(^*c_3) \in \mathbb{Z}} \Phi_{c,x} \exp(-\frac{e^2}{2} \| d^* k \|^2) \tag{8.11.17}$$

after transforming back to the original compact QED. From there one arrives at a Coulomb gas representation of the correlation function, which makes the creation and annihilation of monopoles explicit.

When one performs the duality transformation again one goes back to the original compact lattice QED. The expectation value of the monopole field then takes the form

$$\begin{aligned} \langle \Phi_{c,x} \rangle \quad = \quad & \frac{1}{Z} \prod_{c_1} \int_{-\pi}^{\pi} d\theta(c_1) \prod_{c_2} \sum_{n(c_2) \in \mathbb{Z}} \\ & \exp(-\frac{1}{2e^2} \| d\theta + 2\pi n + 2\pi \delta \Delta^{-1} (B - \omega) \|^2), \end{aligned} \tag{8.11.18}$$

The integer valued 3-form $\omega \in \mathbb{Z}$ is a remnant of the Dirac string which emanates from the monopole position x. It obeys

$$^*(d\omega) = \delta_x \tag{8.11.19}$$

such that $d(B-\omega)=0$. B and ω give rise to a plaquette shift $2\pi\delta\Delta^{-1}(B-\omega)$ which acts as an external background field.

In the present form it is unclear why eq.(8.11.18) describes the creation of monopoles. Therefore we now rewrite the theory as a Coulomb gas of monopole world lines. First we perform a Hodge decomposition of n

$$n = d\Delta^{-1}\delta n + \delta\Delta^{-1}dn = d\Delta^{-1}\delta n + \delta\Delta^{-1}m, \tag{8.11.20}$$

where the monopoles are described by the 3-form $m = dn$. The dual 1-form $^*m$ describes monopole world lines on the dual links, which form closed loops because of the continuity equation for magnetic charge

$$\delta^* m = {}^*dm = {}^*d^2 n = 0. \tag{8.11.21}$$

When n is shifted to $n' = n + dl$, where $l \in \mathbb{Z}$ is a 1-form we still have

$$m' = dn' = dn + d^2 l = dn = m. \tag{8.11.22}$$

One may thus eliminate n from the path integral in favor of m and l

$$\begin{aligned}\langle \Phi_{c,x} \rangle = \frac{1}{Z} \prod_{c_3} \sum_{m(c_3)\in\mathbb{Z}, dm=0} \prod_{c_1} \sum_{l(c_1)\in\mathbb{Z}} \prod_{c_1} \int_{-\pi}^{\pi} d\varphi(c_1) \\ \exp(-\frac{1}{2e^2}\|d(\varphi + 2\pi l + 2\pi\delta\Delta^{-1}n) + 2\pi\delta\Delta^{-1}(m+B-\omega)\|^2).\end{aligned} \tag{8.11.23}$$

Introducing a noncompact gauge field $A = \varphi + 2\pi l + 2\pi\delta\Delta^{-1}n$ one now writes

$$\begin{aligned}\langle \Phi_{c,x} \rangle &= \frac{1}{Z} \prod_{c_3} \sum_{m(c_3)\in\mathbb{Z}, dm=0} \prod_{c_1} \int_{-\infty}^{\infty} dA(c_1) \\ &\quad \exp(-\frac{1}{2e^2}\|dA + 2\pi\delta\Delta^{-1}(m+B-\omega)\|^2).\end{aligned} \tag{8.11.24}$$

Using partial integration and $d(m+B-\omega)=0$ one finds

$$\langle \Phi_{c,x} \rangle = \frac{1}{Z} \prod_{c_3} \sum_{m(c_3)\in\mathbb{Z}, dm=-dB} \exp(-\frac{2\pi^2}{e^2}(m+B, \Delta^{-1}(m+B))) \tag{8.11.25}$$

with

$$Z = \prod_{c_3} \sum_{m(c_3)\in\mathbb{Z}, dm=0} \exp(-\frac{2\pi^2}{e^2}(m, \Delta^{-1}m)). \tag{8.11.26}$$

The expectation value is expressed as a ratio of two partition functions. Z in the denominator describes an ensemble of closed monopole world lines. The monopoles interact with each other via long-range Coulomb forces represented by the inverse Laplacian. In the numerator there is, in addition, one open monopole world line that starts at x. Now it is clear that a monopole has indeed been created at the point x. This shows explicitly that the expectation value, which was originally constructed in the dual scalar QED, does indeed describe the creation of the topological excitations of compact QED. Note that the invisibility of the Dirac string is now obvious, because ω has disappeared from the final expression.

Because of the Dirac quantization condition strong electric couplings e correspond to weak magnetic couplings g. Hence, the strong coupling limit of the original compact QED corresponds to the weak coupling limit of the dual noncompact QED. We can make use of this to show that monopoles are indeed condensed in the confined phase. For this purpose we investigate the monopole field expectation value $\langle\Phi_{c,x}\rangle$ in the strong coupling $e = \infty$ limit, which corresponds to the weak coupling $g = 0$ limit in the dual formulation. In this limit only the configurations with zero dual action contribute to the path integral. Then $k = 0$ and consequently

$$\langle\Phi_{c,x}\rangle = 1, \tag{8.11.27}$$

such that monopoles indeed condense in the confined phase.

8.12 The Monte Carlo Method

The most interesting questions in lattice field theory — especially those that may eventually lead to a solution of QCD — cannot be answered analytically. For example, the strong coupling expansion does not converge well in the weak coupling scaling region, in which we want to take the continuum limit of lattice QCD. Fortunately, the close analogy with classical statistical mechanics allows us to use other techniques developed there. A powerful numerical technique to solve problems in statistical mechanics is the so-called Monte Carlo method. The idea is to compute the partition function (do the path integral) by generating field configurations numerically. Of course, the path integral is an extremely high dimensional integral, such that doing it with standard numerical integration techniques is completely hopeless. In the Monte Carlo method predominantly

those field configurations are generated that have the largest contribution to the path integral. In fact, the Boltzmann factor $\exp(-S[\Phi])$ is used as the probability to generate the field configuration Φ.

In a Monte Carlo simulation one generates a sequence of field configurations

$$\Phi^{(1)} \to \Phi^{(2)} \to ... \to \Phi^{(N)}, \tag{8.12.1}$$

which form a so-called Markow chain, by applying an algorithm that turns the configuration $\Phi^{(i)}$ into $\Phi^{(i+1)}$. The initial configuration $\Phi^{(1)}$ is either picked at random or selected otherwise. Ultimately, nothing should depend on this choice. After a (possibly large) number M of Monte Carlo iterations (applications of the algorithm) an equilibrium is reached, and the system has forgotten about the initial configurations. Only the configurations generated after equilibration are used in the actual calculation. To estimate the expectation value of some observable one averages its values over all configurations of the Monte Carlo sample

$$\langle \mathcal{O}[\Phi] \rangle \approx \frac{1}{N-M} \sum_{i=M+1}^{N} \mathcal{O}[\Phi^{(i)}]. \tag{8.12.2}$$

In the limit $N \to \infty$ the approximation becomes exact. At finite $N - M$ one makes a calculable numerical error that decreases proportional to $1/\sqrt{N-M}$. Hence, to increase the numerical accuracy by a factor of two one must run the Monte Carlo algorithm four times as long. The Boltzmann factor $\exp(-S[\Phi])$ is not explicitly included in the above sum. It is implicitly included, because the configurations in the Markow chain occur with probability $\exp(-S[\Phi])$.

To demonstrate that a particular Monte Carlo algorithm converges to the correct equilibrium distribution it is sufficient to show that it is ergodic and obeys detailed balance. Ergodicity means that starting from an arbitrary initial configuration the algorithm can in principle reach any other field configuration. This condition is obviously necessary, because the correct value for the path integral can be obtained only if all field configurations (of finite action) are included. Detailed balance means that

$$\exp(-S[\Phi]) w[\Phi, \Phi'] = \exp(-S[\Phi']) w[\Phi', \Phi]. \tag{8.12.3}$$

Here $w[\Phi, \Phi']$ is the transition probability for the algorithm to turn the configuration Φ into Φ'. A Monte Carlo algorithm is completely characterized by its $w[\Phi, \Phi']$. Since the algorithm definitely generates a new configuration the proper normalization is

$$\int \mathcal{D}\Phi' \; w[\Phi, \Phi'] = 1. \tag{8.12.4}$$

When the Monte Carlo algorithm converges to an equilibrium distribution $p[\Phi]$ of field configurations, this distribution is an eigenvector of $w[\Phi, \Phi']$ with eigenvalue 1

$$\int \mathcal{D}\Phi \; p[\Phi] w[\Phi, \Phi'] = p[\Phi']. \tag{8.12.5}$$

Now we want to show that the canonical Boltzmann distribution

$$p[\Phi] = \exp(-S[\Phi]) \tag{8.12.6}$$

is indeed an eigenvector of $w[\Phi, \Phi']$ if the algorithm obeys detailed balance. We find

$$\begin{aligned} \int \mathcal{D}\Phi \; \exp(-S[\Phi]) w[\Phi, \Phi'] &= \int \mathcal{D}\Phi \; \exp(-S[\Phi']) w[\Phi', \Phi] \\ &= \exp(-S[\Phi']) \int \mathcal{D}\Phi w[\Phi', \Phi] \\ &= \exp(-S[\Phi']). \end{aligned} \tag{8.12.7}$$

Assuming ergodicity one can show that only one eigenvector with eigenvalue 1 exists, and that the equilibrium distribution is therefore unique.

A simple example of an algorithm that is ergodic and obeys detailed balance is the so-called Metropolis algorithm. In this algorithm a new configuration Φ' is randomly chosen in the vicinity of the old configuration Φ. If the action of the new configuration is smaller than the action of the old configuration, the new configuration is accepted, i.e.

$$S[\Phi'] < S[\Phi] \Rightarrow w[\Phi, \Phi'] = 1. \tag{8.12.8}$$

On the other hand, if the new action is larger, the new configuration is accepted only with a certain probability, i.e.

$$S[\Phi'] > S[\Phi] \Rightarrow w[\Phi, \Phi'] = \exp(-S[\Phi'] + S[\Phi]). \tag{8.12.9}$$

Otherwise the old configuration is kept. This algorithm obeys detailed balance. Let us consider two configurations Φ and Φ'. We can assume that $S[\Phi'] < S[\Phi]$ such that $w[\Phi, \Phi'] = 1$. Then of course, $S[\Phi] > S[\Phi']$ such that $w[\Phi', \Phi] = \exp(-S[\Phi] + S[\Phi'])$, and hence

$$\begin{aligned} \exp(-S[\Phi]) w[\Phi, \Phi'] &= \exp(-S[\Phi]) = \exp(-S[\Phi']) \exp(-S[\Phi] + S[\Phi']) \\ &= \exp(-S[\Phi']) w[\Phi', \Phi]. \end{aligned} \tag{8.12.10}$$

The Metropolis algorithm is particularly simple, but not very efficient. It turns out that subsequent configurations in the Markow chain are correlated with

each other. Hence, to generate a new statistically independent field configuration may require a large number of Monte Carlo iterations. The autocorrelation time τ of the Metropolis algorithm actually increases when one approaches a second order phase transition (or equivalently the continuum limit). At a second order phase transition the correlation length ξ diverges (or equivalently the particle mass in lattice units vanishes). One finds so-called critical slowing down

$$\tau \propto \xi^z, \tag{8.12.11}$$

where z is a dynamical critical exponent characterizing the efficiency of a Monte Carlo algorithm. For the Metropolis algorithm one finds $z \approx 2$, which leads to a very bad critical slowing down behavior. In lattice gauge theory the best algorithm that is presently known (the so-called overrelaxation algorithm) has $z \approx 1$. For simpler spin models so-called cluster algorithms exist which have $z \approx 0$, and which hence eliminate critical slowing down. These algorithms are extremely efficient, and allow one to perform very accurate numerical simulations in these models. Unfortunately, for lattice gauge theory no algorithm with $z \approx 0$ is known.

The inclusion of quarks poses additional very hard problems in numerical simulations of QCD. As a matter of fact, computers cannot deal with Grassmann numbers directly. Therefore, one integrates out the fermions analytically and obtains a fermion determinant, that is a very complicated function of the gluon link variables. One includes the determinant in the effective gluon action, which then becomes highly nonlocal. To evaluate the change in action, which is necessary for the Metropolis step, then is a very time-consuming procedure. Simulations with dynamical quarks are therefore orders of magnitude more complicated than simulations of the pure gluon theory. It is a matter of ongoing research to find better ways to handle quarks numerically. Due to the tremendous difficulties to simulate dynamical quarks, people often use the so-called quenched approximation, in which the fermion determinant is simply put to 1. It is unclear how this very drastic approximation affects the physics. Especially for light quarks one expects that the quenched approximation should be bad. On the other hand, for somewhat heavier quarks the approximation seems to work reasonably well.