

# Accidental Symmetry in Quantum Physics

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May 13, 2008

# Abstract

In this thesis we will study examples of accidental symmetry in quantum physics. An indicator for the existence of conserved quantities is that all bound classical orbits are closed in the corresponding classical system. In analogy of Kepler problem we study accidental symmetry by constructing vectors that play the role of the Runge-Lenz vector. First we consider a particle moving on a cone and bound to its tip by  $1/r$  or  $r^2$  potentials. When the deficit angle of the cone divided by  $2\pi$  is a rational number, all bound classical orbits are closed. Correspondingly, the quantum system has accidental degeneracies for the discrete energy spectrum. An accidental  $SU(2)$  symmetry is generated by the rotations around the tip of the cone as well as by a Runge-Lenz vector. Remarkably, some of the corresponding multiplets have fractional “spin” and unusual degeneracies in both potentials.

A classical particle in a constant magnetic field moves in cyclotron motion on a circular orbit. At the quantum level, again here, all classical orbits are closed, and this gives rise to degeneracies in the spectrum. It is well-known that the spectrum of a charged particle in a constant magnetic field consists of infinitely degenerate Landau levels. Here, one also expects some hidden accidental symmetry with infinite-dimensional representations. The position of the center of the cyclotron circle plays the role of a Runge-Lenz vector. After identifying the corresponding accidental symmetry algebra, we re-analyze the system in a finite periodic volume. Interestingly, similar to the quantum mechanical breaking of CP invariance due to the  $\theta$ -vacuum angle in non-Abelian gauge theories, quantum effects due to two self-adjoint extension parameters  $\theta_x$  and  $\theta_y$  explicitly break the continuous translation invariance of the classical theory. This reduces the symmetry to a discrete magnetic translation group and leads to finite degeneracy. Similar to a particle moving on a cone, a particle in a constant magnetic field shows a very peculiar realization of accidental symmetry in quantum mechanics.

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# Chapter 1

## Introduction

Symmetry is a fundamental feature of the physical world. It plays a pivotal role in studying classical systems since the beginning of the development of classical mechanics because it leads to the identification of conserved quantities of classical systems. Geometrical symmetries of space itself lead to conserved dynamical quantities. For example, the assumption that space is homogeneous or, in other words, has translation symmetry, leads to the conservation of linear momentum of a closed isolated system, and isotropy of space leads to conservation of angular momentum of an isolated system. On the other hand, if the system has time translation symmetry, then the total energy is conserved [1]. For a particle in an external potential the status of the space symmetry is decided by the symmetry of the potential under which the particle is moving. Therefore the symmetry of the potential dictates the conserved quantities. For example, a particle moving under the influence of a spherically symmetric potential has a conserved angular momentum. The importance of the relation between symmetry and conserved quantities led to the development of theorems like the Noether theorem that allow us to obtain these conserved quantities from the knowledge of symmetry.

In quantum mechanics symmetry also leads to conserved quantities which manifest themselves by good quantum numbers. States of different quantum numbers with the same energy lead to degeneracy. We can understand degeneracy in classical mechanical terms as different possible trajectories that belong to the same energy. For example, rotating the plane of motion of a particle moving under the influence of a spherically symmetric potential (a  $1/r$  potential, for example) by an angle  $\alpha$  does not change the energy. Changing the plane of rotation means changing the direction of the angular momentum vector  $\vec{L}$  by an angle  $\alpha$  without changing the length of  $\vec{L}$ . There is an infinite number of trajectories that belong to the same energy. Analogously, in quantum mechanics the value of the  $z$ -component of the

angular momentum  $L_z = m\hbar$ , with  $m$  being the integer magnetic quantum number, characterizes different degenerate states. Different values of  $L_z$  do not affect the energy since  $L_z$  commutes with the Hamiltonian operator  $H$  for spherically symmetric potentials. In contrast to the classical case, in quantum mechanics the degeneracy is a finite number, equal to  $2l + 1$  where  $l$  is the integer-valued angular momentum quantum number.

There is another kind of symmetry in addition to the geometrical symmetry of space itself. This is a dynamical symmetry that results from special features of the Lagrangian of the system, and not from homogeneity or isotropy of space. This type of symmetry is also known as accidental symmetry. It is called accidental because the system possesses such a symmetry only due to special values of some parameters in the Lagrangian of the system, or due to a special functional form of the potential of the system. For example, for spherically symmetric potentials of the form  $r^\alpha$ , only for the special values  $\alpha = -1$  and  $\alpha = 2$  there is accidental symmetry. Another example is a particle moving on a cone. Only for special values of the deficit angle  $\delta$  of the cone there is accidental symmetry. Interestingly, there is a relation between accidental symmetry and the fact that the bound classical orbits are closed. The uniqueness of the  $1/r$  and  $r^2$  potentials was realized by Bertrand in 1873. He proved that they are the only spherically symmetric scalar potentials in Euclidean space for which all bound orbits are closed [2]. Conserved quantities due to geometrical symmetry in discrete systems can usually be easily identified. It is sufficient to find the cyclic canonical coordinate in the Lagrangian. Then the conserved quantity is just the conjugate canonical momentum, i.e.

$$\frac{\partial L}{\partial q_c} = \dot{p}_c = 0, \quad (1.0.1)$$

which implies that  $p_c$  is constant. In the case of accidental symmetry, the conserved quantities are not always so easy to identify. For the Kepler problem, after some calculations involving Newton's second law, it is relatively easy to find a conserved quantity that is called the Runge-Lenz vector [3]. It is the three-vector pointing from the center of the force towards the perihelion of the elliptic orbit of the particle. On the other hand, for other systems with accidental symmetries, finding the conserved quantities may be a complicated process. For example, this is the case for the problem of a particle moving under the influence of a  $1/r$  potential (the Kepler problem), or an  $r^2$  potential (the isotropic harmonic oscillator) modified by an angular momentum-dependent potential of the form

$$V(r) = \frac{L^2(\lambda^2 - 1)}{2Mr^2}, \quad (1.0.2)$$

where  $M$  is the mass of the particle, and  $\lambda$  is a constant. It can be proved that for  $\lambda = p/q$  a rational number, all classical orbits are closed. Finding the conserved



quantities for a modified Kepler problem is highly nontrivial, and was studied in [4], [5]. In fact, in [4] it was argued that the conserved quantities are tensors of ranks depending on the value of the constant  $\lambda$ . Other later attempts investigated the possibility of finding a general scheme to identify accidental symmetry from the expression of the Hamiltonian [6]. Here I refer to unpublished work by Wiese in 1982 who studied the problem of the modified Kepler problem quantum mechanically [7].

Any problem of a particle moving under the influence of a spherically symmetric potential modified by an angular momentum-dependent potential of the form in Eq.(1.0.2) in 2-dimensions is equivalent to a particle moving on a 2-dimensional cone with the center of force at the tip of the cone. The problem of a particle moving on a cone under the influence of  $1/r$  and  $r^2$  potentials centered at the tip of the cone was studied previously. A solution of the Schrödinger equation for this problem was given for both cases of  $1/r$  and  $r^2$  potentials [8]. However, the accidental symmetry of such systems has never been studied. In this thesis, from which some results have already been published [9], the accidental symmetry of the problem will be investigated classically and quantum mechanically. The first part of the work is to construct the conserved quantities. They are identified as two components of what is equivalent to a Runge-Lenz vector, as well as one component of the angular momentum vector. From the commutation relations between these quantities, one finds that they generate an  $SU(2)$  symmetry for both potentials. What makes this problem particularly interesting is the value of the Casimir operator which behaves like a spin operator. Unlike the usual value of an integer or a half-integer of the Casimir spin that one finds in all previously studied quantum systems, the Casimir spin for this problem can take any value. However, in order to have extra degeneracies one must have  $s = p/q = 1 - \delta/2\pi$  as a rational number. The Casimir spin for the Kepler problem is  $S = n_r + |m|/s$ , where  $n_r = 0, 1, 2, \dots$  is the radial quantum number, and  $\delta$  is the deficit angle of the cone. Accordingly, in this case the Casimir spin can be a fractional number. This is unusual for an  $SU(2)$  representation. What makes this  $SU(2)$  representation so special? We can understand this by recalling the basic procedure by which the spectrum of other systems was derived. Take, for example, a general spherically symmetric potential problem in quantum mechanics. If we take the angular momentum components as generators, then the Casimir operator for this case is nothing but  $\vec{L}^2$ . Deriving the spectrum of this operator involves obtaining the matrix elements of the raising operator  $L_+ = L_x + iL_y$ , and the lowering operator  $L_- = L_x - iL_y$ , and using eigenfunctions of  $L_z$  as a basis. The whole treatment is based on the assumption that the functions are normalizable and constitute a complete set. (This will be explained in more details in section (2.3)). For motion on a cone the situation is different. The generators  $R_x$  and  $R_y$  have different features because the Runge-Lenz vector is Hermitian in a domain  $\mathcal{D}[\vec{R}]$  which is in general different from the domain of the Hamiltonian  $\mathcal{D}[H]$ . Therefore the raising and lowering operators  $R_{\pm} = R_x \pm iR_y$

can take a wave function outside  $\mathcal{D}[H]$ . Thus the usual treatment for finding the spectrum of the Casimir operator and the degeneracies does not apply. The issue of the domains of the operators in a corresponding Hilbert space, and the issue of Hermiticity versus self-adjointness plays a role in this thesis. In a usual  $SU(2)$  representation the number of wave functions in the multiplet is finite and can be found by a purely algebraic treatment (see section (2.3)). Here it is not possible to do that. Counting the number of wave functions in a multiplet requires a different technique. The counting is done by direct application of the explicit form of the raising and lowering operators on the wave function. After lengthy calculations, it can be shown that the multiplet terminates for both  $1/r$  and  $r^2$  potentials when  $S$  is an integer or a half-integer, while it never terminates otherwise, and then the multiplet contains an infinite number of wave functions. Not all the wave functions in the multiplet are physical. To count the degeneracies we must count only the physical wave functions. Therefore counting degeneracies is also a different process than the one in the usual  $SU(2)$  representations. In our case, it requires finding a rule by induction that relates the degeneracy  $g$  to the Casimir spin  $S$ , as well as to  $p$  and  $q$  which are the parameters of  $s = p/q$ . The same argument applies to the case of the isotropic harmonic oscillator.

Another important case of accidental symmetry is given by a particle moving under the influence of a constant magnetic field undergoing cyclotron motion on a circular orbit. Here the classical bound orbits are also closed, and again this leads to conserved quantities. The conserved quantity is a vector that is pointing from the origin to the center of the circular orbit. Since the coordinates of the center of the circle do not change with time, they are conserved quantities and play the role of the Runge-Lenz vector. In the case of infinite space we have an additional geometrical symmetry, the rotational symmetry that leads to a conserved one-component angular momentum. The Poisson brackets (as well as the commutator) for these three generators do not give an  $SU(2)$  symmetry because  $\{R_x, R_y\} = c$ , where  $R_x$ ,  $R_y$  are the components of the Runge-Lenz vector, and  $c$  is a constant (see subsection(2.2.2)). The generators constitute the Heisenberg group [10]. The radius of the circle  $r$  is a constant of motion since it is proportional to the total energy  $E = Mr^2\omega^2/2$ . Quantum mechanically the fact that  $R_x$ , and  $R_y$  do not commute implies that we cannot measure the components of the center of the circle simultaneously with absolute precision, although the radius of the circle can be measured with absolute precision. In the cyclotron problem, translation invariance disguises itself as an “accidental” symmetry. As a consequence, the symmetry multiplets — i.e. the Landau levels — are infinitely degenerate. In order to further investigate the nature of the accidental symmetry, in [11] the charged particle in the magnetic field was coupled to the origin by an  $r^2$  harmonic oscillator potential. This explicitly breaks translation invariance and thus reduces the degeneracy to a finite amount, while rotation invariance remains intact. We put the system on a torus

by imposing a boundary condition over a rectangular region  $L_x \times L_y$ , which leads to a quantization condition for the magnetic flux. This leads to interesting results regarding the degeneracy of the system, which turns out to be given by  $n_\Phi$ , the number of magnetic flux quanta. This explicitly breaks rotation invariance, while leaving translation invariance (and hence the accidental symmetry) intact. Remarkably, the Polyakov loops, which are a consequence of the non-trivial holonomies of the torus, give rise to non-trivial Aharonov-Bohm phases which are observable at the quantum level but not at the classical level. In analogy to the quantum mechanical breaking of CP invariance due to the  $\theta$ -vacuum angle in non-Abelian gauge theories, here two self-adjoint extension parameters  $\theta_x$  and  $\theta_y$  explicitly break the continuous translation invariance of the classical system down to a discrete magnetic translation group. This reduces the degeneracy to a finite number, and allows us to further investigate the nature of the accidental symmetry. In particular, just like for motion on a cone, symmetry manifests itself in a rather unusual way in this quantum system. In particular, due to its relevance to the quantum Hall effect, the Landau level problem has been studied very extensively (for a recent review see [12]). For example, the problem has already been investigated on a torus in [13, 14], however, without elaborating on the accidental symmetry aspects. In the second part of this thesis we concentrate exactly on those aspects, thus addressing an old and rather well-studied problem from an unconventional point of view.



# Chapter 2

## An Introduction to Symmetry in Classical and Quantum Physics

In this chapter we will highlight the importance of symmetry in studying classical and quantum mechanics. Symmetry plays a pivotal role in understanding quantum mechanics. In most cases the symmetry of a quantum system is the same as the one of the corresponding classical system. That is aside from anomalies which break some of the symmetries of the quantum system. However, such cases are beyond the scope of this thesis.

In the beginning of this chapter we will discuss symmetry in classical physics, and briefly discuss the symmetry of some classical systems. After that we discuss in more details the symmetries of quantum systems. Two systems will be studied in detail, the Kepler problem and the Landau level problem. Through the discussions, we will distinguish between two kinds of symmetries, dynamical symmetry and geometrical symmetry.

The last section concentrates on one of the important benefits of studying symmetry. We will show that the knowledge of symmetry leads to the energy spectrum and degeneracies without even solving the Schrödinger equation.

### 2.1 Symmetry in Classical Mechanics

A continuous symmetry of a classical system is generated by a number of symmetry generators. A generator is a mechanical variable that can be written in terms of canonical momenta  $p_k$  and coordinates  $q_k$ , and has a vanishing Poisson bracket with the classical Hamilton function  $H$ . Let us consider a classical system with  $n$

generators, and let us denote the  $i$ th generator by  $G_i(p_k, q_k)$ . Then

$$\{H, G_i\} = \frac{dG_i}{dt} = \sum_k \left( \frac{\partial H}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial H}{\partial p_k} \frac{\partial G}{\partial q_k} \right) = 0, \quad (2.1.1)$$

which implies that  $G_i$  is a constant of motion.

The Poisson bracket of two generators of some classical system may constitute a Lie group bracket relation of the form [3]

$$\{G_i, G_j\} = \sum_{k=1}^n f_{ijk} G_k, \quad (2.1.2)$$

where  $f_{ijk}$  are the so-called structure constants, and  $n$  is the number of generators of the classical system. (From now on we will use the Einstein summation convention unless stated otherwise). For the rotation group  $SO(3)$  we have  $f_{ijk} = \epsilon_{ijk}$ .

Consider a particle with mass  $M$  moving under the influence of spherically symmetric potential in three dimensions. In this case, we have three generators that represent the three components of the angular momentum vector  $\vec{L}$ . The generators constitute an  $SO(3)$  algebra. In the case of the  $1/r$  potential, the so-called Kepler problem, it is easy to prove that the three components of the vector

$$\vec{R} = \vec{p} \times \vec{L} - \frac{M\kappa\vec{r}}{r}, \quad (2.1.3)$$

are also conserved. Here  $\kappa$  is the strength of the  $1/r$  potential. This vector is known as the Runge-Lenz vector and it will play a major role in this thesis.

Let us re-scale  $\vec{R}$  to  $\tilde{R}$ , where

$$\tilde{R} = \frac{1}{\sqrt{-2ME}} \vec{R}. \quad (2.1.4)$$

Here  $E$  is the total energy of the particle. It is straightforward to obtain the following Lie group brackets

$$\{L_i, L_j\} = \epsilon_{ijk} L_k, \quad \{\tilde{R}_i, L_j\} = \epsilon_{ijk} \tilde{R}_k, \quad \{\tilde{R}_i, \tilde{R}_j\} = \epsilon_{ijk} \tilde{R}_k. \quad (2.1.5)$$

The structure constants in the above equations can be identified as the ones of  $SO(4)$ . Here it should be noted that this is only valid for the bound Kepler problem in which the total energy is negative. For the unbound Kepler problem  $E$  is positive and  $\tilde{R}$  is a vector with purely imaginary components. This changes the structure constants in the above equations.

Another important example is the isotropic harmonic oscillator. In this case the potential is given by  $V(r) = M\omega^2 r^2/2$ , where  $\omega$  is the angular frequency of the oscillation. Here, in addition to the conserved components of the angular momentum, there are other conserved quantities. For example, in two dimensions they are the components of a second-rank tensor given by

$$Q_{ij} = \frac{1}{2M}(p_i p_j + M^2 \omega^2 x_i x_j). \quad (2.1.6)$$

From the components of this tensor we construct the following conserved quantities

$$G_1 = \frac{Q_{12} + Q_{21}}{2\omega}, \quad G_2 = \frac{Q_{22} - Q_{11}}{2\omega}, \quad G_3 = \frac{L}{2} = \frac{xp_y - yp_x}{2}. \quad (2.1.7)$$

After using Eqs.(2.1.6) and Eqs.(2.1.7), it can be proved that the values of the Poisson brackets are given by the following relation

$$\{G_i, G_j\} = \epsilon_{ijk} G_k. \quad (2.1.8)$$

The structure constants of the above Lie bracket identify the symmetry group as  $SO(3)$ . It is worth noting that  $SO(3)$  is homomorphic to the group  $SU(2)$ .

### 2.1.1 The Casimir Operator of the Symmetry Group

In general, for a semi-simple Lie algebra with  $n$  generators one can define a quadratic Casimir operator as

$$C = \sum_{i=1}^n a_i G_i^2, \quad (2.1.9)$$

where  $a_i$  are constants that can be chosen such that  $C$  commutes with all generators  $G_i$ . The number of independent Casimir operators is given by the rank of the group, i.e. by the number of commuting generators. For example, in the case of spherically symmetric potentials in three dimensions with a geometric  $SO(3)$  symmetry only, the only Casimir operator is the square of the length of the angular momentum vector  $\vec{L}^2 = L_x^2 + L_y^2 + L_z^2$ .

## 2.2 The Symmetries of Some Quantum Systems

In this section we will discuss the symmetries of some important quantum systems by finding the conserved quantities for each of these systems. This leads to the definition of the generators of the symmetry group. For a quantum system the

classical dynamical variables will be replaced by the corresponding operators. The symmetry will be studied here by replacing the Poisson brackets by commutators and the classical generators will be replaced by their corresponding operators. For example, the Poisson brackets of the symmetry group of a particle moving in a spherically symmetric potential in the first equation of Eqs.(2.1.5) will be written for the corresponding quantum mechanical system as

$$[L_i, L_j] = i\epsilon_{ijk}L_k. \quad (2.2.1)$$

In this thesis we will use natural units in which  $\hbar = 1, c = 1$ .

### 2.2.1 The Kepler Problem

For the Kepler problem the Hamiltonian takes the form

$$H = -\frac{\Delta}{2M} - \frac{\kappa}{r} = \frac{-1}{2M} \left( \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \frac{\cot^2\theta}{r^2}\partial_\theta + \frac{\csc^2\theta}{r^2}\partial_\varphi^2 \right) - \frac{\kappa}{r}. \quad (2.2.2)$$

The expression of the Runge-Lenz vector as an operator can be obtained from Eq.(2.1.3). There is an ordering ambiguity in the first term. We can overcome this difficulty by writing

$$\vec{R} = \frac{1}{2} \left( \vec{p} \times \vec{L} - \vec{L} \times \vec{p} \right) - \frac{M\kappa\vec{r}}{r}. \quad (2.2.3)$$

In order to verify that the components of the above operator are constants of motion we must prove that they commute with the Hamiltonian in Eq.(2.2.2). This can be done, for example, by writing the components of the Runge-Lenz vector in terms of explicit expression for the components of  $\vec{L}$  and  $\vec{p}$  in spherical coordinates, and then proving that the commutator is zero for all three components of  $\vec{R}$ . The calculation either can be done by hand, or by a Mathematica program that we constructed which calculates the commutator for any two operators. The components of  $\vec{L}$  in spherical coordinates are

$$\begin{aligned} L_x &= i(\cot\theta \cos\varphi\partial_\varphi + \sin\varphi\partial_\theta), \\ L_y &= i(\cot\theta \sin\varphi\partial_\varphi - \cos\varphi\partial_\theta), \\ L_z &= -i\partial_\varphi, \end{aligned} \quad (2.2.4)$$

and of  $\vec{p}$  in spherical coordinates are

$$\begin{aligned} p_x &= -i \left( \frac{\cos\varphi}{r\sin\theta}\partial_\varphi + \sin\theta \sin\varphi\partial_r + \frac{\cos\theta \sin\varphi}{r}\partial_\theta \right), \\ p_y &= -i \left( -\frac{\sin\varphi}{r\sin\theta}\partial_\varphi + \sin\theta \cos\varphi\partial_r + \frac{\cos\theta \cos\varphi}{r}\partial_\theta \right), \\ p_z &= -i \left( \cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta \right). \end{aligned} \quad (2.2.5)$$



Substituting the values of the components of  $\vec{L}$  and  $\vec{p}$  in Eq.(2.2.3) one then calculates the commutator with the Hamiltonian. The result is zero for all components. This proves that the components of  $\vec{R}$  are indeed constants of motion. In addition to this, one can prove that the components of  $\vec{L}$  commute with the Hamiltonian of Eq.(2.2.2). Accordingly, in total we have six generators. This gives the following commutation relations

$$[L_i, L_j] = i\epsilon_{ijk}L_k, \quad (2.2.6)$$

$$[\tilde{R}_i, L_j] = i\epsilon_{ijk}\tilde{R}_k, \quad (2.2.7)$$

$$[\tilde{R}_i, \tilde{R}_j] = i\epsilon_{ijk}\tilde{R}_k, \quad (2.2.8)$$

where we again have

$$\tilde{\tilde{R}} = \frac{1}{\sqrt{-2ME}}\tilde{R}. \quad (2.2.9)$$

It is also clear here that the symmetry group is  $SO(4)$ . This can be realized by labelling  $\tilde{R}_x = L_{14}$ ,  $\tilde{R}_y = L_{24}$ ,  $\tilde{R}_z = L_{34}$ , and  $\vec{L} = (L_{23}, L_{31}, L_{12})$ . The extension of the definition of the components of the angular momentum  $L_{ij} = x_i p_j - x_j p_i$  to  $i, j = 1, 2, 3, 4$  gives six generators of the group of proper rotations in four dimensions. However, we must keep in mind that unlike the purely geometrical  $SO(3)$  symmetry, the  $SO(4)$  symmetry is an accidental dynamical symmetry because  $r_4$  and  $p_4$  are not a real coordinate and momentum of the system in a fourth dimension. Therefore this symmetry is considered to be a dynamical symmetry instead of a geometrical one.

## 2.2.2 Particle Moving on a Plane under the Influence of a Constant Magnetic Field

Consider a particle moving on an infinite 2-dimensional plane under the influence of a constant magnetic field. When the direction of the magnetic field is perpendicular to the plane of motion, the Hamiltonian of the system takes the form

$$H = -\frac{1}{2M} (\partial_x^2 + \partial_y^2 + 2ieBx\partial_y - (eBx)^2). \quad (2.2.10)$$

Here we have used the asymmetric gauge with  $\vec{A} = (0, Bx, 0)$ . (This problem will be discussed in more details in chapter 4). It can be proved that there are three conserved quantities that commute with the above Hamiltonian. They are

$$L = x(p_y + x\frac{eB}{2}) - y(p_x + y\frac{eB}{2}), \quad (2.2.11)$$

$$R_x = -\frac{p_y}{Be}, \quad R_y = y + \frac{p_x}{eB}, \quad (2.2.12)$$

where  $B$  is the magnetic field. As it is clear from Eq.(2.2.11),  $L$  is the component of a conserved quantity that can be identified with the angular momentum. The quantities  $R_x$  and  $R_y$  represent two components of a vector that is pointing towards the center of what is classically a circular orbit of the particle on the plane (as we will explain later in chapter 4). The commutation relations for these three operators are given by

$$[L, \tilde{R}_x] = -i\tilde{R}_y, \quad [L, \tilde{R}_y] = i\tilde{R}_x, \quad [\tilde{R}_x, \tilde{R}_y] = i, \quad (2.2.13)$$

where  $\tilde{R}_x = -\sqrt{1/Be}R_x$  and  $\tilde{R}_y = \sqrt{1/Be}R_y$ . From the structure constants in the above equations we can identify the group as the Heisenberg group.

## 2.3 Energy Levels and Degeneracy from just Symmetry

One of the useful aspects of studying the symmetry of a quantum system is that it may enable us to find the energy levels and degeneracies without solving the Schrödinger equation [1]. To understand this, consider a Lie algebra with three generators, each of which being a Hermitian dynamical variable, say the components of the angular momentum  $L_x$ ,  $L_y$ , and  $L_z$ . As we saw in section (2.2) in such a case these three generators constitute an  $SU(2)$  algebra. The Casimir operator for this case is just

$$L^2 = L_x^2 + L_y^2 + L_z^2. \quad (2.3.1)$$

Let us define two operators as

$$\begin{aligned} L_+ &= L_x + iL_y, \\ L_- &= L_x - iL_y. \end{aligned} \quad (2.3.2)$$

Using the above equations as well as Eq.(2.2.6), we obtain the following commutation relations

$$[L_z, L_+] = L_+, \quad [L_z, L_-] = -L_-, \quad [L_+, L_-] = 2L_z. \quad (2.3.3)$$

An eigenfunction of  $L_z$  with eigenvalue  $m$  can at the same time be chosen as an eigenfunction of  $L^2$  since these operators commute. Then we have

$$\langle l, m | L^2 | l', m' \rangle = f(l)\delta_{mm'}\delta_{ll'}, \quad \langle l, m | L_z | l', m' \rangle = m\delta_{mm'}\delta_{ll'}, \quad (2.3.4)$$

where  $l$  is the quantum number associated with the eigenvalues of  $L^2$ . If we assume that the states  $|l, m\rangle$  are normalized and form a complete set, then the first of the Eqs.(2.3.3) leads to

$$\begin{aligned} \langle l, m|L_z|l', m'\rangle\langle l', m'|L_+|l'', m''\rangle - \langle l, m|L_+|l', m'\rangle\langle l', m'|L_z|l'', m''\rangle \\ = \langle l, m|L_+|l'', m''\rangle. \end{aligned} \quad (2.3.5)$$

The above equation together with Eqs.(2.3.3) gives

$$(m - m'' - 1)\langle l, m|L_+|l, m''\rangle = 0. \quad (2.3.6)$$

It is clear from the above equation that  $\langle l, m|L_+|l, m''\rangle$  vanishes unless  $m = m'' + 1$ . Repeating the same process with the second commutator of Eqs.(2.3.3) we obtain a vanishing  $\langle l, m|L_-|l, m''\rangle$  unless  $m = m'' - 1$ . This means that  $L_+$  is indeed a raising operator, while  $L_-$  is a lowering operator. The only non-vanishing matrix elements of these two operators are

$$\langle l, m + 1|L_+|l, m\rangle = \lambda_m, \quad \langle l, m|L_-|l, m + 1\rangle = \lambda_m^*. \quad (2.3.7)$$

The above Eq.(2.3.7) and the third commutation relation in Eqs.(2.3.3) give

$$|\lambda_{m-1}|^2 - |\lambda_m|^2 = 2m. \quad (2.3.8)$$

The above equation is a first order difference equation. The general solution is

$$|\lambda_m|^2 = C - m(m + 1), \quad (2.3.9)$$

where  $C$  is a constant. Since  $|\lambda_m|^2$  must be positive or zero and  $m$  is an integer, we have two values of  $m = m_1$  and  $m = m_2$  at which  $|\lambda_m|^2 = 0$ , while it is positive in between these two values. The solution of the equation  $C - m(m + 1) = 0$  gives

$$m_1 = -\frac{1}{2} + \frac{1}{2}(1 + 4C)^{\frac{1}{2}}, \quad m_2 = -\frac{1}{2} - \frac{1}{2}(1 + 4C)^{\frac{1}{2}}. \quad (2.3.10)$$

From the above equations we obtain  $m_2 + 1 = -m_1$ , which implies that  $m_1$  changes to  $-m_1$  by an integer number of steps. This leads to  $m_1$  either being an integer or a half-integer ( $m_1 = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$ ).

In order to evaluate  $f(l)$  we use the relation

$$L^2 = \frac{1}{2}(L_+L_- + L_-L_+) + L_z^2. \quad (2.3.11)$$

Using the above equation as well as Eqs.(2.3.4) gives

$$f(l) = \frac{1}{2}(|\lambda_{m-1}|^2 + |\lambda_m|^2) + m^2 = C = m_1(m_1 + 1) = l(l + 1). \quad (2.3.12)$$

Since  $m_1$  is the largest value of  $m$ , we can identify it with  $l$ . Accordingly, the eigenvalue of  $L^2$  is  $l(l + 1)$ , and for each value of  $l$  we have  $2l + 1$  values of  $m$ .

As we saw here, just knowing the symmetry, we found the spectrum of the operators  $L_z$ ,  $L^2$  and the degeneracy of states with quantum number  $l$  without solving any eigenvalue equation.

### 2.3.1 The Kepler Problem

The Kepler problem with  $\kappa = Ze^2$  is the case of the hydrogen atom. We can define two quantities using  $\vec{L}$  and  $\vec{R}$  [1].

$$\vec{I} = \frac{1}{2}(\vec{L} + \vec{R}), \quad \vec{K} = \frac{1}{2}(\vec{L} - \vec{R}). \quad (2.3.13)$$

It is easy to see that the components of  $\vec{I}$  and  $\vec{K}$  constitute separately an  $SU(2)$  algebra. In addition, these vectors commute with the Hamiltonian of Eq.(2.2.2) and they have a vanishing commutator with each other. Therefore we have

$$[I_i, I_j] = i\epsilon_{ijk}I_k, \quad [K_i, K_j] = i\epsilon_{ijl}K_l. \quad (2.3.14)$$

From the previous section we know that when generators constitute an  $SU(2)$  algebra, then the spectrum of the Casimir operator is  $l(l+1)$  (see Eq.(2.3.12)). Therefore the spectrum of  $\vec{I}^2$  is  $i(i+1)$  and the spectrum of  $\vec{K}^2$  is  $k(k+1)$ . As we said, the symmetry group for this system is  $SO(4) = SU(2) \otimes SU(2)$  accordingly, there are two Casimir operators, they are

$$C = \vec{I}^2 + \vec{K}^2 = \frac{1}{2}(\vec{L}^2 + \vec{R}^2), \quad C' = \vec{I}^2 - \vec{K}^2 = \vec{L} \cdot \vec{R}. \quad (2.3.15)$$

It can be proved from the expression of  $\vec{R}$  in Eq.(2.2.3) and  $\vec{L}$  that  $\vec{L} \cdot \vec{R} = 0$ . Accordingly,  $C' = 0$ , and this gives  $\vec{I}^2 = \vec{K}^2$ , which implies  $i = k$ . This result together with the first of Eqs.(2.3.15) lead to

$$C = \vec{I}^2 + \vec{K}^2 = 2k(k+1), \quad k = 0, \frac{1}{2}, 1, \dots \quad (2.3.16)$$

Substituting for the value of  $\vec{R}$  from Eq.(2.1.4) and using the definition of  $\vec{R}$  from Eq.(2.2.3) we get

$$C = \frac{1}{2}(\vec{L}^2 - \frac{M}{2E}\vec{R}^2) = -\frac{M\kappa^2}{4E} - \frac{1}{2}. \quad (2.3.17)$$

From the above and Eq.(2.3.15) we obtain the relation between the total energy  $E$  and  $k$  as

$$E = -\frac{M\kappa^2}{2(2k+1)^2} = -\frac{M\kappa^2}{2n^2}, \quad (2.3.18)$$

where  $n = 1, 2, 3, \dots$ . From the symmetry approach we know from section (2.3) that the eigenvalues of  $\vec{L}^2$  are  $l(l+1)$  with  $l = 0, \frac{1}{2}, 1, \dots$ . By adding an additional physical restriction that makes  $l$  just an integer, and knowing from Eqs.(2.3.13) that  $\vec{L} = \vec{I} + \vec{K}$ , this yield  $l$  with any value between  $i+k = 2k-1 = n-1$  down to  $|i-k| = 0$ . As we found in section (2.3), it was proved using symmetry only that for

each value of  $l$  there are  $2l + 1$  values for  $L_z = m$ . Accordingly, the total degeneracy of an energy level with quantum number  $n$  is

$$g = \sum_{l=0}^{n-1} (2l + 1) = n^2. \quad (2.3.19)$$

As we saw, it is possible to find the energy levels, the spectrum of the operator  $\vec{L}^2$ , the spectrum of the operator  $L_z$ , and the total degeneracy of an energy level  $g$  without solving the Schrödinger equation.

### 2.3.2 Particle under the Influence of a Magnetic Field

Another important example of using symmetry to calculate energy levels and degeneracies is the case of a particle moving on an infinite plane under the action of a magnetic field perpendicular to the plane.

As we mentioned before, in this case the Hamiltonian of the system is given by Eq.(2.2.10) and the conserved quantities are the angular momentum in Eq.(2.2.11) and the components of the Runge-Lenz vector given by Eqs.(2.2.12). The Hamiltonian in Eq.(2.2.10) can be written as

$$H = \frac{\omega}{2} (2L + \omega(R_x^2 + R_y^2)). \quad (2.3.20)$$

The energy levels and their degeneracies can be calculated using a similar approach as the one that has been used in section (2.3). To do this, we first define the raising and lowering operators

$$R_+ = R_x + iR_y, \quad R_- = R_x - iR_y. \quad (2.3.21)$$

Using the commutation relations in Eqs.(2.2.13) and the above equations give

$$[L, R_+] = R_+, \quad [L, R_-] = -R_-, \quad [R_+, R_-] = \frac{2M}{\omega}. \quad (2.3.22)$$

Using basis states that are eigenfunctions of  $L$  gives the matrix element

$$\langle m|L|m'\rangle = m\delta_{mm'}. \quad (2.3.23)$$

Assuming that the basis states are normalized, the first commutator in Eqs.(2.3.22) gives

$$\langle m|L|m'\rangle\langle m'|R_+|m''\rangle - \langle m|R_+|m'\rangle\langle m'|L|m''\rangle = \langle m|R_+|m''\rangle. \quad (2.3.24)$$

From the above equation and Eq.(2.3.23) we get

$$(m - m'' - 1)\langle m|R_+|m''\rangle = 0. \quad (2.3.25)$$

This means that the only non-vanishing matrix elements are the ones with  $m'' = m - 1$ , and  $R_+$  is indeed a raising operator. Using the same method for  $R_-$ , we can prove that it is indeed a lowering operator. Accordingly, the only non-vanishing matrix elements are the ones with  $m'' = m + 1$ . We write the non-vanishing matrix elements as

$$\langle m + 1|R_+|m\rangle = \lambda_m, \quad \langle m|R_-|m + 1\rangle = \lambda_m^*. \quad (2.3.26)$$

The third commutator of Eqs.(2.3.22) gives

$$\langle m|R_+|m'\rangle\langle m'|R_-|m\rangle - \langle m|R_-|m'\rangle\langle m'|R_+|m\rangle = \frac{2M}{\omega}. \quad (2.3.27)$$

Substituting the matrix elements from Eqs.(2.3.26) into the above equation gives

$$|\lambda_{m-1}|^2 - |\lambda_m|^2 = \frac{2M}{\omega}. \quad (2.3.28)$$

The solution of the above difference equation is

$$|\lambda_m|^2 = -m\frac{2M}{\omega} + C. \quad (2.3.29)$$

Since  $|\lambda_m|^2 \geq 0$  we obtain  $C = 2nM/\omega$ , where  $n$  is the upper value for  $m \geq 0$ . Unlike the case of the  $SU(2)$  symmetry, the solution of the difference equation Eq.(2.3.28) gives no condition on the value of  $m$  whether it is an integer, half-integer, or not. On the other hand, if we assume that there is a state in which  $m = 0$ , then definitely all the values of  $m \in \mathbb{Z}$  occur, since the state with  $m = 0$  can be raised or lowered by unit steps using  $R_+$  or  $R_-$ . It is obvious that we have an infinite degeneracy with  $m \in [-\infty, n]$ , where  $n$  is a positive integer.

In order to calculate the energy levels we write the Hamiltonian as

$$H = \frac{\omega}{2} \left( 2L + \omega \frac{1}{2} (R_+ R_- + R_- R_+) \right). \quad (2.3.30)$$

Since  $H$  and  $L$  commute they can have the same eigenfunctions. Accordingly the matrix representing the above operator is diagonal in the  $m$  representation and from the above Eq.(2.3.30) and Eq.(2.3.26) we get

$$\begin{aligned} E_m = \langle m|H|m\rangle &= \frac{1}{2M} \left( 2M\omega m + \omega^2 M \frac{1}{2} (|\lambda_{m-1}|^2 + |\lambda_m|^2) \right) \\ &= \omega \left( n + \frac{1}{2} \right). \end{aligned} \quad (2.3.31)$$

It is worth noting that there is another solution of Eq.(2.3.28) with negative  $n$  and negative  $m \leq n$ . However this solution must be discarded because it leads to negative energy according to the above equation. Since  $m \in [-\infty, n]$  we can write

$$n = \frac{m + |m|}{2} + k, \quad k = 0, 1, 2, \dots \quad (2.3.32)$$

Here the degeneracy is infinite but countable. This is because we are working in the  $L$  representation. Working in a different representation like the  $p_x$  or  $p_y$  representation also leads to infinite degeneracy, however, in this case it is continuous rather than countable. This will be discussed later in chapter 3.

To show that the previous results are correct, we will now obtain the energy levels and degeneracies by solving the Schrödinger equation for this problem. For technical reasons the calculations in the following will be carried out using the symmetric gauge. We must stress here that the choice of gauge does not change the physics. Nevertheless it will simplify the solution of the Schrödinger equation. The Hamiltonian in symmetric gauge  $\vec{A} = (-By/2, Bx/2, 0)$  takes the following form.

$$H = -\frac{1}{2m} \left( \partial_x^2 + \partial_y^2 + ieBx\partial_y - ieBy\partial_x - \left( \frac{eBx}{2} \right)^2 - \left( \frac{eBy}{2} \right)^2 \right). \quad (2.3.33)$$

In this gauge the angular momentum takes the form

$$L = xp_y - yp_x. \quad (2.3.34)$$

After writing the Hamiltonian in polar coordinates, it can be proved that a solution can be chosen which is simultaneously an eigenfunction of the Hamiltonian in Eq.(2.3.33) as well as of the angular momentum operator of Eq.(2.3.34). It has the following general form

$$\psi(r, \varphi) = \exp(im\varphi)\psi(r). \quad (2.3.35)$$

Accordingly, the Schrödinger equation for this problem can be written as

$$(2m + M\omega r^2)^2\psi(r) - 16r^2\partial_{r^2}\psi(r) - 16r^4\partial_{r^2}^2\psi(r) = 8MEr^2\psi(r). \quad (2.3.36)$$

By making the substitutions

$$\rho = \frac{ME}{2}r^2 = \alpha^2r^2, \quad \mu = \frac{2\omega}{E}, \quad \psi(r) = \exp\left(-\frac{\rho\mu}{4}\right)F(\rho), \quad (2.3.37)$$

Eq.(2.3.36) takes the form

$$\left( \frac{m^2}{4} + \left( \frac{\mu(m+1)}{4} - 1 \right) \rho \right) F(\rho) + \left( \rho \frac{\mu}{2} - 1 \right) \rho \partial_\rho F(\rho) - \rho^2 \partial_\rho^2 F(\rho) = 0. \quad (2.3.38)$$

The solution can be found using Frobenius' method, by assuming that it is a series of the form

$$F(\rho) = \sum_{n=0}^{\infty} a_n \rho^{s+n}. \quad (2.3.39)$$

Substituting the above solution in Eq.(2.3.38), and after that expressing the value of the coefficients  $a_n$  and  $s$ , we find that  $s = \pm|m|/2$ . Here the  $a_n$ 's are the coefficients of the confluent hypergeometric function. Therefore we can write

$$\psi(r) = A(r^2 \alpha^2)^{|m|/2} \exp\left(-\frac{r^2 \mu \alpha}{4}\right) {}_1F_1\left(\frac{m}{2} + \frac{|m|}{2} + \frac{1}{2} - \frac{2}{\mu}, |m| + 1, \alpha^2 r^2\right), \quad (2.3.40)$$

where the solution with  $s = -|m|/2$  has been discarded because it diverges at the origin. The confluent hypergeometric function diverges as  $r \rightarrow \infty$  unless the series terminates (see chapter 3). This can happen only if the first argument of  ${}_1F_1(a, b, x)$  is a negative integer. Accordingly we get

$$-k = \frac{m}{2} + \frac{|m|}{2} + \frac{1}{2} - \frac{2}{\mu}. \quad (2.3.41)$$

However, in Eqs.(2.3.37) we defined  $\mu = 2\omega/E$ . Therefore the above equation leads to

$$E = \omega \left( \frac{m + |m|}{2} + \frac{1}{2} + k \right), \quad k = 0, 1, 2, 3, \dots \quad (2.3.42)$$

This is exactly the same result that was reached in the beginning of this section by investigating just the symmetry.



# Chapter 3

## Runge-Lenz Vector and Accidental $SU(2)$ Symmetry for Motion on a Cone

### 3.1 Introduction to Motion on a Cone

Among countless classical systems the  $1/r$  and harmonic oscillator potentials are exceptional because in addition to rotation invariance they have accidental dynamical symmetries as was explained in chapters 1 and 2. There are other conserved quantities in addition to the angular momentum vector that arises from spherical symmetry. In the case of the Kepler potential the additional quantity is the three-component Runge-Lenz vector, and in the case of the isotropic harmonic oscillator it is a rank two tensor.

At the classical level the accidental symmetries imply that all bound orbits are closed, while at the quantum level they lead to additional degeneracies in the discrete energy spectrum. In general we can say that the  $SO(d)$  rotational symmetry of the  $d$ -dimensional  $1/r$  potential (the Coulomb potential for  $d = 3$ ) is enlarged to the accidental symmetry  $SO(d + 1)$ . The additional conserved quantities form the  $d$ -components of the Runge-Lenz vector. Similarly, the  $d$ -dimensional harmonic oscillator has an  $SO(d)$  rotational symmetry which is contained as a subgroup in an accidental  $SU(d)$  symmetry.

The uniqueness of the  $1/r$  and  $r^2$  potentials was realized by Bertrand in 1873. He proved that they are the only spherically symmetric scalar potentials in Euclidean space for which all bound orbits are closed [2]. On the other hand, there exist

a number of other systems with accidental symmetries involving vector potentials, direction-dependent potentials or non-Euclidean spaces. For example, a free particle confined to the surface of the  $d$ -dimensional hyper-sphere  $S^d$  moves along a great circle (which obviously is closed). Indeed the rotational  $SO(d+1)$  symmetry of this system corresponds to the accidental symmetry of the  $1/r$  potential. At the quantum mechanical level it was first realized by Fock in 1935 that the hydrogen atom possesses “hyper-spherical” symmetry [15]. Based on this work, Bargmann [16] has shown that the generators of the accidental symmetry are the components of the Runge-Lenz vector [17]

$$\vec{R} = \frac{1}{2M} (\vec{p} \times \vec{L} - \vec{L} \times \vec{p}) - \kappa \vec{e}_r. \quad (3.1.1)$$

Accidental symmetry may also involve vector potentials, for example, the symmetry involving the vector potential in cyclotron motion [18, 19]. In all these cases, there is a deep connection between the fact that all bound classical orbits are closed and additional degeneracies in the discrete energy spectrum of the corresponding quantum system. The subject of accidental symmetry has been reviewed, for example, by McIntosh [20].

In order to further investigate the phenomenon of accidental symmetries in this chapter, we study a particle confined to the surface of a cone. A cone is obtained from the plane by removing a wedge of deficit angle  $\delta$  and gluing the open ends back together. As a consequence, the polar angle  $\chi$  no longer extends from 0 to  $2\pi$ , but only to  $2\pi - \delta$ . The geometry of the cone is illustrated in figure (3.1). The surface of a cone can be mapped on a 2-dimensional plane with points defined by polar coordinates  $r$  and the angle  $\varphi$  such that it again covers the full interval, i.e.

$$\varphi = \frac{\chi}{s} \in [0, 2\pi], \quad (3.1.2)$$

with the scale factor

$$s = 1 - \frac{\delta}{2\pi}. \quad (3.1.3)$$

The kinetic energy of a particle of mass  $M$  then takes the form

$$T = \frac{M}{2} (\dot{r}^2 + r^2 \dot{\chi}^2) = \frac{M}{2} (\dot{r}^2 + r^2 s^2 \dot{\varphi}^2). \quad (3.1.4)$$

The radial component of the momentum  $p_r$  is canonically conjugate to  $r$ , i.e.

$$p_r = \frac{\partial T}{\partial \dot{r}} = M\dot{r}. \quad (3.1.5)$$

Similarly, the canonically conjugate momentum corresponding to the rescaled angle  $\varphi$  is given by the angular momentum

$$L = \frac{\partial T}{\partial \dot{\varphi}} = Mr^2 s^2 \dot{\varphi}, \quad (3.1.6)$$

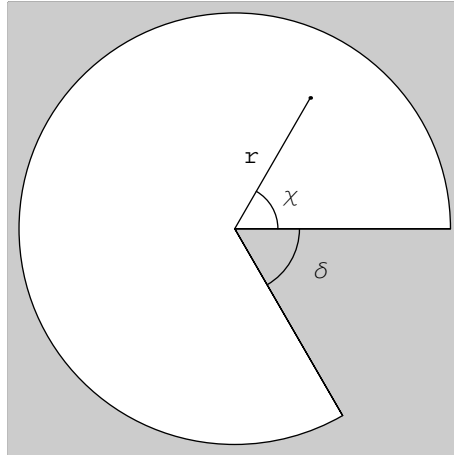


Figure 3.1: We can think of a cone as a result of cutting a wedge of deficit angle  $\delta$  out of the 2-dimensional plane, and by gluing the open ends back together. A point on the cone is defined by the distance  $r$  from the tip and an angle  $\chi$  which varies between 0 and  $2\pi - \delta$ . Unlike the cone in the figure (3.1), the actual cone considered in this work extends over the whole range  $r \in [0, \infty]$ .

such that

$$T = \frac{1}{2M} \left( p_r^2 + \frac{L^2}{r^2 s^2} \right). \quad (3.1.7)$$

From Eq.(3.1.7) we can see that a positive deficit angle  $\delta$  (i.e.  $s < 1$ ) leads to an increase of the centrifugal barrier, while a negative deficit angle ( $s > 1$ ) leads to its reduction. As usual, upon canonical quantization (and again using natural units in which  $\hbar = 1$ ) the angular momentum conjugate to the rescaled angle  $\varphi$  is represented by the operator

$$L = -i\partial_\varphi. \quad (3.1.8)$$

## 3.2 The Domains of Operators and Hermiticity

The domains of the operators in a corresponding Hilbert space, Hermiticity and self-adjointness play an important role in this work. For some mathematical background we refer to [21, 22]. The Hilbert space can be denoted as  $\mathcal{H} = L_2((0, \infty) \times (0, 2\pi); r)$ . For a particle moving on a cone it consists of the square-integrable functions  $\Psi(r, \varphi)$  with  $r \in (0, \infty)$ ,  $\varphi \in [0, 2\pi]$ , and with the norm  $\langle \Psi | \Psi \rangle < \infty$ . The norm here can be evaluated using the scalar product which is defined by

$$\langle \Phi | \Psi \rangle = \int_0^\infty dr r \int_0^{2\pi} d\varphi \Phi(r, \varphi)^* \Psi(r, \varphi). \quad (3.2.1)$$

As it is obvious from the definition of the Hilbert space, the functions in that space need not be continuous nor differentiable, they even need not be periodic. In the following argument we will use the angular momentum operator as an example. In order to completely define a quantum mechanical operator  $O$ , the domain of wave functions on which the operator acts must be defined, say  $\mathcal{D}[O] \subset \mathcal{H}$ . This is because  $L$  acts on a wave function that must be differentiable at least once with respect to  $\varphi$ .

An operator  $O$  is defined to be Hermitean or called symmetric in other mathematical terms if

$$\langle O\Phi|\Psi\rangle = \langle\Phi|O\Psi\rangle, \quad (3.2.2)$$

for all wave functions  $\Phi, \Psi \in \mathcal{D}[O]$ . To examine the Hermiticity of  $L$ , let us perform a partial integration that gives

$$\begin{aligned} \langle O\Phi|\Psi\rangle &= \int_0^\infty dr r \int_0^{2\pi} d\varphi [-i\partial_\varphi\Phi(r, \varphi)]^*\Psi(r, \varphi) \\ &= \int_0^\infty dr r \int_0^{2\pi} d\varphi \Phi(r, \varphi)^*[-i\partial_\varphi\Psi(r, \varphi)] \\ &+ i \int_0^\infty dr r \Phi(r, \varphi)^*\Psi(r, \varphi)\Big|_{\varphi=0}^{\varphi=2\pi} \\ &= \langle\Phi|O\Psi\rangle + i \int_0^\infty dr r \Phi(r, \varphi)^*\Psi(r, \varphi)\Big|_{\varphi=0}^{\varphi=2\pi}. \end{aligned} \quad (3.2.3)$$

Thus, the operator  $L$  is Hermitean if

$$\Phi(r, \varphi)^*\Psi(r, \varphi)\Big|_{\varphi=0}^{\varphi=2\pi} = 0. \quad (3.2.4)$$

Understanding self-adjointness and Hermiticity is very important here because Hermiticity alone does not guarantee a real-valued spectrum. On the other hand, self-adjointness means that the operator indeed has a real-valued spectrum. An operator  $O$  is self-adjoint (i.e.  $O = O^\dagger$ ) if it is Hermitean and the domain of its adjoint  $O^\dagger$  coincides with the domain of  $O$ , i.e.  $\mathcal{D}[O^\dagger] = \mathcal{D}[O]$ , for all  $\Psi \in \mathcal{D}[O]$ . For example, let us consider the operator  $L$  in the domain of differentiable functions  $\Psi \in \mathcal{H}$  (with  $L\Psi \in \mathcal{H}$ ). There are two possibilities for the operator  $L$  to be Hermitean. It is either that  $\Psi(r, 0) = \Psi(r, 2\pi) = 0$  then  $\mathcal{D}[L]$  is smaller than  $\mathcal{D}[L^\dagger]$  because  $\Phi$  is unrestricted on the boundary, or  $\Phi(r, 0) = \Phi(r, 2\pi) = 0$  then  $\mathcal{D}[L]$  is bigger than  $\mathcal{D}[L^\dagger]$ . For neither case the operator  $L$  is self-adjoint restricted to such domains. Self-adjointness of  $L$  can be reached by choosing a different domain for  $L$ . Let us impose the boundary condition

$$\Psi(r, 2\pi) = z^*\Psi(r, 0), \quad z \in \mathbb{C}. \quad (3.2.5)$$

Then the condition in Eq.(3.2.4) implies that

$$\Phi(r, 2\pi)^* \Psi(r, 2\pi) - \Phi(r, 0)^* \Psi(r, 0) = [\Phi(r, 2\pi)^* z - \Phi(r, 0)^*] \Psi(r, 0) = 0, \quad (3.2.6)$$

such that

$$\Phi(r, 2\pi) = \frac{1}{z^*} \Phi(r, 0). \quad (3.2.7)$$

For  $L$  to be self-adjoint (i.e. to have  $\mathcal{D}[L^\dagger] = \mathcal{D}[L]$ ) the functions  $\Phi \in \mathcal{D}[L^\dagger]$  must obey the same condition as  $\Psi \in \mathcal{D}[L]$ . This implies  $z = 1/z^* = \exp(i\theta)$ . The angle  $\theta$  characterizes a one-parameter family of self-adjoint extensions of the operator  $L$  to the domain of differentiable functions obeying the boundary condition

$$\Psi(r, 2\pi) = \exp(i\theta) \Psi(r, 0). \quad (3.2.8)$$

Since the coordinates  $\varphi = 0$  and  $\varphi = 2\pi$  describe the same physical point on the cone, this requires a single-valued wave function. Therefore, for wave functions on the cone, the domain  $\mathcal{D}[L] \in \mathcal{H}$  consists of the periodic differentiable functions  $\Psi$  (with  $L\Psi \in \mathcal{H}$ ) which obey

$$\Psi(r, 2\pi) = \Psi(r, 0). \quad (3.2.9)$$

### 3.2.1 The Hermiticity and Self-Adjointness of $T$

Let us consider the operator of the kinetic energy,

$$T = -\frac{1}{2M} \left( \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2 s^2} \partial_\varphi^2 \right). \quad (3.2.10)$$

Since  $\partial_\varphi = s\partial_\chi$ , this operator seems to be identically the same as the standard one operating on wave functions on the plane. For a complete definition of  $T$ , its domain  $\mathcal{D}[T]$  must be identified. The wave functions of  $\mathcal{D}[T]$  should again obey Eq.(3.2.9), which means that they should be periodic in the rescaled angle  $\varphi$  (not in the original polar angle  $\chi$  of the full plane). Writing

$$\Psi(r, \varphi) = \psi(r) \exp(im\varphi), \quad (3.2.11)$$

Eq.(3.2.9) leads to  $m \in \mathbb{Z}$ , and we have

$$T = -\frac{1}{2M} \left( \partial_r^2 + \frac{1}{r} \partial_r \right) + \frac{m^2}{2Mr^2 s^2}. \quad (3.2.12)$$

The radial wave functions belong to the radial Hilbert space  $\mathcal{H}_r = L_2((0, \infty); r)$ . It is well-known that the operator  $-i\partial_r$  is not Hermitean in  $\mathcal{H}_r$ . This follows from

$$\begin{aligned}
\langle \phi | \partial_r \psi \rangle &= \int_0^\infty dr r \phi(r)^* \partial_r \psi(r) \\
&= - \int_0^\infty dr \partial_r [r \phi(r)^*] \psi(r) + r \phi(r)^* \psi(r) \Big|_0^\infty \\
&= - \int_0^\infty dr [r \partial_r \phi(r)^* + \phi(r)^*] \psi(r) + r \phi(r)^* \psi(r) \Big|_0^\infty \\
&= \int_0^\infty dr r \left[ -\partial_r \phi(r)^* - \frac{1}{r} \phi(r)^* \right] \psi(r) + r \phi(r)^* \psi(r) \Big|_0^\infty \\
&= \langle \partial_r^\dagger \phi | \psi \rangle + r \phi(r)^* \psi(r) \Big|_0^\infty.
\end{aligned} \tag{3.2.13}$$

Therefore the Hermitean conjugate of  $\partial_r$  is

$$\partial_r^\dagger = -\partial_r - \frac{1}{r}. \tag{3.2.14}$$

The above equation is correct if the boundary term vanishes (i.e.  $r \phi(r)^* \psi(r) \Big|_0^\infty = 0$ ). On the other hand, it can be proved that the operator  $D_r$  given below is Hermitean in the domain  $\mathcal{D}[D_r]$  of differentiable functions  $\psi(r)$  (with  $D_r \psi \in \mathcal{H}_r$ ) obeying  $\psi(0) = 0$ , with

$$D_r = -i \left( \partial_r + \frac{1}{2r} \right) = -i \frac{1}{\sqrt{r}} \partial_r \sqrt{r}. \tag{3.2.15}$$

However, that does not mean that it represents a physical observable because it is not self-adjoint. On the other hand,

$$D_r^2 = - \left( \partial_r + \frac{1}{2r} \right)^2 = -\partial_r^2 - \frac{1}{r} \partial_r + \frac{1}{4r^2}, \tag{3.2.16}$$

which is closely related to the kinetic energy operator  $T$ , possesses a family of self-adjoint extensions. Eq.(3.2.14) also seems to readily imply Hermiticity of the kinetic energy operator  $T$  because, at least formally,

$$\begin{aligned}
\left( \partial_r^2 + \frac{1}{r} \partial_r \right)^\dagger &= \partial_r^{\dagger 2} + \partial_r^\dagger \frac{1}{r} = \left( \partial_r + \frac{1}{r} \right)^2 - \left( \partial_r + \frac{1}{r} \right) \frac{1}{r} \\
&= \partial_r^2 + \frac{2}{r} \partial_r - \frac{1}{r} \partial_r = \partial_r^2 + \frac{1}{r} \partial_r.
\end{aligned} \tag{3.2.17}$$

In more details

$$\begin{aligned}
\langle \phi | \left( \partial_r^2 + \frac{1}{r} \partial_r \right) \psi \rangle &= \int_0^\infty dr r \phi(r)^* \left( \partial_r^2 + \frac{1}{r} \partial_r \right) \psi(r) \\
&= - \int_0^\infty dr \partial_r [r \phi(r)^*] \partial_r \psi(r) + r \phi(r)^* \partial_r \psi(r) \Big|_0^\infty \\
&\quad - \int_0^\infty dr \partial_r \phi(r)^* \psi(r) + \phi(r)^* \psi(r) \Big|_0^\infty \\
&= \int_0^\infty dr \partial_r^2 [r \phi(r)^*] \psi(r) - \partial_r [r \phi(r)^*] \psi(r) \Big|_0^\infty \\
&\quad - \int_0^\infty dr \partial_r \phi(r)^* \psi(r) + [r \phi(r)^* \partial_r \psi(r) + \phi(r)^* \psi(r)] \Big|_0^\infty \\
&= \int_0^\infty dr r \left[ \left( \partial_r^2 + \frac{2}{r} \partial_r - \frac{1}{r} \partial_r \right) \phi(r)^* \right] \psi(r) \\
&\quad + [r \phi(r)^* \partial_r \psi(r) - r \partial_r \phi(r)^* \psi(r)] \Big|_0^\infty \\
&= \left\langle \left( \partial_r^2 + \frac{1}{r} \partial_r \right) \phi | \psi \right\rangle + [r \phi(r)^* \partial_r \psi(r) - r \partial_r \phi(r)^* \psi(r)] \Big|_0^\infty .
\end{aligned} \tag{3.2.18}$$

However, this imposes a delicate condition so that  $T$  is Hermitean. The condition is

$$[r \phi(r)^* \partial_r \psi(r) - r \partial_r \phi(r)^* \psi(r)] \Big|_0^\infty = 0. \tag{3.2.19}$$

The condition implies that the component of the probability current density  $j_r$  along  $\vec{r}$  must vanish at  $\infty$ . This become clear when we replace  $\phi(r)$  by  $\psi(r)$ . The condition admits a one-parameter family of self-adjoint extensions. The self-adjoint extensions of  $T$  have been studied in [23]. It turns out that the tip of the cone is a singular point that may be endowed with non-trivial physical properties. These properties are described by a real-valued parameter that defines a family of self-adjoint extensions. Physically speaking, the different self-adjoint extensions correspond to properly renormalized  $\delta$ -function potentials of different strengths located at the tip of the cone. In this paper, we limit ourselves to the case without  $\delta$ -function potentials, which corresponds to the so-called Friedrichs extension [21] characterized by the boundary condition

$$\lim_{r \rightarrow 0} r \partial_r \psi(r) = 0. \tag{3.2.20}$$

If we impose this condition on  $\psi \in \mathcal{D}[T]$ , and also want to satisfy Eq.(3.2.19), the function  $\phi \in \mathcal{D}[T^\dagger]$  must also obey Eq.(3.2.20). As a result,  $\mathcal{D}[T^\dagger] = \mathcal{D}[T]$ , such that  $T = T^\dagger$  is indeed self-adjoint.

While the cone is as flat as the plane, its singular tip and its deficit angle  $\delta$  have crucial effects on the dynamics. In the following, we will consider a particle

moving on a cone and bound to its tip by a  $1/r$  or  $r^2$  potential. Interestingly, when the deficit angle divided by  $2\pi$  (or equivalently  $s$ ) is a rational number, all bound classical orbits are again closed and once more there are additional degeneracies in the discrete spectrum of the Hamilton operator  $H$ . Just like in the plane, the  $1/r$  and  $r^2$  potentials on a cone have accidental  $SU(2)$  symmetries. However, unlike in the plane, the corresponding multiplets may now have fractional “spin” and unusual degeneracies. This unusual behavior arises because, in this case, the Runge-Lenz vector  $\vec{R}$  — although Hermitean in its appropriate domain  $\mathcal{D}[\vec{R}]$  — does not act as a Hermitean operator in the domain  $\mathcal{D}[H]$  of the Hamiltonian and thus does not represent a proper physical observable.

The problem of space-times with a conical singularity was investigated by 't Hooft [24] and by Deser and Jackiw [25] concerning  $(2 + 1)$ -dimensional Einstein gravity. In this context, the Klein-Gordon and the Dirac equation have also been studied [26]. Conical space-times also arise in the study of cosmic strings and are related to the Aharonov-Bohm effect [27]. Indeed,  $1/r$  and  $r^2$  potentials have already been considered in this context [8, 28], however, without discussing accidental symmetries. Furthermore, graphene — a single sheet of graphite, i.e. a honeycomb of carbon hexagons — can be bent to form cones by adding or removing a wedge of carbon atoms and by replacing one hexagon by a carbon hepta- or pentagon [29, 30]. While the low-energy degrees of freedom in graphene are massless Dirac fermions, in this thesis we limit ourselves to studying the Schrödinger equation for motion on a cone.

### 3.3 The $1/r$ Potential on a Cone

In this section we will study a particle moving on the surface of a cone bound to its tip by a  $1/r$  potential

$$V(r) = -\frac{\kappa}{r}. \quad (3.3.1)$$

The corresponding total energy is thus given by

$$H = T + V = \frac{1}{2M} \left( p_r^2 + \frac{L^2}{r^2 s^2} \right) - \frac{\kappa}{r}. \quad (3.3.2)$$

#### 3.3.1 Classical Solutions

The classical orbit can be found by solving Newton’s equation of motion for this problem. The Newton orbit equation is

$$\frac{d^2 u}{d^2 \theta} + u = -\frac{f(u^{-1})}{h^2 u^2}, \quad (3.3.3)$$



where  $u = 1/r$ ,  $h = r^2\dot{\theta}$ ,  $f(u^{-1})$  is component of the central force in the  $e_r$ -direction. The above equation is valid only for central potentials, i.e,  $f(\vec{r}) = f(r)$ . The solution of this equation for the  $1/r$  potential is

$$\frac{1}{r} = \frac{M\kappa s^2}{L^2} [1 + e \cos(s(\varphi - \varphi_0))], \quad (3.3.4)$$

with the eccentricity given by

$$e = \sqrt{1 + \frac{2EL^2}{M\kappa^2 s^2}}, \quad (3.3.5)$$

where  $E < 0$  is the energy and  $L$  is the angular momentum. The radial component of the momentum takes the form

$$p_r = \frac{M\kappa s}{L} e \sin(s(\varphi - \varphi_0)). \quad (3.3.6)$$

Here  $\varphi_0$  is the angle between the positive  $x$ - direction and a vector pointing towards the perihelion. From Eq.(3.3.4) we see that the classical orbit is closed only when  $s = p/q$  is a rational number (with  $p, q \in \mathbb{N}$  not sharing a common divisor). For  $s = p/q$ , after  $q$  revolutions around the tip of the cone, both  $r$  and  $p_r$  return to their initial values. Some examples of classical orbits are shown in figure (3. 2). Without this condition the orbit never closes which means that both  $r$  and  $p_r$  never return to their initial value after any length of time.

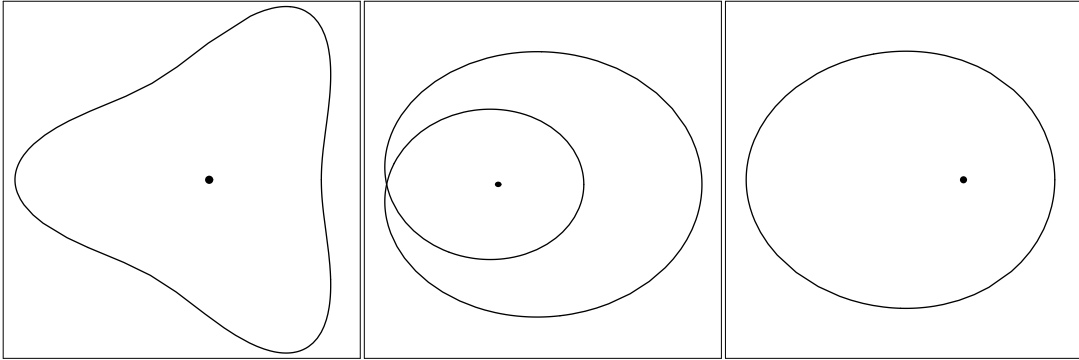


Figure 3.2: *Examples of bound classical orbits for the  $1/r$  potential with  $s = 3$  (left),  $s = \frac{1}{2}$  (middle), and  $s = 1$  (right). The latter case represents a standard Kepler ellipse. The orbits are shown in the  $x$ - $y$ -plane with  $(x, y) = r(\cos \varphi, \sin \varphi)$  where  $\varphi = \chi/s \in [0, 2\pi]$  is the rescaled polar angle.*

### 3.3.2 Semi-classical Bohr-Sommerfeld Quantization

Let us consider Bohr-Sommerfeld quantization. The quantization condition for the angular momentum takes the form

$$\oint d\varphi L = 2\pi L = 2\pi m, \quad (3.3.7)$$

such that  $L = m \in \mathbb{Z}$ . Similarly, the quantization condition for the radial motion is given by

$$\oint dr p_r = 2\pi \left( n_r + \frac{1}{2} \right), \quad n_r \in \{0, 1, 2, \dots\}. \quad (3.3.8)$$

The factor  $1/2$ , which is sometimes not taken into account in Bohr-Sommerfeld quantization, arises for librations but is absent for rotations. Using Eqs.(3.3.4), (3.3.5), and (3.3.6) and integrating over the period  $2\pi/s$  it is straightforward to obtain

$$\oint dr p_r = \int_0^{2\pi/s} d\varphi \frac{|L|e^2 \sin^2(s(\varphi - \varphi_0))}{(1 + e \cos(s(\varphi - \varphi_0)))^2} = 2\pi \left( \sqrt{-\frac{M\kappa^2}{2E}} - \frac{|L|}{s} \right), \quad (3.3.9)$$

which leads to

$$E = -\frac{M\kappa^2}{2 \left( n_r + \frac{|m|}{s} + \frac{1}{2} \right)^2}. \quad (3.3.10)$$

The above result is exact and not just limited to the semi-classical regime.

### 3.3.3 Solution of the Schrödinger Equation

The radial Schrödinger equation takes the form

$$\left[ -\frac{1}{2M} \left( \partial_r^2 + \frac{1}{r} \partial_r \right) + \frac{m^2}{2Mr^2 s^2} - \frac{\kappa}{r} \right] \psi(r) = E\psi(r), \quad (3.3.11)$$

which can be solved using Frobenius' method [8].

$$\psi_{n_r, m}(r) = A \exp\left(-\frac{\alpha r}{2}\right) (\alpha r)^{|m|/s} {}_1F_1\left(-n_r, \frac{2|m|}{s} + 1, \alpha r\right), \quad (3.3.12)$$

with

$$\alpha = \sqrt{8M|E|}, \quad (3.3.13)$$

where  ${}_1F_1$  is a confluent hyper-geometric function. The corresponding quantized energy values are given by Eq.(3.3.10). Now  $n_r$  is the number of nodes of the radial

wave function and  $m \in \mathbb{Z}$  is the angular momentum quantum number. Parity symmetry together with the  $SO(2)$  rotational symmetry ensures the degeneracy of states with quantum numbers  $m$  and  $-m$ . The relation between a closed bound orbit and additional accidental degeneracies in the spectrum can be realized here from Eq.(3.3.4). When the scale factor  $s$  is a rational number the orbit is closed. Equivalently, Eq.(3.3.10) gives additional degeneracies.

It is important to note that Eq.(3.3.11) has other solutions that diverge at the origin. These solutions will play a pivotal role in counting the degeneracies of different states. For example, let us consider the case  $s = \frac{1}{2}$  corresponding to the deficit angle  $\delta = \pi$ . In that case, a wave function without nodes (i.e. with  $n_r = 0$ ) and with angular momentum  $m = \pm 1$  is degenerate in energy with a wave function with two nodes ( $n_r = 2$ ) and with  $m = 0$ . As another example, let us consider  $s = 2$  which corresponds to the negative deficit angle  $\delta = -2\pi$ . In this case, one builds a “cone” by cutting two planes open and gluing them together in the same way as the double-layered Riemann surface of the complex square root. This effectively lowers the centrifugal barrier by a factor of  $s^2 = 4$ . In this case, a wave function without nodes ( $n_r = 0$ ) and with  $m = \pm 2$  is degenerate with a wave function with one node ( $n_r = 1$ ) and with  $m = 0$ . Similarly, for  $s = n \in \mathbb{N}$ , one glues  $n$  cut planes to a “cone” in the same way as the multi-layered Riemann surface of the complex  $n$ -th root. Now, a wave function without nodes ( $n_r = 0$ ) and with  $m = \pm n$  is degenerate with a wave function with one node ( $n_r = 1$ ) and with  $m = 0$ . Some features of the energy spectrum are illustrated in figure (3.3).

### 3.3.4 Runge-Lenz Vector and $SU(2)$ Algebra

When we realize that all classical orbits are closed, we expect that there must be a hidden conserved quantity. Quantum mechanically this leads to a Hamilton operator with a discrete spectrum and accidental degeneracies, and this suggests that there must be a corresponding accidental symmetry. From the Kepler problem in 2-d (with  $s = 1$ ) we know that the corresponding conserved quantity is the Runge-Lenz vector and the accidental symmetry is an  $SU(2)$  symmetry. The Runge-Lenz vector can be constructed for any  $s = p/q$ . However, it should be pointed out that, due to the conical geometry, the resulting object no longer transforms as a proper vector. We still continue to refer to it as the “Runge-Lenz vector”.

At the classical level, we can use eqs.(3.3.4) and (3.3.6) to write

$$\begin{aligned} \kappa e \cos(s(\varphi - \varphi_0)) &= \kappa e [\cos(s\varphi) \cos(s\varphi_0) + \sin(s\varphi) \sin(s\varphi_0)] = \frac{L^2}{Mr s^2} - \kappa, \\ \kappa e \sin(s(\varphi - \varphi_0)) &= \kappa e [\sin(s\varphi) \cos(s\varphi_0) - \cos(s\varphi) \sin(s\varphi_0)] = \frac{p_r L}{Ms}, \end{aligned} \quad (3.3.14)$$

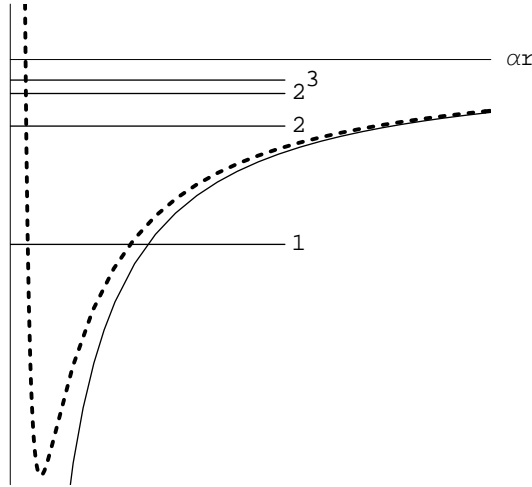


Figure 3.3: The  $1/r$  potential (solid curve) together with an effective potential including the centrifugal barrier with  $m = \pm 1$  (dashed curve) for  $s = 3$ . The energies of the ground state and the first three excited states are indicated by horizontal lines. The numbers besides the lines specify the degree of degeneracy. The ground state (with  $n_r = 0, m = 0$ ) is non-degenerate, while the first and second excited states (with  $n_r = 0, m = \pm 1$  and  $n_r = 0, m = \pm 2$ , respectively) are two-fold degenerate due to parity symmetry. The third excited level has an accidental three-fold degeneracy and consists of the states with  $n_r = 0, m = \pm 3$  and  $n_r = 1, m = 0$ .

such that

$$\begin{aligned} R_x &= \kappa e \cos(s\varphi_0) = \left( \frac{L^2}{Mr s^2} - \kappa \right) \cos(s\varphi) + \frac{p_r L}{Ms} \sin(s\varphi), \\ R_y &= \kappa e \sin(s\varphi_0) = \left( \frac{L^2}{Mr s^2} - \kappa \right) \sin(s\varphi) - \frac{p_r L}{Ms} \cos(s\varphi), \end{aligned} \quad (3.3.15)$$

are indeed independent of time or constants of motion. It is easy to see that for  $s = 1$ ,  $R_x$  and  $R_y$  are just the components of the familiar Runge-Lenz vector. There are two ways to verify that these two components are constants of motion, either by proving that they have a vanishing Poisson bracket with the Hamilton function or by direct substitution of Eq.(3.3.4) and Eq.(3.3.6) in the above expression of  $R_x$  and  $R_y$ . If the result is independent of  $\varphi$  then they are constants of motion. It should be noted that, for non-integer values of  $s$ , the quantities  $R_x$  and  $R_y$  are not conserved quantities in the usual sense. In particular, they are not single-valued functions of the coordinates  $x = r \cos \varphi$  and  $y = r \sin \varphi$ , but depend on the angle  $\varphi$  itself. As a consequence, the values of  $R_x$  and  $R_y$  depend on the history of the motion, i.e. on the number of revolutions around the tip of the cone. However, quantities that are

“conserved” only because they refer back to the initial conditions, do not qualify as proper physical constants of motion. To further clarify this issue, it is useful to construct the complex variables

$$R_{\pm} = R_x \pm iR_y = \left( \frac{L^2}{Mr s^2} - \kappa \mp i \frac{p_r L}{Ms} \right) \exp(\pm i s \varphi). \quad (3.3.16)$$

For rational values  $s = p/q$  (with  $p, q \in \mathbb{N}$ ) the quantities

$$R_{\pm}^q = \left( \frac{L^2}{Mr s^2} - \kappa \mp i \frac{p_r L}{Ms} \right)^q \exp(\pm i p \varphi) \quad (3.3.17)$$

are single-valued functions of  $x = r \cos \varphi$  and  $y = r \sin \varphi$ , and hence qualify as proper conserved quantities.

The length of the Runge-Lenz vector is given by

$$\begin{aligned} R^2 &= R_x^2 + R_y^2 = \left( \frac{L^2}{Mr s^2} - \kappa \right)^2 + \left( \frac{p_r L}{Ms} \right)^2 \\ &= 2 \frac{L^2}{Ms^2} \left( \frac{p_r^2}{2M} + \frac{L^2}{2Mr^2 s^2} - \frac{\kappa}{r} \right) + \kappa^2 = \frac{2HL^2}{Ms^2} + \kappa^2. \end{aligned} \quad (3.3.18)$$

In the quantum mechanical treatment it will turn out to be useful to introduce the rescaled variables

$$\tilde{R}_x = \sqrt{-\frac{M}{2H}} R_x, \quad \tilde{R}_y = \sqrt{-\frac{M}{2H}} R_y, \quad \tilde{L} = \frac{L}{s}, \quad (3.3.19)$$

which makes sense for bound orbits with negative energy. We then obtain

$$C = \tilde{R}_x^2 + \tilde{R}_y^2 + \tilde{L}^2 = -\frac{M\kappa^2}{2H} \Rightarrow H = -\frac{M\kappa^2}{2C}. \quad (3.3.20)$$

It will turn out that the quantum analogue of  $C$  is the Casimir operator of an accidental  $SU(2)$  symmetry.

### 3.4 Runge-Lenz Vector at the Quantum Level

To treat the problem quantum mechanically, the components of the Runge-Lenz vector must be turned into operators. The procedure that is presented here is useful not just for this particular problem, but for any problem involving finding a conserved quantity generating a symmetry group of a quantum mechanical system. The steps of the procedure are as follows:

1) An ansatz must be written down based on the classical form of the operator with unknown coefficient functions of coordinates in front of the derivatives that compose the operator. For example, in our case, there are functions of  $\varphi$  such that

$$R = \frac{1}{r}A_1(\varphi)\partial_\varphi^2 + \frac{1}{r}A_2(\varphi)\partial_\varphi + A_3(\varphi) + A_4(\varphi)\partial_\varphi\partial_r + A_5(\varphi)\partial_r. \quad (3.4.1)$$

2) Commuting the proposed operator with the Hamiltonian, we get coefficients of  $\psi(r, \varphi)$ ,  $\partial_r\psi(r, \varphi)$ ,  $\partial_\varphi\psi(r, \varphi)$ ,  $\partial_\varphi^2\psi(r, \varphi)$ ,  $\partial_\varphi^2\psi(r, \varphi)$  etc. If the proposed operator shall commute with the Hamiltonian, these coefficients of  $\psi(r, \varphi)$  and its derivatives must equal zero. This gives a set of differential equations. In our case, these equations are,

$$\begin{aligned} 2kms^2A_5(\varphi) + \partial_\varphi^2A_3(\varphi) &= 0, \\ s^2A_2(\varphi) + \partial_\varphi^2A_2(\varphi) &= 0, \\ kms^2A_4(\varphi) + \partial_\varphi A_3(\varphi) &= 0, \\ s^2A_5(\varphi) + \partial_\varphi^2A_5(\varphi) &= 0, \\ -2s^2A_2(\varphi) + s^2A_4(\varphi) + 2\partial_\varphi A_5(\varphi) + \partial_\varphi^2A_4(\varphi) &= 0, \\ -s^2A_1(\varphi) + \partial_\varphi A_4(\varphi) &= 0, \\ s^2A_1(\varphi) + 2A_5(\varphi) + 2\partial_\varphi A_2(\varphi) + \partial_\varphi^2A_1(\varphi) &= 0. \end{aligned} \quad (3.4.2)$$

3) The differential equations are solved using the appropriate boundary conditions. In our case the solution must be  $s\varphi$ -periodic. Solving Eqs.(3.4.2) gives two solutions for which all  $A$ 's are non-zero functions of  $\varphi$ . These two solutions represent the two components of the Runge-Lenz operator. In addition to that, we have a third solution with all the  $A$ 's equal to zero, except  $A_2$  equal to a constant. This solution represents the one-component angular momentum operator. We then obtain

$$\begin{aligned} R_x &= -\frac{1}{Mrs^2} \cos(s\varphi)\partial_\varphi^2 + \frac{1}{2Mrs} \sin(s\varphi)\partial_\varphi - \kappa \cos(s\varphi) \\ &\quad - \frac{1}{Ms} \sin(s\varphi)\partial_r\partial_\varphi - \frac{1}{2M} \cos(s\varphi)\partial_r, \\ R_y &= -\frac{1}{Mrs^2} \sin(s\varphi)\partial_\varphi^2 - \frac{1}{2Mrs} \cos(s\varphi)\partial_\varphi - \kappa \sin(s\varphi) \\ &\quad + \frac{1}{Ms} \cos(s\varphi)\partial_r\partial_\varphi - \frac{1}{2M} \sin(s\varphi)\partial_r. \end{aligned} \quad (3.4.3)$$

The result can be checked by commuting the components of the Runge-Lenz vector with the Hamiltonian, and indeed the result is zero, i.e.

$$[R_x, H] = [R_y, H] = 0. \quad (3.4.4)$$

It can be proved that  $R_x$ ,  $R_y$ , and  $L$  obey the following algebra

$$[R_x, R_y] = -i\frac{2HL}{Ms}, \quad [R_x, L] = -isR_y, \quad [R_y, L] = isR_x. \quad (3.4.5)$$

Applying the rescaling of Eq.(3.3.19), this leads to

$$[\tilde{R}_x, \tilde{R}_y] = i\tilde{L}, \quad [\tilde{R}_y, \tilde{L}] = i\tilde{R}_x, \quad [\tilde{L}, \tilde{R}_x] = i\tilde{R}_y. \quad (3.4.6)$$

Hence,  $\tilde{R}_x$ ,  $\tilde{R}_y$ , and  $\tilde{L}$  generate an  $SU(2)$  algebra. The raising and lowering operators are defined by

$$\tilde{R}_\pm = \tilde{R}_x \pm i\tilde{R}_y. \quad (3.4.7)$$

### 3.4.1 Casimir Operator

The Casimir operator for the  $SU(2)$  algebra is

$$C = \tilde{R}_x^2 + \tilde{R}_y^2 + \tilde{L}^2 = -\frac{M\kappa^2}{2H} - \frac{1}{4}, \quad (3.4.8)$$

such that

$$H = -\frac{M\kappa^2}{2(C + \frac{1}{4})} = -\frac{M\kappa^2}{2(S + \frac{1}{2})^2}. \quad (3.4.9)$$

As it was explained in chapter 2, we have one Casimir operator for  $SU(2)$  with spectrum  $S(S+1)$  where  $S$  is either an integer or a half-integer as we explained in chapter 2. Accordingly we have

$$C + \frac{1}{4} = S(S+1) + \frac{1}{4} = \left(S + \frac{1}{2}\right)^2. \quad (3.4.10)$$

By comparison with Eq.(3.3.10) for the energy spectrum, we thus identify

$$S = n_r + \frac{|m|}{s}. \quad (3.4.11)$$

This result is unexpected because for  $2|m|/s \notin \mathbb{N}$  the abstract spin  $S$  is not an integer or a half-integer. For a general scale factor  $s$  corresponding to a general deficit angle  $\delta$ , the abstract spin is continuous. Even for general rational  $s$ , for which all bound classical orbits are closed and there are accidental degeneracies in the discrete spectrum of the Hamiltonian, the spin  $S$  is not always an integer or a half-integer. The treatment that was considered in chapter 2 assumed that the components of the Runge-Lenz vector form raising and lowering operators that will never take the wave function outside the Hilbert space. This leads to integer or half-integer  $S$ . Here we are facing a different situation because the raising and lowering operators can take one outside the Hilbert space and the treatment in section (2.3) is no longer valid.

### 3.4.2 The Raising and Lowering Operator Acting on the Wave Function

In order to understand the degeneracy of the energy levels and other aspects like domains of operators and Hermiticity, it is important to understand the effect of the raising and lowering operator on the wave function. For our problem this is possible by direct application of the raising and lowering operators on the wave function. The explicit form of the lowering operators can be found by substituting the expressions for  $R_x$  and  $R_y$  from Eq.(3.4.3) into Eq.(3.4.7). One then applies the lowering operator on the wave function of Eq.(3.3.12). The result depends on whether  $|m|/s$  is an integer or half-integer or not. We will study both cases and prove that, upon repeated application of  $R_-$  the multiplet terminates when  $|m|/s$  is an integer or a half-integer, while it does not terminate otherwise. One application of  $R_-$  on the wave function with  $m > 0$  gives

$$\begin{aligned}
R_- \psi_{n_r, m}(r, \varphi) &= -A \sqrt{\frac{-\alpha^2}{8HM}} (\alpha r)^{\frac{|m|}{s}-1} \exp\left(-\frac{\alpha r}{2}\right) \exp(i\varphi(m-s)) \\
&\times \left( \alpha r n_r \frac{\left(\frac{2|m|}{s}-1\right)}{\left(\frac{2|m|}{s}+1\right)} {}_1F_1\left(1-n_r, \frac{2|m|}{s}+2, \alpha r\right) \right. \\
&\left. + \left( \alpha r \left(\frac{2|m|}{s}+n_r\right) - \left(\frac{2|m|}{s}-1\right) \frac{2|m|}{s} \right) {}_1F_1\left(-n_r, \frac{2|m|}{s}+1, \alpha r\right) \right).
\end{aligned} \tag{3.4.12}$$

The confluent hypergeometric function is defined by

$${}_1F_1(a, b, x) = 1 + \frac{a}{b}x + \frac{a(a+1)}{b(b+1)} \frac{x^2}{2!} + \dots + \frac{a \dots (a+k)}{b \dots (b+k)} \frac{x^{k+1}}{(k+1)!} = \sum_{k=0}^{k=\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!}. \tag{3.4.13}$$

The above series terminates only if  $a$  is zero or a negative integer. For such cases the wave function in Eq.(3.3.12) goes to zero as  $r$  goes to infinity. Therefore the only physically acceptable cases are  $a$  equal zero or a negative integer. Two identities, proved using the definition in Eq.(3.4.13), will play an important role in our work later,

$$\lim_{b \rightarrow 0} b {}_1F_1(a, b, x) = ax {}_1F_1(a+1, 2, x), \tag{3.4.14}$$

$$\lim_{b \rightarrow -1} (b+1) {}_1F_1(a, b, x) = -a(a+1) \frac{x^2}{2} {}_1F_1(a+2, 3, x). \tag{3.4.15}$$

Using well-known recurrence relations [31], after lengthy calculations the following identity can be proved

$$\begin{aligned}
(2-3b+b^2+x(1+a-b)) {}_1F_1(a, b, x) &- x \frac{(2-b)a}{b} {}_1F_1(a+1, b+1, x) \\
&= (1-b)(2-b) {}_1F_1(a-1, b-2, x).
\end{aligned} \tag{3.4.16}$$



Using the above identity, it is easy to see that Eq.(3.4.12) reduces to

$$\begin{aligned}
R_- \psi_{n_r, m}(r, \varphi) &= -A \sqrt{\frac{-\alpha^2}{8HM}} (\alpha r)^{\frac{|m|}{s}-1} \exp\left(-\frac{\alpha r}{2}\right) \exp(i\varphi(m-s)) \\
&\times \left(\frac{2|m|}{s} - 1\right) \frac{2|m|}{s} {}_1F_1\left(-1 - n_r, \frac{2|m|}{s} - 1, \alpha r\right) \\
&= B \psi_{n_r+1, m-s}(r, \varphi),
\end{aligned} \tag{3.4.17}$$

where  $B$  is a constant that results from one application of  $R_-$  or  $R_+$  on wave function. The explicit form or value of this constant is not important in our argument. We must keep in mind that when  $B$  appears in different equations, it may not have the same value, but this also does not affect our arguments. A general rule was found for  $k$  applications of  $R_-$  on the wave function, that is

$$\begin{aligned}
R_-^k \psi_{n_r, m}(r, \varphi) &= A \prod_{j=1}^k (-1)^j \left(\frac{2|m|}{s} + 2 - 2j\right) \left(\frac{2|m|}{s} + 1 - 2j\right) \\
&\times \left(\frac{-\alpha^2}{8HM}\right)^{\frac{k}{2}} (\alpha r)^{\frac{|m|}{s}-k} \exp\left(-\frac{\alpha r}{2}\right) \exp(i\varphi(m-ks)) \\
&\times {}_1F_1\left(-k - n_r, \frac{2|m|}{s} + 1 - 2k, \alpha r\right) \\
&= B \psi_{n_r+k, m-sk}(r, \varphi).
\end{aligned} \tag{3.4.18}$$

Here we must stress that during successive applications of the operator  $R_-$  we may get non-integer  $m - sk$  depending on the values of  $s$  and  $k$ . Therefore this operator may take the physical wave function to an unphysical one. For the case of  $m < 0$ , the same procedure can be applied by using a well-known identity for the confluent hypergeometric function [31]. The result of  $k$  applications of  $R_-$  on the wave function is

$$\begin{aligned}
R_-^k \psi_{n_r, -|m|}(r, \varphi) &= A \prod_{j=1}^k (-1)^j \left(\frac{n_r + \frac{2|m|}{s} + j}{\frac{2|m|}{s} + 2j - 1}\right) \left(\frac{n_r - j + 1}{\frac{2|m|}{s} + 2j}\right) \\
&\times \left(\frac{-\alpha^2}{8HM}\right)^{\frac{k}{2}} (\alpha r)^{\frac{|m|}{s}+k} \exp\left(-\frac{\alpha r}{2}\right) \exp(i\varphi(m-ks)) \\
&\times {}_1F_1\left(k - n_r, \frac{2|m|}{s} + 1 + 2k, \alpha r\right) \\
&= B \psi_{n_r-k, m-sk}(r, \varphi).
\end{aligned} \tag{3.4.19}$$

The case of  $k$  applications of  $R_+$  on the wave function with  $m \geq 0$  can be studied similarly and the result is

$$\begin{aligned}
R_+^k \psi_{n_r, m}(r, \varphi) &= A \prod_{j=1}^k (-1)^j \left( \frac{n_r + \frac{2|m|}{s} + j}{\frac{2|m|}{s} + 2j - 1} \right) \left( \frac{n_r - j + 1}{\frac{2|m|}{s} + 2j} \right) \\
&\times \left( \frac{-\alpha^2}{8HM} \right)^{\frac{k}{2}} (\alpha r)^{\frac{|m|}{s} + k} \exp\left(-\frac{\alpha r}{2}\right) \exp(i\varphi(m + ks)) \\
&\times {}_1F_1\left(k - n_r, \frac{2|m|}{s} + 1 + 2k, \alpha r\right) \\
&= B \psi_{n_r - k, m + sk}(r, \varphi). \tag{3.4.20}
\end{aligned}$$

For the case of  $k$  application of  $R_+$  on the wave function with  $m < 0$  we get

$$\begin{aligned}
R_+^k \psi_{n_r, -|m|}(r, \varphi) &= A \prod_{j=1}^k (-1)^j \left( \frac{2|m|}{s} + 2 - 2j \right) \left( \frac{2|m|}{s} + 1 - 2j \right) \\
&\times \left( -\frac{\alpha^2}{8HM} \right)^{\frac{k}{2}} (\alpha r)^{\frac{|m|}{s} - k} \exp\left(-\frac{\alpha r}{2}\right) \exp(i\varphi(m + ks)) \\
&\times {}_1F_1\left(-k - n_r, \frac{2|m|}{s} + 1 - 2k, \alpha r\right) \\
&= B \psi_{n_r + k, m - sk}(r, \varphi). \tag{3.4.21}
\end{aligned}$$

To count the number of wave functions in the multiplet we begin with  $n_r = 0$ . Then, Eq.(3.4.11) gives  $S = m_{max}/s$  with  $m = m_{max}$ . The value of  $m = m_{max}$  can not be raised further because applying  $R_+$  gives zero according to Eq.(3.4.20). Now applying the lowering operator  $R_-$  a number of  $k_0$  times, with  $k_0 = m_{max}/s + 1/2$ , using Eq.(3.4.18) gives

$$\begin{aligned}
R_-^{k_0} \psi_{0, m_{max}}(r, \varphi) &= A \prod_{j=1}^{k_0} (-1)^j \left( \frac{2|m_{max}|}{s} + 2 - 2j \right) \left( \frac{2|m_{max}|}{s} + 1 - 2j \right) \\
&\times \left( \frac{-\alpha^2}{8HM} \right)^{\frac{k_0}{2}} (\alpha r)^{\left(\frac{|m_{max}|}{s} - k_0\right)} \exp\left(-\frac{\alpha r}{2}\right) \exp(i\varphi(m_{max} - k_0 s)) \\
&\times {}_1F_1\left(-k_0, \frac{2|m_{max}|}{s} + 1 - 2k_0, \alpha r\right). \tag{3.4.22}
\end{aligned}$$

Here we have two cases in which the multiplet terminates.

### 1) Half-integer $m_{max}/s$

Applying the raising operator  $R_+$  once gives zero according to Eq.(3.4.20). On the other hand, applying the lowering operator  $R_-$  a number of  $k_0 = m_{max}/s + 1/2$  times,

and after that using the identity in Eq.(3.4.14), then Eq.(3.4.22) can be written as

$$\begin{aligned} R_-^{k_0} \psi_{0, m_{max}}(r, \varphi) &= C(\alpha r)^{\frac{1}{2}} \exp\left(-\frac{\alpha r}{2}\right) \exp(-i\varphi \frac{s}{2}) {}_1F_1\left(-\frac{|m_{max}|}{s} + \frac{1}{2}, 2, \alpha r\right) \\ &= B \psi_{\frac{|m_{max}|}{s} - \frac{1}{2}, -s/2}(r, \varphi), \end{aligned} \quad (3.4.23)$$

where  $C$  is just another constant that does not influence the argument. As we can see, the resulting wave function has  $m = -s/2 < 0$ . The operator  $R_-$  can be applied further  $k'_0$  times. However, we must use the appropriate relation Eq.(3.4.19) and we obtain

$$\begin{aligned} R_-^{k'_0} (R_-^{k_0} \psi_{0, m_{max}}(r, \varphi)) &= B \prod_{j=1}^{k'_0} (-1)^j \left( \frac{\frac{|m_{max}|}{s} + \frac{1}{2} + j}{2j} \right) \left( \frac{\frac{|m_{max}|}{s} + \frac{1}{2} - j}{2j + 1} \right) \\ &\times \left( \frac{-\alpha^2}{8HM} \right)^{\frac{k'_0}{2}} (\alpha r)^{\frac{1}{2} + k'_0} \exp\left(-\frac{\alpha r}{2}\right) \exp(-i\varphi (\frac{1}{2} + k'_0)s) \\ &\times {}_1F_1\left(k_0 - \frac{|m_{max}|}{s} + \frac{1}{2}, 2k'_0 + 2, \alpha r\right). \end{aligned} \quad (3.4.24)$$

It is obvious from the above equation that applying  $R_-$  a number of  $|m_{max}|/s + 1/2$  times gives zero. Therefore the number of wave functions in the multiplet is

$$N = k_0 + k'_0 = 2 \frac{|m_{max}|}{s} + 1. \quad (3.4.25)$$

It is worth noting that for  $s = 1$  one obtains  $N = 2|m_{max}| + 1$  which is the number of states in a multiplet of the Kepler problem in two dimensions.

## 2) Integer $m_{max}/s$

Applying the raising operator  $R_+$  gives zero according to Eq.(3.4.20). On the other hand, applying the lowering operator  $R_-$  a number of  $k_0 = m_{max}/s + 1$  times, and after using the identity in Eq.(3.4.15), then Eq.(3.4.22) can be written as

$$\begin{aligned} R_-^{k_0} \psi_{0, m_{max}}(r, \varphi) &= C(\alpha r)^1 \exp\left(-\frac{\alpha r}{2}\right) \exp(-i\varphi s) {}_1F_1\left(-\frac{|m_{max}|}{s} + 1, 3, \alpha r\right) \\ &= B \psi_{\frac{|m_{max}|}{s} - 1, -s}(r, \varphi). \end{aligned} \quad (3.4.26)$$

As we can see, the resulting wave function has  $m = -s < 0$ . The operator  $R_-$  can be applied further  $k'_0$  times. However, we must use the appropriate relation for this

case, that is Eq.(3.4.19), and we obtain

$$\begin{aligned}
R_-^{k'_0}(R_-^{k_0}\psi_{0,m_{max}}(r,\varphi)) &= A \prod_{j=1}^{k'_0} (-1)^j \left( \frac{|m_{max}|/s + 1 + j}{2j + 1} \right) \left( \frac{|m_{max}|/s - j}{2j + 2} \right) \\
&\times \left( \frac{-\alpha^2}{8HM} \right)^{\frac{k'_0}{2}} (\alpha r)^{1+k'_0} \exp\left(-\frac{\alpha r}{2}\right) \exp(-i\varphi(1+k'_0)s) \\
&\times {}_1F_1\left(k'_0 - \frac{|m_{max}|}{s} + 1, 2k'_0 + 3, \alpha r\right). \tag{3.4.27}
\end{aligned}$$

It is obvious from the above equation that applying a number of  $R_-$   $|m_{max}|/s$  times gives zero. Therefore the number of wave functions in the multiplet is again

$$N = k_0 + k'_0 = 2 \frac{|m_{max}|}{s} + 1. \tag{3.4.28}$$

### Not integer or half-integer $m_{max}/s$

Applying the lowering operator  $R_-$  any number of times will not make identity (3.4.14) or identity (3.4.15) applicable, since neither  $(2|m_{max}|/s+1-2j)$  nor  $(2|m_{max}|/s+2-2j)$  in Eq.(3.4.22) is equal to 0 or  $-1$  for any value of  $j \in \mathbb{Z}$ . After  $k_0 > |m_{max}|/s$  applications of  $R_-$ , the power of  $(\alpha r)^{(|m_{max}|/s)-k_0}$  is negative, and the resulting wave function diverges at the origin, although it is still a solution of the Schrödinger equation. The multiplet does not terminate for any number of applications of  $R_-$ , and the number of wave functions in such a multiplet is infinite. The multiple application of  $R_-$  transforms a well-behaved wave function into one that diverges at the origin after  $k_0 > |m_{max}|/s$  applications.

The same argument can be repeated for the case  $m = -|m_{max}|$ , this time by using the raising operator  $R_+$ , as well as Eq.(3.4.20) and Eq.(3.4.21). The same result is reached regarding the relation between  $|m_{max}|/s$  and the number of wave functions in the multiplet.

### 3.4.3 Domains and Hermiticity of the Components of the Runge-Lenz Vector

Let us now address the questions of Hermiticity and of the domains of the various operators. Once it is endowed with an appropriate extension, the Hermitean kinetic energy operator  $T$  becomes self-adjoint and thus qualifies as a physical observable. The same is true for the full Hamiltonian including the potential. In this case,

we assume the standard Friedrichs extension [21], which implies that there is no  $\delta$ -function potential located at the tip of the cone.

Using  $\partial_r^\dagger = -\partial_r - 1/r$  as well as  $\partial_\varphi^\dagger = -\partial_\varphi$ , it is straightforward to show that, at least formally,  $\tilde{R}_x^\dagger = \tilde{R}_x$  and  $\tilde{R}_y^\dagger = \tilde{R}_y$ , which implies  $\tilde{R}_\pm^\dagger = \tilde{R}_\mp$ . However, Hermiticity also requires appropriate boundary conditions, which restrict the domains of the corresponding operators. It is interesting to note that, using  $s\varphi = \chi$ , the operators  $R_x$  and  $R_y$  of Eq.(3.4.3) formally agree with the components of the standard Runge-Lenz vector for the plane from Eq.(3.1.1). The Runge-Lenz vector for the plane is a Hermitean and even self-adjoint operator acting in a domain  $\mathcal{D}[\vec{R}]$  that contains the domain of the Hamiltonian. This domain contains smooth functions which are  $2\pi$ -periodic in the polar angle  $\chi$  of the plane. The operators  $\tilde{R}_x$  and  $\tilde{R}_y$ , on the other hand, act on the Hilbert space of square-integrable wave functions on the cone. In this case, the domain of the Hamiltonian  $\mathcal{D}[H]$  contains smooth functions which are  $2\pi$ -periodic in the rescaled angle  $\varphi$  and obey the boundary condition of Eq.(3.2.20). While  $\tilde{R}_x$  and  $\tilde{R}_y$  on the cone are still Hermitean in their appropriate domain, in contrast to the case of the plane, they are not Hermitean in the domain  $\mathcal{D}[H]$  of the Hamiltonian. In particular, for  $s \neq 1$  the operators  $\tilde{R}_x$  and  $\tilde{R}_y$  map  $2\pi$ -periodic physical wave functions onto functions outside  $\mathcal{D}[H]$ , because they contain multiplications with the  $2\pi/s$ -periodic functions  $\cos(s\varphi)$  and  $\sin(s\varphi)$ . Proper symmetry generators should map wave functions from the domain of the Hamiltonian back into  $\mathcal{D}[H]$ . Hence, for  $s \notin \mathbb{N}$ , the operators  $\tilde{R}_x$  and  $\tilde{R}_y$  do not represent proper symmetry generators.

It is interesting to consider the case of rational  $s = p/q$  with  $p, q \in \mathbb{N}$ . In this case, a single application of

$$\tilde{R}_\pm = \tilde{R}_x \pm i\tilde{R}_y \tag{3.4.29}$$

may take us out of the domain of the Hamiltonian, but a  $q$ -fold application of these operators brings us back into  $\mathcal{D}[H]$ . Indeed, just as for rational  $s$  the classical object  $R_\pm^q$  represents a proper conserved physical quantity,  $\tilde{R}_\pm^q$  (but not  $\tilde{R}_\pm$  itself) qualifies as a proper symmetry generator. The case of integer  $s = n$  is also interesting, because in that case  $\cos(s\varphi)$  and  $\sin(s\varphi)$  are indeed  $2\pi$ -periodic. Hence, by acting with  $\tilde{R}_\pm$  we might expect to stay within  $\mathcal{D}[H]$ , although for  $n \geq 3$  the abstract spin  $S = n_r + |m|/s = n_r + |m|/n$  is still quantized in unusual fractional units. However, we already saw this above, another subtlety arises because  $\tilde{R}_\pm$  may turn a physical wave function that is regular at the origin (and thus obeys the boundary condition of Eq.(3.2.20) into a singular one. This further limits the domain of the operators  $\tilde{R}_\pm$ . The unusual (not properly quantized) value of the Casimir spin can be traced back to the mathematical fact that the Runge-Lenz vector — although Hermitean in its appropriate domain — does not act as a Hermitean operator in the domain of the Hamiltonian. Hence, in retrospect the  $SU(2)$  commutation relations

of Eq.(3.4.6) are rather formal. In fact, they are satisfied for functions  $\Psi(r, \varphi)$  with  $\varphi \in \mathbb{R}$ , but not for the periodic functions in  $\mathcal{D}[H]$  for which  $\varphi \in [0, 2\pi]$ . This is another indication that the accidental “ $SU(2)$ ” symmetry of Eq.(3.4.6) is rather unusual.

### 3.4.4 Unusual Multiplets

In this subsection we study the multiplets by determining the Casimir spectrum, and examine carefully the implication of the results of subsection (3.4.2) on the multiplets.

As we saw in section (2.3) for the case of an  $SU(2)$  algebra, the spectrum of the Casimir operator and the matrix elements of the raising and lowering operators were derived under the assumption that the wave functions are normalizable and form complete set. On the other hand, we have shown that this is not always the case since  $R_-$  or  $R_+$  may map the wave function onto another one that is divergent at the origin and therefore outside  $\mathcal{D}[H]$ . However, one can derive the spectrum of the Casimir operator without having to worry about the normalizability of the wave functions. This will help to further understand the puzzling result that the Casimir spin  $S$  is not always quantized in integer or half-integer units. Acting with  $\tilde{R}_\pm$  on a  $2\pi$ -periodic wave function

$$\langle r, \varphi | n_r, m \rangle = \psi_{n_r, m}(r) \exp(im\varphi) \quad (3.4.30)$$

one changes both  $n_r \in \mathbb{N}$  and  $m \in \mathbb{Z}$ . For  $m > 0$  one obtains

$$\tilde{R}_+ |n_r, m\rangle \propto |n_r - 1, m + s\rangle, \quad \tilde{R}_- |n_r, m\rangle \propto |n_r + 1, m - s\rangle, \quad (3.4.31)$$

and for  $m < 0$  one finds

$$\tilde{R}_+ |n_r, m\rangle \propto |n_r + 1, m + s\rangle, \quad \tilde{R}_- |n_r, m\rangle \propto |n_r - 1, m - s\rangle. \quad (3.4.32)$$

Finally, for  $m = 0$  we have

$$\tilde{R}_+ |n_r, 0\rangle \propto |n_r - 1, s\rangle, \quad \tilde{R}_- |n_r, 0\rangle \propto |n_r - 1, -s\rangle. \quad (3.4.33)$$

These relations follow from the  $SU(2)$  algebra which implies that  $\tilde{R}_\pm$  are raising and lowering operators for  $\tilde{L} = L/s$ . Hence, by acting with  $\tilde{R}_\pm$ , the eigenvalue  $m$  of  $L$  is shifted by  $\pm s$ . Using the fact that the eigenvalue of the Casimir operator, which is determined by  $S = n_r + |m|/s$ , does not change under applications of  $\tilde{R}_\pm$ , one immediately obtains the effects of  $\tilde{R}_\pm$  on the radial quantum number  $n_r$ . Eqs.(3.4.31), (3.4.32), and (3.4.33) also follow directly by applying the explicit forms

of  $R_{\pm} = R_x \pm iR_y$  from Eq.(3.4.3) to the wave functions of Eq.(3.3.12) and using the relations in section(3.4.2). Moreover, from the results of section(3.4.2) we obtain

$$\begin{aligned}\tilde{R}_+^{n_r+1}|n_r, m \geq 0\rangle &\propto \tilde{R}_+|0, m + n_r s \geq 0\rangle = 0, \\ \tilde{R}_-^{n_r+1}|n_r, m \leq 0\rangle &\propto \tilde{R}_-|0, m - n_r s \leq 0\rangle = 0.\end{aligned}\quad (3.4.34)$$

Hence, depending on the sign of  $m$ , by acting  $n_r + 1$  times either with  $\tilde{R}_+$  or with  $\tilde{R}_-$  we reach zero, and thus the multiplet naturally terminates. This allows us to confirm the value of the Casimir spin  $S = n_r + |m|/s$  by evaluating

$$\begin{aligned}C|0, m + n_r s \geq 0\rangle &= \left[ \frac{1}{2}(\tilde{R}_+ \tilde{R}_- + \tilde{R}_- \tilde{R}_+) + \tilde{L}^2 \right] |0, m + n_r s \geq 0\rangle \\ &= \left[ \frac{1}{2}([\tilde{R}_+, \tilde{R}_-] + 2\tilde{R}_- \tilde{R}_+) + \tilde{L}^2 \right] |0, m + n_r s \geq 0\rangle \\ &= (\tilde{L} + \tilde{L}^2)|0, m + n_r s \geq 0\rangle \\ &= \left( \frac{m}{s} + n_r \right) \left( \frac{m}{s} + n_r + 1 \right) |0, m + n_r s \geq 0\rangle \\ &= S(S+1)|0, m + n_r s \geq 0\rangle, \\ C|0, m - n_r s \leq 0\rangle &= \left[ \frac{1}{2}(\tilde{R}_+ \tilde{R}_- + \tilde{R}_- \tilde{R}_+) + \tilde{L}^2 \right] |0, m - n_r s \leq 0\rangle \\ &= \left[ \frac{1}{2}(\tilde{R}_+ \tilde{R}_- + 2[\tilde{R}_-, \tilde{R}_-]) + \tilde{L}^2 \right] |0, m - n_r s \leq 0\rangle \\ &= (-\tilde{L} + \tilde{L}^2)|0, m - n_r s \leq 0\rangle \\ &= \left( -\frac{m}{s} + n_r \right) \left( -\frac{m}{s} + n_r + 1 \right) |0, m - n_r s \leq 0\rangle \\ &= S(S+1)|0, m - n_r s \leq 0\rangle.\end{aligned}\quad (3.4.35)$$

The multiplet of degenerate states with the same value of  $S$  can now be obtained by  $n$  repeated applications of either  $\tilde{R}_+$  or  $\tilde{R}_-$ . It is important to note that, if  $s$  is not an integer,  $m \pm ns$  may also not be an integer and thus the corresponding state may be outside  $\mathcal{D}[H]$ . Despite this, its radial wave function is still defined by Eq.(3.3.12) and it still solves the radial Schrödinger equation.

Let us first consider the generic case of irrational  $s$ . In that case, the classical orbits are not closed, there are no accidental degeneracies in the discrete spectrum of the Hamiltonian, and the Casimir spin  $S = n_r + |m|/s$  is irrational. Acting with  $\tilde{R}_{\pm}$  on the  $2\pi$ -periodic wave function  $|n_r, m \in \mathbb{Z}\rangle$  an arbitrary number of times, one generates functions which are not  $2\pi$ -periodic and thus outside  $\mathcal{D}[H]$ . As a consequence of parity symmetry, for  $m \neq 0$  the two levels with the quantum numbers  $m$  and  $-m$  are still degenerate. However, that two-fold degeneracy is not accidental.

Next, let us discuss the case of rational  $s = p/q$  in which all classical orbits are closed and there are accidental degeneracies in the discrete spectrum of the Hamiltonian. First, we consider the case  $2|m|/s = 2|m|q/p \in \mathbb{N}$  for which the Casimir spin  $S$  is an integer or a half-integer. Only in that case, the set of degenerate wave functions terminates on both ends, i.e.

$$\tilde{R}_-^{2S+1}|0, m + n_r s \geq 0\rangle = 0, \tilde{R}_+^{2S+1}|0, m - n_r s \leq 0\rangle = 0. \quad (3.4.36)$$

This follows by applying the operators of Eq.(3.4.3) to the wave function given in Eq.(3.3.12) that we discussed in the previous subsection (3.4.2).

The multiplets are even more unusual in the case of rational  $s = p/q$  with the Casimir spin  $S$  neither being an integer nor a half-integer. In that case, the set of degenerate wave functions only terminates on one end, but not on the other. In particular, while still  $\tilde{R}_+|0, m + n_r s \geq 0\rangle = 0$ ,  $\tilde{R}_-^k|0, m + n_r s \geq 0\rangle$  does not vanish, even for arbitrarily large  $k$  (see the end of subsection (3.4.2)). Since an infinite number of values  $m + (n_r - k)s$  are integers, one might think that the multiplet of degenerate states inside  $\mathcal{D}[H]$  should contain an infinite number of states. Interestingly, this is not the case for a rather unusual reason. For  $S$  neither being an integer nor a half-integer, the states  $\tilde{R}_-^k|0, m + n_r s \geq 0\rangle$  with  $m + (n_r - k)s < 0$  are outside  $\mathcal{D}[H]$  because the corresponding wave function is singular at the origin. This again follows from applying the operators of Eq.(3.4.3) to the wave function of Eq.(3.3.12). Although they do not qualify as physical states, the divergent wave functions still are mathematical solutions of the Schrödinger differential equation which take the form

$$\psi(r) = A \exp\left(-\frac{\alpha r}{2}\right) (\alpha r)^{-|m|/s} {}_1F_1\left(-n_r, -\frac{2|m|}{s} + 1, \alpha r\right). \quad (3.4.37)$$

The singularity may or may not make the wave function non-normalizable. Even if it remains normalizable, the corresponding singular wave function does not belong to  $\mathcal{D}[H]$  because it does not obey the boundary condition of Eq.(3.2.20). For  $S$  neither being an integer nor a half-integer, the states with positive and negative  $m$  have the same energy as a consequence of parity symmetry, but they are not related to one another by applications of the raising and lowering operators  $\tilde{R}_\pm$ . Remarkably, in this case, by acting with a symmetry generator  $\tilde{R}_x$  or  $\tilde{R}_y$  on a wave function inside  $\mathcal{D}[H]$ , one may generate a physically unacceptable wave function outside  $\mathcal{D}[H]$ . A sequence of physical and unphysical wave functions is illustrated in figure (3.4). To summarize, for  $s \neq 1$  different types of unusual multiplets arise. First, even for integer or half-integer  $S = n_r + |m|/s$ , the degeneracy of the physical multiplet is not  $2S + 1$  because  $m \pm ns$  may not be an integer in which case the corresponding wave function is not  $2\pi$ -periodic. When  $S = n_r + |m|/s$  is neither an integer nor a half-integer, there is an infinite number of degenerate solutions of the Schrödinger equation. However, only a finite number of them obeys the boundary condition of Eq.(3.2.20) and thus belongs to  $\mathcal{D}[H]$ .



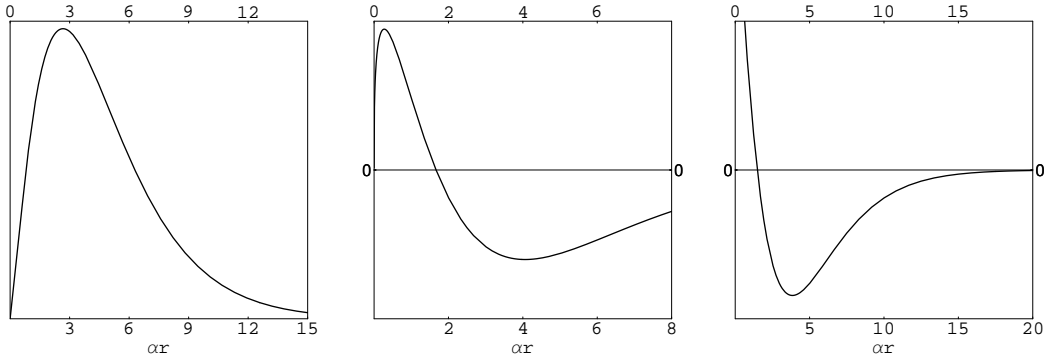


Figure 3.4: A sequence of wave functions for the  $1/r$  potential with  $s = 3$  obtained from repeated applications of  $\tilde{R}_-$ . The quantum numbers are  $n_r = 0$ ,  $m = 4$  (left),  $n_r = 1$ ,  $m = 4 - s = 1$  (middle), and  $n_r = 2$ ,  $m = 4 - 2s = -2$  (right). The third state in the sequence is outside the domain of the Hamiltonian because the corresponding wave function does not obey the boundary condition of Eq.(3.2.20) and the state is thus unphysical.

### 3.4.5 Counting the Degeneracies in the Kepler Case

Our experience with  $SU(2)$  algebras would suggest that there are  $2S + 1$  degenerate states. However, we should not forget that a single application of the raising and lowering operators  $\tilde{R}_\pm$  may take us outside  $\mathcal{D}[H]$ , and only  $q$  applications of  $\tilde{R}_\pm$  take us back into  $\mathcal{D}[H]$ . As it is obvious, not all of the members of the multiplet represent physical wave functions, because not all of them are  $2\pi$ -periodic and are regular at the origin. The  $\varphi$ -dependent part for each member of the multiplet is  $\exp(i\varphi(m_{max} - sk))$ . One way to realize that is from studying the action of the lowering operator  $R_-$  on the wave function in Eq.(3.4.22) for  $k$  applications. For the function to be  $2\pi$ -periodic,  $(m_{max} - sk)$  must be an integer. As we proved before,  $k$  could be finite for the case of  $|m_{max}|/s$  an integer or a half-integer. Then  $k = 0, 1, \dots, 2|m_{max}|/s$  or infinite when  $|m_{max}|/s$  is a fraction. A rule can be derived by induction for the degeneracy  $g$ . When we begin with  $m_{max}$  as an integer, the rule is

$$g = \left[ \frac{2|m_{max}|}{p} \right] + 1 = \left[ \frac{2S}{q} \right] + 1. \quad (3.4.38)$$

We must note here that it is possible that  $|m_{max}|$  has a non-integer value. For example, consider the case when  $S = n_r + |m|/s = 4$ , with  $s = 1/3$ ,  $m = 1$ , and  $n_r = 1$ . According to Eq.(3.4.11), applying the raising operator on the wave function gives  $n_r = 0$ ,  $m = m_{max} = m + s = 1 + 1/3 = 4/3$ . For such case, the wave function is unphysical although the multiplet has a finite number of members when  $|m_{max}|/s$  is an integer or a half-integer. Counting the degeneracies needs more elaboration.

Let us assume that  $m_{max} = P/Q$  is not an integer. Now let us assume that  $i_1$  applications of  $R_-$  are needed until one reaches the closest integer to  $P/Q$  say  $b$ . One then obtains

$$\frac{P}{Q} - i_1 \frac{p}{q} = b, \quad (3.4.39)$$

which implies

$$i_1 = \frac{q}{pQ}(P - Qb). \quad (3.4.40)$$

The rule of finding the degeneracy for  $b$  an integer is given by Eq.(3.4.38). Accordingly we obtain

$$g = \left[ \frac{2b}{p} \right] + 1. \quad (3.4.41)$$

The previous argument shows that we found an unusual representation for the unconventional “ $SU(2)$ ” algebra.

## 3.5 The $r^2$ Potential on a Cone

Let us now turn to the problem of a particle moving on a cone and bound to its tip by a harmonic oscillator potential

$$V(r) = \frac{1}{2}M\omega^2 r^2. \quad (3.5.1)$$

The Hamiltonian is then given by

$$H = T + V = \frac{1}{2M} \left( p_r^2 + \frac{L^2}{r^2 s^2} \right) + \frac{1}{2}M\omega^2 r^2. \quad (3.5.2)$$

### 3.5.1 Classical Solutions

Using the corresponding classical equations of motion one obtains the classical orbits

$$\frac{1}{r^2} = \frac{MEs^2}{L^2} [1 + f \cos(2s(\varphi - \varphi_0))], \quad (3.5.3)$$

with  $E$  and  $L$  again denoting energy and angular momentum and with

$$f = \sqrt{1 - \frac{\omega^2 L^2}{E^2 s^2}}. \quad (3.5.4)$$

The radial component of the momentum is given by

$$\frac{p_r}{r} = \frac{MEs}{L} f \sin(2s(\varphi - \varphi_0)). \quad (3.5.5)$$

All classical orbits are closed as long as  $2s = p/q$  is a rational number (with  $p, q \in \mathbb{N}$  again not sharing a common divisor). Some examples of classical orbits are shown in figure (3.5).

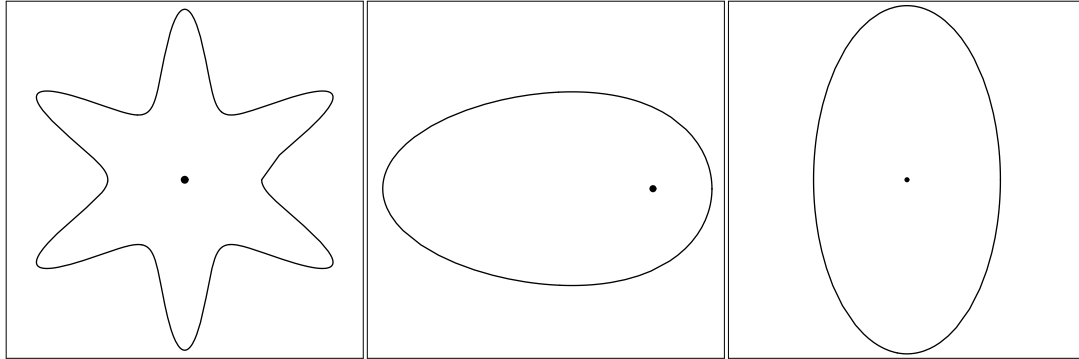


Figure 3.5: Examples of bound classical orbits for the  $r^2$  potential with  $s = 3$  (left),  $s = \frac{1}{2}$  (middle), and  $s = 1$  (right). The latter case represents an elliptic orbit of the standard harmonic oscillator. The orbits are shown in the  $x$ - $y$ -plane with  $(x, y) = r(\cos \varphi, \sin \varphi)$  where  $\varphi = \chi/s \in [0, 2\pi]$  is the rescaled polar angle.

### 3.5.2 Semi-classical Bohr-Sommerfeld Quantization

As in the case of the  $1/r$  potential, the semi-classical quantization condition for the angular momentum is again given by  $L = m \in \mathbb{Z}$ . For the harmonic oscillator the quantization condition for the radial motion takes the form

$$\oint dr p_r = \int_0^{\pi/s} d\varphi \frac{|L|f^2 \sin^2(2s(\varphi - \varphi_0))}{(1 + f \cos(2s(\varphi - \varphi_0)))^2} = \pi \left( \frac{E}{\omega} - \frac{|L|}{s} \right) = 2\pi \left( n_r + \frac{1}{2} \right), \quad (3.5.6)$$

such that

$$E = \omega \left( 2n_r + \frac{|m|}{s} + 1 \right). \quad (3.5.7)$$

Again, it will turn out that the semi-classical result exactly reproduces the one of the full quantum theory.

### 3.5.3 Solution of the Schrödinger Equation

For the particle on the cone with harmonic oscillator potential the radial Schrödinger equation takes the form

$$\left[ -\frac{1}{2M} \left( \partial_r^2 + \frac{1}{r} \partial_r \right) + \frac{m^2}{2Mr^2s^2} + \frac{1}{2} M\omega^2 r^2 \right] \psi(r) = E\psi(r). \quad (3.5.8)$$

In this case, the solution is given by [8]

$$\psi_{n_r, m}(r) = A \exp\left(-\frac{\alpha^2 r^2}{2}\right) (\alpha r)^{|m|/s} {}_1F_1\left(-n_r, \frac{|m|}{s} + 1, \alpha^2 r^2\right), \alpha = \sqrt{M\omega}. \quad (3.5.9)$$

The corresponding quantized energy values are given by Eq.(3.5.7). There are accidental degeneracies if  $2s = p/q$  is a rational number, which thus again arise exactly when all classical orbits are closed. Some features of the energy spectrum are illustrated in figure (3.6).

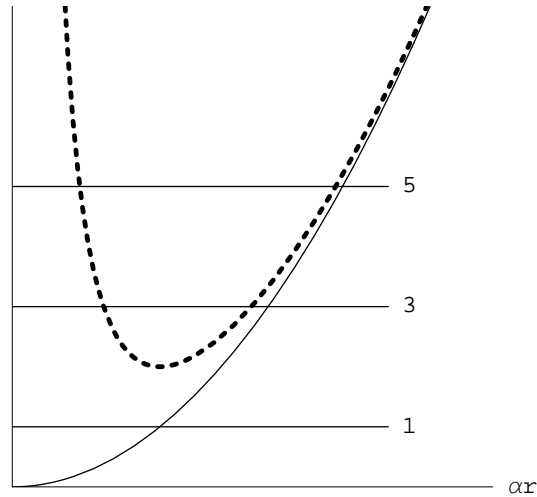


Figure 3.6: The  $r^2$  potential (solid curve) together with an effective potential including the centrifugal barrier with  $m = \pm 1$  (dashed curve) for  $s = \frac{1}{2}$ . The energies of the ground state and the first two excited states are indicated by horizontal lines. The numbers besides the lines specify the degree of degeneracy. The ground state (with  $n_r = 0, m = 0$ ) is non-degenerate, while the first excited level (consisting of the states with  $n_r = 0, m = \pm 1$  and  $n_r = 1, m = 0$ ) and the second excited level (consisting of the states with  $n_r = 0, m = \pm 2, n_r = 1, m = \pm 1$ , and  $n_r = 2, m = 0$ ) are accidentally three-fold, respectively five-fold degenerate.

### 3.5.4 Runge-Lenz Vector

The accidental degeneracies for rational  $s$  again point to the existence of a conserved Runge-Lenz vector. At the classical level, we can use Eqs.(3.5.3) and (3.5.5) to write

$$\begin{aligned} Ef \cos(2s(\varphi - \varphi_0)) &= Ef [\cos(2s\varphi) \cos(2s\varphi_0) + \sin(2s\varphi) \sin(2s\varphi_0)] = \frac{L^2}{Mr^2s} - H, \\ Ef \sin(2s(\varphi - \varphi_0)) &= Ef [\sin(2s\varphi) \cos(2s\varphi_0) - \cos(2s\varphi) \sin(2s\varphi_0)] = \frac{p_r L}{Mrs} \end{aligned} \quad (3.5.10)$$

such that

$$\begin{aligned} R_x &= Ef \cos(2s\varphi_0) = \left( \frac{L^2}{Mr^2s^2} - H \right) \cos(2s\varphi) + \frac{p_r L}{Mrs} \sin(2s\varphi), \\ R_y &= Ef \sin(2s\varphi_0) = \left( \frac{L^2}{Mr^2s^2} - H \right) \sin(2s\varphi) - \frac{p_r L}{Mrs} \cos(2s\varphi). \end{aligned} \quad (3.5.11)$$

It should again be pointed out that  $R_x$  and  $R_y$  are proper conserved quantities only if  $2s$  is an integer. Otherwise the Runge-Lenz vector is not a  $2\pi$ -periodic function of the angle  $\varphi$ , and its value depends on the number of revolutions of the particle around the tip of the cone. As before, it is useful to introduce the complex quantities

$$R_{\pm} = R_x \pm iR_y = \left( \frac{L^2}{Mr^2s^2} - H \mp i \frac{p_r L}{Mrs} \right) \exp(\pm 2is\varphi). \quad (3.5.12)$$

For rational values  $2s = p/q$  (with  $p, q \in \mathbb{N}$ ) the quantities

$$R_{\pm}^q = \left( \frac{L^2}{Mr^2s^2} - H \mp i \frac{p_r L}{Mrs} \right)^q \exp(\pm ip\varphi) \quad (3.5.13)$$

are again single-valued functions of  $x = r \cos \varphi$  and  $y = r \sin \varphi$ , and are hence proper conserved quantities.

For the harmonic oscillator, the length of the Runge-Lenz vector is given by

$$\begin{aligned} R^2 &= R_x^2 + R_y^2 = \left( \frac{L^2}{Mr^2s^2} - H \right)^2 + \left( \frac{p_r L}{Mrs} \right)^2 \\ &= \left( \frac{p_r^2}{2M} - \frac{L^2}{2Mr^2s^2} + \frac{1}{2}M\omega^2 r^2 \right)^2 + \left( \frac{p_r L}{Mrs} \right)^2 = H^2 - \left( \omega \frac{L}{s} \right)^2 \end{aligned} \quad (3.5.14)$$

As in the case of the  $1/r$  potential, it is useful to introduce rescaled variables which now take the form

$$\tilde{R}_x = \frac{1}{2\omega} R_x, \quad \tilde{R}_y = \frac{1}{2\omega} R_y, \quad \tilde{L} = \frac{L}{2s}. \quad (3.5.15)$$

We thus obtain

$$C = \tilde{R}_x^2 + \tilde{R}_y^2 + \tilde{L}^2 = \left( \frac{H}{2\omega} \right)^2 \Rightarrow H = 2\omega\sqrt{C}. \quad (3.5.16)$$

Once again, it will turn out that the quantum analogue of  $C$  is the Casimir operator of an accidental  $SU(2)$  symmetry.

### 3.5.5 The Runge-Lenz Vector as a Quantum Mechanical Operator

We are using the same technique that has been used in the case of the  $1/r$  potential. The classical expression of the Runge-Lenz vector suggests the following ansatz

$$R = \frac{1}{r^2}A_1(\varphi)\partial_\varphi^2 + \frac{1}{r^2}A_2(\varphi)\partial_\varphi + \frac{1}{r}A_3(\varphi)\partial_\varphi\partial_r + \frac{1}{r}A_4(\varphi)\partial_r + A_5(\varphi)\partial_r^2 + r^2A_6(\varphi). \quad (3.5.17)$$

As before the commutators of the above operator with the Hamiltonian give a system of differential equations as follows

$$\begin{aligned} 4n^2A_6(\varphi) + 2kMn^2A_4(\varphi) + 2kMn^2A_5(\varphi) + \partial_\varphi^2A_6(\varphi) &= 0, \\ 4n^2A_2(\varphi) + \partial_\varphi^2A_2(\varphi) &= 0, \\ kMn^2A_3(\varphi) + \partial_\varphi A_6(\varphi) &= 0, \\ 4n^2A_1(\varphi) + 2A_4(\varphi) - 6A_5(\varphi) + 2\partial_\varphi A_2(\varphi) + \partial_\varphi A_1(\varphi) &= 0, \\ A_3(\varphi) + \partial_\varphi A_1(\varphi) &= 0, \\ 2n^2A_4(\varphi) - 2n^2A_5(\varphi) + \partial_\varphi^2A_4(\varphi) &= 0, \\ A_6(\varphi) + kMA_5(\varphi) &= 0, \\ -4n^2A_2(\varphi) + 2n^2A_3(\varphi) + 2\partial_\varphi A_4(\varphi) + \partial_\varphi^2A_3(\varphi) &= 0, \\ -2n^2A_1(\varphi) + 2A_5(\varphi) + \partial_\varphi A_3(\varphi) &= 0, \\ -2n^2A_4(\varphi) + 2n^2A_5(\varphi) + \partial_\varphi^2A_5(\varphi) &= 0, \\ -n^2A_3(\varphi) + \partial_\varphi A_5(\varphi) &= 0. \end{aligned} \quad (3.5.18)$$

Solving the differential equations for this case is more complicated than in the case of the  $1/r$  potential. The differential Eqs.(3.5.18) give two solutions for which all  $A$ 's are non-zero functions of  $\varphi$ . These two solutions represent the two components of the Runge-Lenz operator. In addition to that, we have a third solution with all the  $A$ 's equal to zero except  $A_2$  equal to a constant. This solution represents the one-component angular momentum operator. At the quantum level the Runge-Lenz

vector now takes the form

$$\begin{aligned}
R_x &= \frac{1}{2M} \cos(2s\varphi) \partial_r^2 - \frac{1}{2Mr^2s^2} \cos(2s\varphi) \partial_\varphi^2 + \frac{1}{Mr^2s} \sin(2s\varphi) \partial_\varphi \\
&\quad - \frac{1}{2} M\omega^2 r^2 \cos(2s\varphi) - \frac{1}{Mr_s} \sin(2s\varphi) \partial_r \partial_\varphi - \frac{1}{2Mr} \cos(2s\varphi) \partial_r, \\
R_y &= \frac{1}{2M} \sin(2s\varphi) \partial_r^2 - \frac{1}{2Mr^2s^2} \sin(2s\varphi) \partial_\varphi^2 - \frac{1}{Mr^2s} \cos(2s\varphi) \partial_\varphi \\
&\quad - \frac{1}{2} M\omega^2 r^2 \sin(2s\varphi) + \frac{1}{Mr_s} \cos(2s\varphi) \partial_r \partial_\varphi - \frac{1}{2Mr} \sin(2s\varphi) \partial_r. \quad (3.5.19)
\end{aligned}$$

One can show that the Runge-Lenz vector as well as the angular momentum  $L$  commute with the Hamiltonian, and that these operators obey the algebra

$$[R_x, R_y] = 2i\omega \frac{L}{s}, \quad [R_x, L] = -2isR_y, \quad [R_y, L] = 2isR_x. \quad (3.5.20)$$

Applying the rescaling of Eq.(3.5.15), this leads to

$$[\tilde{R}_x, \tilde{R}_y] = i\tilde{L}, \quad [\tilde{R}_y, \tilde{L}] = i\tilde{R}_x, \quad [\tilde{L}, \tilde{R}_x] = i\tilde{R}_y, \quad (3.5.21)$$

which again represents an  $SU(2)$  algebra.

### 3.5.6 Casimir Operator

The Casimir operator for the harmonic oscillator on the cone takes the form

$$C = \tilde{R}_x^2 + \tilde{R}_y^2 + \tilde{L}^2 = \left( \frac{H}{2\omega} \right)^2 - \frac{1}{4}, \quad (3.5.22)$$

which implies

$$H = 2\omega \sqrt{C + \frac{1}{4}} = 2\omega \left( S + \frac{1}{2} \right). \quad (3.5.23)$$

Comparing with Eq.(3.5.7) for the energy spectrum, we now identify

$$S = n_r + \frac{|m|}{2s}. \quad (3.5.24)$$

### 3.5.7 The Raising and Lowering Operators Acting on the Wave Function

The explicit form of the lowering operator can be found by substituting the expressions for  $R_x$  and  $R_y$  from Eq.(3.5.19) into Eq.(3.4.7). After that one applies the

lowering operator to the explicit form of the wave function in Eq.(3.5.9). The result depends on whether  $|m|/s$  is an integer or a half-integer or not. We will study both cases and prove that the multiplet terminates under repeated applications of  $R_-$  when  $|m|/s$  is an integer or a half-integer, while it does not terminate otherwise. One application of  $R_-$  on the wave function with  $m \geq 0$  gives

$$\begin{aligned}
R_- \psi_{n_r, m}(r, \varphi) &= A \frac{2\alpha^2}{M(\frac{|m|}{s} + 1)(\frac{|m|}{s} + 2)} (\alpha r)^{\frac{|m|}{s} - 2} \exp(-\frac{\alpha^2 r^2}{2}) \exp(i\varphi(m - 2s)) \\
&\times \left( \frac{|m|}{s} (\frac{|m|}{s} + 1) (\frac{|m|}{s} + 2) (\frac{|m|}{s} - 1 - \alpha^2 r^2) {}_1F_1(-n_r, \frac{|m|}{s} + 1, \alpha^2 r^2) \right. \\
&\times \left. + n_r \left( -2 \frac{|m|^2}{s^2} + 2\alpha^2 r^2 + \frac{|m|}{s} (-4 + \alpha^2 r^2) \right) r^2 \alpha^2 \right. \\
&\times \left. {}_1F_1(1 - n_r, \frac{|m|}{s} + 2, \alpha^2 r^2) + n_r (n_r - 1) \alpha^4 r^4 \right. \\
&\times \left. {}_1F_1(2 - n_r, \frac{|m|}{s} + 3, \alpha^2 r^2) \right). \tag{3.5.25}
\end{aligned}$$

Another important identity can be derived using the recurrence relations of the confluent hypergeometric functions [31], and after a lengthy calculation we can prove that

$$\begin{aligned}
&b(b^2 - 1)(b - 2 - x) {}_1F_1(a, b, x) + (1 + a)ax^2 {}_1F_1(a + 2, b + 2, x) \\
&- ax(2 - 2b^2 + x + bx) {}_1F_1(a + 1, b + 1, x) \\
&= b(b^2 - 1)(b - 2) {}_1F_1(a - 1, b - 2, x). \tag{3.5.26}
\end{aligned}$$

Using the above identity, Eq.(3.5.25) can be written as

$$\begin{aligned}
R_- \psi_{n_r, m}(r, \varphi) &= A \frac{2\alpha^2}{M} (\alpha r)^{\frac{|m|}{s} - 2} \exp(-\frac{\alpha^2 r^2}{2}) \exp(i\varphi(m - 2s)) \\
&\times \frac{|m|}{s} (\frac{|m|}{s} - 1) {}_1F_1(-1 - n_r, \frac{|m|}{s} - 1, \alpha^2 r^2) \\
&= B \psi_{n_r + 1, m - 2s}(r, \varphi), \tag{3.5.27}
\end{aligned}$$

where  $B$  is a constant that results from one application of  $R_-$  or  $R_+$  on the wave function. The value of this constant is not important in our argument. We must keep in mind that when  $B$  appears in different equations it may not have the same value, but that will not affect the argument. A number of  $k$  applications of  $R_-$  to



the wave function with  $m \geq 0$  gives

$$\begin{aligned}
R_-^k \psi_{n_r, m}(r, \varphi) &= A \left( \frac{2\alpha^2}{M} \right)^k (\alpha r)^{\frac{|m|}{s} - 2k} \exp\left(-\frac{\alpha^2 r^2}{2}\right) \exp(i\varphi(m - 2sk)) \\
&\times \prod_{j=1}^k \left( \frac{|m|}{s} + 2 - 2j \right) \left( \frac{|m|}{s} + 1 - 2j \right) \\
&\times {}_1F_1\left(-k - n_r, \frac{|m|}{s} + 1 - 2k, \alpha^2 r^2\right) \\
&= B \psi_{n_r + k, m - 2sk}(r, \varphi). \tag{3.5.28}
\end{aligned}$$

The lowering operator  $R_-$  operating on the wave function with  $m < 0$  gives

$$\begin{aligned}
R_- \psi_{n_r, -|m|}(r, \varphi) &= \frac{2\alpha^2 A}{M \left( \frac{|m|}{s} + 1 \right) \left( \frac{|m|}{s} + 2 \right)} (\alpha r)^{\frac{|m|}{s} + 2} \exp\left(-\frac{\alpha^2 r^2}{2}\right) \\
&\times \exp(i\varphi(m - 2s)) n_r \left( \left( \frac{|m|}{s} + 2 \right) {}_1F_1\left(1 - n_r, \frac{|m|}{s} + 2, \alpha^2 r^2\right) \right. \\
&\left. + (n_r - 1) {}_1F_1\left(2 - n_r, \frac{|m|}{s} + 3, \alpha^2 r^2\right) \right). \tag{3.5.29}
\end{aligned}$$

Using identity (3.5.26), the above equation can be written as

$$\begin{aligned}
R_- \psi_{n_r, m}(r, \varphi) &= A \frac{2\alpha^2}{M \left( \frac{|m|}{s} + 1 \right) \left( \frac{|m|}{s} + 2 \right)} (\alpha r)^{\frac{|m|}{s} + 2} \exp\left(-\frac{\alpha^2 r^2}{2}\right) \\
&\times \exp(i\varphi(m - 2s)) n_r \left( n_r + \frac{|m|}{s} + 1 \right) {}_1F_1\left(1 - n_r, \frac{|m|}{s} + 3, \alpha^2 r^2\right) \\
&= B \psi_{n_r - 1, m - 2s}(r, \varphi). \tag{3.5.30}
\end{aligned}$$

The general rule for  $k$  applications of  $R_-$  on the wave function is

$$\begin{aligned}
R_-^k \psi_{n_r, m}(r, \varphi) &= A \left( \frac{2\alpha^2}{M} \right)^k (\alpha r)^{\frac{|m|}{s} + 2k} \exp\left(-\frac{\alpha^2 r^2}{2}\right) \exp(i\varphi(m - 2sk)) \\
&\times \prod_{j=1}^k \left( \frac{(n_r - j + 1)(n_r + \frac{|m|}{s} + j)}{\left( \frac{|m|}{s} + 2j - 1 \right) \left( \frac{|m|}{s} + 2j \right)} \right) {}_1F_1\left(k - n_r, \frac{|m|}{s} + 1 + 2k, \alpha^2 r^2\right) \\
&= B \psi_{n_r - k, m - 2sk}(r, \varphi). \tag{3.5.31}
\end{aligned}$$

The raising operator  $R_+$  operating on the wave function with  $m \geq 0$  gives

$$\begin{aligned}
R_+ \psi_{n_r, m}(r, \varphi) &= A \frac{2\alpha^2}{M(\frac{|m|}{s} + 1)(\frac{|m|}{s} + 2)} (\alpha r)^{\frac{|m|}{s} + 2} \exp(-\frac{\alpha^2 r^2}{2}) \exp(i\varphi(m + 2s)) \\
&\times n_r \left( \left(\frac{|m|}{s} + 2\right) {}_1F_1(1 - n_r, \frac{|m|}{s} + 2, \alpha^2 r^2) \right. \\
&\left. + (n_r - 1) {}_1F_1(2 - n_r, \frac{|m|}{s} + 3, \alpha^2 r^2) \right). \tag{3.5.32}
\end{aligned}$$

Using the recurrence relations of the confluent hypergeometric function [31] the above equation can be written as

$$\begin{aligned}
R_+ \psi_{n_r, m}(r, \varphi) &= A \frac{2\alpha^2}{M(\frac{|m|}{s} + 1)(\frac{|m|}{s} + 2)} (\alpha r)^{\frac{|m|}{s} + 2} \exp(-\frac{\alpha^2 r^2}{2}) \\
&\times \exp(i\varphi(m + 2s)) n_r (n_r + \frac{|m|}{s} + 1) {}_1F_1(1 - n_r, \frac{|m|}{s} + 3, \alpha^2 r^2) \\
&= B \psi_{n_r - 1, m + 2s}(r, \varphi). \tag{3.5.33}
\end{aligned}$$

The general rule for  $k$  applications of  $R_+$  on the wave function with  $m \geq 0$  is

$$\begin{aligned}
R_+^k \psi_{n_r, m}(r, \varphi) &= A \left(2 \frac{\alpha^2}{M}\right)^k (\alpha r)^{\frac{|m|}{s} + 2k} \exp(-\frac{\alpha^2 r^2}{2}) \exp(i\varphi(m + 2sk)) \\
&\times \prod_{j=1}^k \left( \frac{(n_r - j + 1)(n_r + \frac{|m|}{s} + j)}{(\frac{|m|}{s} + 2j - 1)(\frac{|m|}{s} + 2j)} \right) {}_1F_1(k - n_r, \frac{|m|}{s} + 1 + 2k, \alpha^2 r^2) \\
&= B \psi_{n_r - k, m + 2sk}(r, \varphi). \tag{3.5.34}
\end{aligned}$$

The general rule for  $k$  applications of  $R_+$  on the wave function with  $m < 0$  is

$$\begin{aligned}
R_+^k \psi_{n_r, m}(r, \varphi) &= A \left(2 \frac{\alpha^2}{M}\right)^k (\alpha r)^{\frac{|m|}{s} - 2k} \exp(-\frac{\alpha^2 r^2}{2}) \exp(i\varphi(m + 2sk)) \\
&\times \prod_{j=1}^k \left( \frac{|m|}{s} + 2 - 2j \right) \left( \frac{|m|}{s} + 1 - 2j \right) \\
&\times {}_1F_1(-k - n_r, \frac{|m|}{s} + 1 - 2k, \alpha^2 r^2) \\
&= B \psi_{n_r + k, m + 2sk}(r, \varphi). \tag{3.5.35}
\end{aligned}$$

To count the number of wave functions in the multiplet we begin with  $n_r = 0$ . Then clearly Eq.(3.5.24) gives  $S = m_{max}/s$ . Consider first the case of  $m \geq 0$  with  $m = m_{max} \geq 0$ . The value of  $m = m_{max}$  can not be raised further because applying

$R_+$  gives zero according to Eq.(3.5.34). Now we apply the lowering operator  $R_-$  a number of  $k_0$  times, with  $k_0 = m_{max}/s + 1/2$ . Using Eq.(3.5.28) we obtain

$$\begin{aligned} R_-^{k_0} \psi_{0, m_{max}}(r, \varphi) &= A \left( \frac{2\alpha^2}{M} \right)^{k_0} (\alpha r)^{\frac{|m_{max}|}{s} - 2k_0} \exp\left(-\frac{\alpha^2 r^2}{2}\right) \exp(i\varphi(m_{max} - 2sk_0)) \\ &\times \prod_{j=1}^{k_0} \left( \frac{|m_{max}|}{s} + 2 - 2j \right) \left( \frac{|m_{max}|}{s} + 1 - 2j \right) \\ &\times {}_1F_1\left(-k_0, \frac{|m_{max}|}{s} + 1 - 2k_0, \alpha^2 r^2\right). \end{aligned} \quad (3.5.36)$$

Here we have two cases in which the multiplet terminates:

**Half-integer**  $m_{max}/2s \geq 0$

Applying the raising operator  $R_+$  once gives zero according to Eq.(3.5.34). On the other hand, applying the lowering operator  $R_-$  a number of  $k_0 = (m_{max}/2s) + 1/2$  times with  $m_{max}/s$  an odd number, and using the identity in Eq.(3.4.14), then Eq.(3.5.36) can be written as

$$\begin{aligned} R_-^{k_0} \psi_{0, m_{max}}(r, \varphi) &= C(\alpha r)^1 \exp\left(-\frac{\alpha^2 r^2}{2}\right) \exp(-i\varphi s) {}_1F_1\left(-\frac{|m_{max}|}{2s} + \frac{1}{2}, 2, \alpha^2 r^2\right) \\ &= B \psi_{\frac{|m_{max}|}{2s} - \frac{1}{2}, -s}(r, \varphi), \end{aligned} \quad (3.5.37)$$

where  $C$  is a constant. As we can see, the resulting wave function has  $m = -s < 0$ . The operator  $R_-$  can be applied further  $k'_0$  times. However, we must use the appropriate relation Eq.(3.5.31), and we get

$$\begin{aligned} R_-^{k'_0} (R_-^{k_0} \psi_{0, m_{max}}(r, \varphi)) &= C(\alpha r)^{1+2k'_0} \exp\left(-\frac{\alpha^2 r^2}{2}\right) \exp(-i\varphi s(1 + 2k'_0)) \\ &\times \prod_{j=1}^{k'_0} \left( \frac{(|m_{max}|/2s + \frac{1}{2} - j)(|m_{max}|/2s + \frac{1}{2} + j)}{2j(1 + 2j)} \right) \\ &\times {}_1F_1\left(k'_0 - \frac{|m_{max}|}{2s} + \frac{1}{2}, 2 + 2k'_0, \alpha^2 r^2\right) \\ &= B \psi_{\frac{|m_{max}|}{2s} - \frac{1}{2} - k'_0, -s - 2sk'_0}(r, \varphi). \end{aligned} \quad (3.5.38)$$

It is obvious from the above equation that applying  $R_-$  a number of  $|m_{max}|/2s + 1/2$  times gives zero. Therefore, the number of wave functions in the multiplet is

$$N = k_0 + k'_0 = \frac{|m_{max}|}{s} + 1. \quad (3.5.39)$$

**Integer**  $m_{max}/2s \geq 0$

Applying the raising operator  $R_+$  once gives zero according to Eq.(3.5.34). On the other hand applying the lowering  $R_-$  operator a number of  $k_0 = (m_{max}/2s)+1$  times, and after that using the identity in Eq.(3.4.15), then Eq.(3.5.36) can be written as

$$\begin{aligned} R_-^{k_0} \psi_{0,m_{max}}(r, \varphi) &= C(\alpha r)^2 \exp\left(-\frac{\alpha^2 r^2}{2}\right) \exp(-i\varphi 2s) {}_1F_1\left(-\frac{|m_{max}|}{2s} + 1, 3, \alpha^2 r^2\right) \\ &= B \psi_{\frac{|m_{max}|}{2s}-1, -2s}(r, \varphi). \end{aligned} \quad (3.5.40)$$

As we can see, the resulting wave function has  $m = -2s < 0$ . The operator  $R_-$  can be applied further  $k'_0$  times. However, we must use the appropriate relation Eq.(3.5.31), and we get

$$\begin{aligned} R_-^{k'_0} (R_-^{k_0} \psi_{0,m_{max}}(r, \varphi)) &= C(\alpha r)^{2+2k'_0} \exp\left(-\frac{\alpha^2 r^2}{2}\right) \exp(-i\varphi s(2 + 2k'_0)) \\ &\times \prod_{j=1}^{k'_0} \left( \frac{(\frac{|m_{max}|}{2s} - j)(\frac{|m_{max}|}{2s} + 1 + j)}{(2j + 2)(1 + 2j)} \right) \\ &\times {}_1F_1\left(k'_0 - \frac{|m_{max}|}{2s} + 1, 3 + 2k'_0, \alpha^2 r^2\right) \\ &= B \psi_{\frac{|m_{max}|}{2s}-1-k'_0, -2s-2sk'_0}(r, \varphi). \end{aligned} \quad (3.5.41)$$

It is obvious from the above equation that applying  $R_-$  a number of  $|m_{max}|/2s$  times gives zero. Therefore, the number of wave functions in the multiplet is

$$N = q_0 + q'_0 = \frac{|m_{max}|}{s} + 1. \quad (3.5.42)$$

**Not Integer or half integer**  $m_{max}/s$

Applying the lowering operator  $R_-$  any number of times will not make either identity (3.4.14) or identity (3.4.15) applicable, since neither  $(|m_{max}|/s) + 1 - 2j$  nor  $(|m_{max}|/s) + 2 - 2j$  in Eq.(3.5.28) is equal to 0 or  $-1$  for any value of  $j \in \mathbb{Z}$ . After  $k_0 > |m_{max}|/s$  applications of  $R_-$ ,  $(\alpha r)^{(|m_{max}|/s)-2k_0}$  is negative, and the resulting wave functions diverge at the origin although it is still a solution of the Schrödinger equation. The multiplet does not terminate for any number of applications of  $R_-$ , and the number of wave functions in such a multiplet is infinite. The multiple application of  $R_-$  transforms a well-behaved wave function to one that diverges at the origin after  $k_0 > |m_{max}|/2s$  applications.

The same argument can be repeated for the case  $m = -|m_{max}|$ , this time by using the raising operator  $R_+$ , as well as Eq.(3.5.34) and Eq.(3.5.35). The same

result is reached regarding the relation between  $|m_{max}|/s$  and the number of wave functions in the multiplet.

### 3.5.8 Unusual Multiplets

Let us now consider the unusual multiplets in case of the harmonic oscillator. The discussion is similar to the one of the  $1/r$  potential and will thus not be repeated in all details. For  $m > 0$  one now obtains

$$\tilde{R}_+|n_r, m\rangle \propto |n_r - 1, m + 2s\rangle, \quad \tilde{R}_-|n_r, m\rangle \propto |n_r + 1, m - 2s\rangle, \quad (3.5.43)$$

and for  $m < 0$  one finds

$$\tilde{R}_+|n_r, m\rangle \propto |n_r + 1, m + 2s\rangle, \quad \tilde{R}_-|n_r, m\rangle \propto |n_r - 1, m - 2s\rangle, \quad (3.5.44)$$

while, for  $m = 0$  we have

$$\tilde{R}_+|n_r, 0\rangle \propto |n_r - 1, 2s\rangle, \quad \tilde{R}_-|n_r, 0\rangle \propto |n_r - 1, -2s\rangle. \quad (3.5.45)$$

As before, these relations follow from the  $SU(2)$  algebra which now implies that  $\tilde{R}_\pm$  are raising and lowering operators for  $\tilde{L} = L/2s$ . Hence, by acting with  $\tilde{R}_\pm$  the eigenvalue  $m$  of  $L$  is now shifted by  $\pm 2s$ .

One now confirms the value of the Casimir spin  $S = n_r + |m|/2s$  by evaluating

$$\begin{aligned} C|0, m + 2n_r s \geq 0\rangle &= (\tilde{L} + \tilde{L}^2)|0, m + 2n_r s \geq 0\rangle \\ &= \left(\frac{m}{2s} + n_r\right) \left(\frac{m}{2s} + n_r + 1\right) |0, m + 2n_r s \geq 0\rangle \\ &= S(S + 1)|0, m + 2n_r s \geq 0\rangle, \\ C|0, m - 2n_r s \leq 0\rangle &= (-\tilde{L} + \tilde{L}^2)|0, m - 2n_r s \leq 0\rangle \\ &= \left(-\frac{m}{2s} + n_r\right) \left(-\frac{m}{2s} + n_r + 1\right) |0, m - 2n_r s \leq 0\rangle \\ &= S(S + 1)|0, m - 2n_r s \leq 0\rangle. \end{aligned} \quad (3.5.46)$$

The multiplet of degenerate states with the same value of  $S$  is again obtained by repeated applications of  $\tilde{R}_+$  or  $\tilde{R}_-$ .

As in the case of the  $1/r$  potential, for  $s \neq 1$  different types of unusual multiplets arise. Again, even for integer or half-integer  $S = n_r + |m|/2s$ , the degeneracy of the physical multiplet is not  $2S + 1$  because  $m \pm 2ns$  may not be an integer in which case the corresponding wave function is not  $2\pi$ -periodic. When  $S = n_r + |m|/2s$  is neither an integer nor a half-integer, there is again an infinite number of degenerate solutions of the Schrödinger equation. However, once more, only a finite number of them obeys the boundary condition of Eq.(3.2.20) and thus belongs to  $\mathcal{D}[H]$ . A sequence of physical and unphysical wave functions is illustrated in figure (3.7).

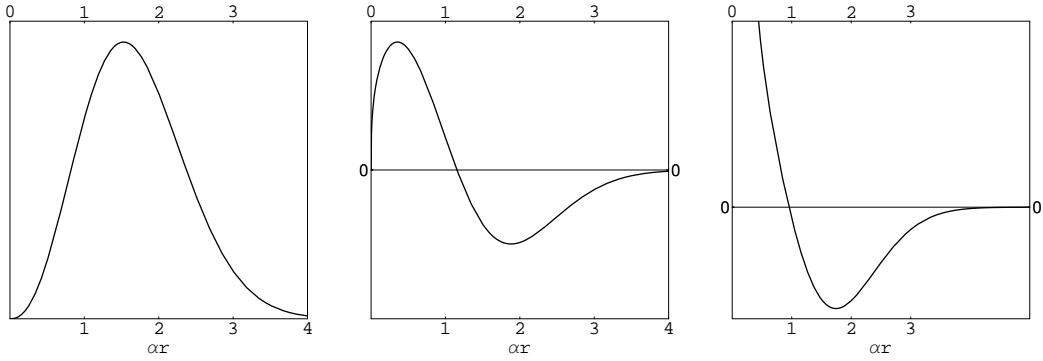


Figure 3.7: A sequence of wave functions for the  $r^2$  potential with  $s = 3$  obtained from repeated applications of  $\tilde{R}_-$ . The quantum numbers are  $n_r = 0$ ,  $m = 7$  (left),  $n_r = 1$ ,  $m = 7 - 2s = 1$  (middle), and  $n_r = 2$ ,  $m = 7 - 4s = -5$  (right). The third state in the sequence is outside the domain of the Hamiltonian because the corresponding wave function does not obey the boundary condition of Eq.(3.2.20) and the state is thus unphysical.

### 3.5.9 Counting Degeneracies in the Case of the Isotropic Harmonic Oscillator

The  $\varphi$ -dependent part of the wave function for each member of the multiplet is  $\exp(i\varphi(m_{max} - 2sk))$ . For the function to be  $2\pi$ -periodic  $(m_{max} - 2sk)$  must be an integer. One way to realize that is by studying the application of the lowering operator  $R_-$  on the wave function a number of  $k$  times. The multiplet can be finite for the case of  $|m_{max}|/2s$  being an integer or a half-integer. Then we have  $k = 0, 1, \dots, (|m_{max}|/s) + 1$  or infinite when  $|m_{max}|/s$  is a fraction. A rule can be derived by induction for the degeneracy  $g$ . When we begin with  $m_{max}$  as an integer, the rule for even  $q$  is

$$g = \left[ \frac{2|m_{max}|}{p} \right] + 1 = \left[ \frac{2S}{q} \right] + 1, \quad (3.5.47)$$

while the rule for odd  $q$  is

$$g = \left[ \frac{|m_{max}|}{p} \right] + 1 = \left[ \frac{S}{q} \right] + 1. \quad (3.5.48)$$

For the case of non-integer  $m_{max} = P/Q$ , we use the same approach that was used in the Kepler case. Let us assume that  $i_1$  applications of  $R_-$  are needed to reach the closest integer to  $P/Q$ , say  $b$ . One then obtains

$$\frac{P}{Q} - 2i_1 \frac{p}{q} = b. \quad (3.5.49)$$

Evaluating  $i_1$  implies

$$i_1 = \frac{q}{2pQ}(P - Qb) \quad (3.5.50)$$

Since the rule of finding the degeneracy for  $m = b$  being an integer is known and given by Eq.(3.5.47), and Eq.(3.5.48), for even  $b$ , one obtains

$$g = \left[ \frac{2b}{p} \right] + 1, \quad (3.5.51)$$

while for odd  $b$  one obtain

$$g = \left[ \frac{b}{p} \right] + 1. \quad (3.5.52)$$





## Chapter 4

# Accidental Symmetries for a Particle in a Constant Magnetic Field on the Torus

### 4.1 Introduction to Cyclotron Motion

In this chapter we will discuss another case of accidental symmetry that emerges as a result of classical orbits being closed. In cyclotron motion a particle moves on a 2-dimensional plane under the influence of a constant magnetic field perpendicular to the plane. Then, the classical orbits are indeed closed and this gives rise to an accidental symmetry [18, 19] as well as a conserved two-component Runge-Lenz vector. When the plane of motion is infinite there is, in addition a conserved angular momentum, as we explained in section (2.2). The angular momentum conservation is due to rotational invariance. This symmetry is lost when the motion takes place on a torus. This leads to less symmetry, and fewer degeneracies in the corresponding quantum problem. The additional conserved quantities arise from the fact that the vector pointing from the origin to the center of the circular cyclotron orbit has conserved, analogous to the Runge-Lenz vector in the Kepler problem [19]. Also, the radius of the cyclotron orbit is a conserved quantity directly related to the energy. Interestingly, quantum mechanically, while the two coordinates of the center of the circle are not simultaneously measurable, the radius of the circle has a sharp value in an energy eigenstate. In the cyclotron problem, translation invariance disguises itself as an “accidental” symmetry. As a consequence, the symmetry multiplets — i.e. the Landau levels — are infinitely degenerate. In order to further investigate the nature of the accidental symmetry, in [11] the charged particle in the magnetic field was coupled to the origin by an  $r^2$  harmonic oscillator potential. This explicitly

breaks translation invariance, and thus reduces the degeneracy to a finite amount, while rotation invariance remains intact. In this chapter, we do the opposite, i.e. we explicitly break rotation invariance, while leaving translation invariance (and hence the accidental symmetry) intact by putting the problem on a torus. Remarkably, the Polyakov loops of the Abelian vector potential, which are a consequence of the non-trivial holonomies of the torus, give rise to non-trivial Aharonov-Bohm phases which are observable at the quantum but not at the classical level. Analogous to the quantum mechanical breaking of CP invariance due to the  $\theta$ -vacuum angle in non-Abelian gauge theories, here two self-adjoint extension parameters  $\theta_x$  and  $\theta_y$  explicitly break the continuous translation invariance of the classical problem down to a discrete magnetic translation group. This reduces the degeneracy to a finite number, and allows us to further investigate the nature of the accidental symmetry. In particular, just like for motion on a cone, symmetry manifests itself in a rather unusual way in this quantum system. In particular, due to its relevance to the quantum Hall effect, the Landau level problem has been studied very extensively (for a recent review see [12]). For example, the problem has already been investigated on a torus in [13, 32], however, without elaborating on the accidental symmetry aspects. In this chapter, we concentrate exactly on those aspects, thus addressing an old and rather well-studied problem from a somewhat unconventional point of view.

## 4.2 Particle in the Infinite Volume

In this section we review more carefully what has been discussed in section (2.3.2), and try to add new understanding to the standard knowledge about a non-relativistic particle moving in a constant magnetic field in the infinite volume. We proceed from a classical to a semi-classical, and finally to a fully quantum mechanical treatment. In particular, we emphasize the symmetry aspects of the problem with a focus on accidental symmetries. This section is a preparation for the case of a finite periodic volume to be discussed in the next section.

### 4.2.1 Classical Treatment

Consider a non-relativistic electron of mass  $M$  and electric charge  $-e$  moving in a constant magnetic field  $\vec{B} = B\vec{e}_z$ . The vector potential can be chosen as

$$A_x(\vec{x}) = 0, \quad A_y(\vec{x}) = Bx, \quad A_z(\vec{x}) = 0. \quad (4.2.1)$$

The motion along the direction of the magnetic field is trivial because the velocity component in the  $z$ -direction is just a constant with vanishing Poisson brackets with

all the other conserved quantities. Therefore we restrict ourselves to 2-dimensional motion in the  $x$ - $y$ -plane. Obviously, this is just standard cyclotron motion. To get started, in this subsection we treat the problem classically. The particle then experiences the Lorentz force

$$\vec{F}(t) = -e\vec{v}(t) \times B\vec{e}_z, \quad (4.2.2)$$

which forces the particle on a circular orbit of some radius  $r$ . It moves along the circle with an angular velocity  $\omega$ , which implies the linear velocity  $v = \omega r$  and the acceleration  $a = \omega^2 r$ . Hence, Newton's equation takes the form

$$M\omega^2 r = e\omega r B \Rightarrow \omega = \frac{eB}{M}, \quad (4.2.3)$$

with the cyclotron frequency  $\omega$  being independent of the radius  $r$ . Obviously, for this system all classical orbits are closed. The same is true for a particle moving in a  $1/r$  or  $r^2$  potential. In those cases, the fact that all bound classical orbits are closed is related to the conservation of the Runge-Lenz vector which generates a hidden accidental dynamical symmetry. Let us now investigate the question of accidental symmetry for the particle in a constant magnetic field. The classical Hamilton function takes the form

$$H = \frac{1}{2M} \left[ \vec{p} + e\vec{A}(\vec{x}) \right]^2 = \frac{1}{2M} \left[ p_x^2 + (p_y + eBx)^2 \right]. \quad (4.2.4)$$

Let us define the following three quantities

$$P_x = p_x + eBy, \quad P_y = p_y, \quad L = x \left( p_y + \frac{eB}{2}x \right) - y \left( p_x + \frac{eB}{2}y \right). \quad (4.2.5)$$

We are familiar with angular momentum section (2.2.2). It is straightforward to convince oneself that  $H$  has vanishing Poisson brackets,  $\{H, P_x\} = \{H, P_y\} = \{H, L\} = 0$ . It is natural to think of  $P_x$ ,  $P_y$ , and  $L$  as the gauge-covariant generators of translations and rotations. In particular, one obtains

$$\{L, P_x\} = P_y, \quad \{L, P_y\} = -P_x, \quad (4.2.6)$$

as one would expect for the rotation properties of the vector  $(P_x, P_y)$ . As is well-known, however, in a magnetic field the two translations  $P_x$  and  $P_y$  do not commute, i.e.

$$\{P_x, P_y\} = eB. \quad (4.2.7)$$

How can these standard symmetry considerations be related to an accidental symmetry due to a Runge-Lenz vector? The Runge-Lenz vector is familiar from the Kepler problem. It points from the center of force to the perihelion position, and is conserved because all bound classical orbits are closed. Similarly, the orbit of

a charged particle in a constant magnetic field is a closed circle with a fixed center. Indeed, in this case the position of this center plays the role of the conserved Runge-Lenz vector and is given by

$$\begin{aligned} R_x &= x - \frac{v_y}{v}r = x - \frac{v_y}{\omega} = x - \frac{1}{M\omega} (p_y + eBx) = -\frac{p_y}{M\omega} = -\frac{P_y}{eB}, \\ R_y &= y + \frac{v_x}{v}r = y + \frac{v_x}{\omega} = y + \frac{p_x}{M\omega} = \frac{P_x}{eB}. \end{aligned} \quad (4.2.8)$$

Interestingly, the position  $(R_x, R_y)$  of the center of the cyclotron circle is, at the same time, proportional to  $(-P_y, P_x)$ , i.e. it is orthogonal to the generators of spatial translations. Consequently, we can write

$$\{R_x, P_x\} = -\frac{1}{eB}\{P_y, P_x\} = 1, \quad (4.2.9)$$

as well as

$$\{R_x, P_y\} = -\frac{1}{eB}\{P_y, P_y\} = 0, \quad \{R_y, P_x\} = \frac{1}{eB}\{P_x, P_x\} = 0. \quad (4.2.10)$$

While Eqs.(4.2.9) and (4.2.10) look like the usual Poisson brackets of position and momentum, one should not forget that  $R_x$  and  $R_y$  are just multiples of  $P_y$  and  $P_x$ , and should hence not be mistaken as independent variables. In particular, one also obtains the relation

$$\{R_x, R_y\} = \frac{1}{eB}. \quad (4.2.11)$$

Hence, just like the two generators of translations, the  $x$ - and  $y$ -components of the Runge-Lenz vector do not have a vanishing Poisson bracket. At the quantum level, this will imply that the  $x$ - and  $y$ -components of the center of a cyclotron circle are not simultaneously measurable with absolute precision. Another conserved quantity is the radius  $r$  of the circular cyclotron orbit which can be expressed as

$$r^2 = (x - R_x)^2 + (y - R_y)^2 = \frac{1}{M^2\omega^2} (p_y + eBx)^2 + \frac{p_x^2}{M^2\omega^2} = \frac{2H}{M\omega^2}. \quad (4.2.12)$$

This shouldn't be a surprise since  $r^2$  is proportional to the energy, therefore it should be conserved.

## 4.2.2 Semi-classical Treatment

Next, we consider the same problem semi-classically, i.e. by using Bohr-Sommerfeld quantization, which is equivalent to the quantization of angular momentum, i.e.  $L = n$ . For a cyclotron orbit of radius  $r$ , it is easy to convince oneself that

$$L = \frac{eB}{2}r^2 = n \Rightarrow r = \sqrt{\frac{2n}{eB}}. \quad (4.2.13)$$

Consequently, in the semi-classical treatment the allowed radii of cyclotron orbits are now quantized. Using Eq.(4.2.4) one finds for the energy

$$E = H = \frac{1}{2}M\omega^2 r^2 = n\omega. \quad (4.2.14)$$

As is well-known, up to a constant  $\omega/2$ , the semi-classically quantized energy values are those of a harmonic oscillator with the cyclotron frequency  $\omega$ .

### 4.2.3 Quantum Mechanical Treatment

Finally, we consider the problem fully quantum mechanically. In this case the Schrödinger equation takes the form

$$-\frac{1}{2M} [\partial_x^2 + (\partial_y + ieBx)^2] \Psi(\vec{x}) = E\Psi(\vec{x}). \quad (4.2.15)$$

We now make the factorization ansatz

$$\Psi(\vec{x}) = \psi(x) \exp(ip_y y), \quad (4.2.16)$$

and we obtain

$$\left[ -\frac{\partial_x^2}{2M} + \frac{1}{2}M\omega^2 \left( x + \frac{p_y}{M\omega} \right)^2 \right] \psi(x) = E\psi(x). \quad (4.2.17)$$

This is nothing but the Schrödinger equation of a shifted harmonic oscillator. Hence, the quantum mechanical energy spectrum takes the form

$$E = \omega \left( n + \frac{1}{2} \right). \quad (4.2.18)$$

The energy of the charged particle is completely independent of the transverse momentum  $p_y$ . We can understand this from the classical solution of this problem. The value of  $p_y = M\dot{y} + eA_y$  depends on the choice of the gauge which is arbitrary. While the total energy equal to  $M(\dot{x}^2 + \dot{y}^2)/2$  is gauge-independent. The fact that the energy is completely independent of the transverse momentum  $p_y$  leads to quantized Landau levels with continuous infinite degeneracy. The energy eigenstates are shifted one-dimensional harmonic oscillator wave functions  $\psi_n(x)$ , i.e.

$$\langle \vec{x} | n p_y \rangle = \psi_n \left( x + \frac{p_y}{M\omega} \right) \exp(ip_y y). \quad (4.2.19)$$

Here  $\psi_n$  can be obtained by substituting the above Eq.(4.2.19) into Eq.(4.2.15), and solving the differential equation. The solution is

$$\psi_n \left( x + \frac{p_y}{M\omega} \right) = H_n \left( \alpha \left( x + \frac{p_y}{M\omega} \right) \right) \exp \left( -\frac{\alpha^2}{2} \left( x + \frac{p_y}{M\omega} \right)^2 \right). \quad (4.2.20)$$

Similarly, one can construct eigenstates of the generator  $P_x = -i\partial_x + eBy$  of infinitesimal translations (up to gauge transformations) in the  $x$ -direction

$$\langle \vec{x} | np_x \rangle = \psi_n \left( y - \frac{p_x}{M\omega} \right) \exp(ip_x x) \exp(-ieBxy), \quad (4.2.21)$$

where  $\alpha^2 = eB$ .

$$\psi_n \left( y - \frac{p_x}{M\omega} \right) = H_n \left( \alpha \left( y - \frac{p_x}{M\omega} \right) \right) \exp \left( -\frac{\alpha^2}{2} \left( y - \frac{p_x}{M\omega} \right)^2 \right). \quad (4.2.22)$$

In the following we will prove that the two sets of eigenstates  $\langle \vec{x} | np_y \rangle$  and  $\langle \vec{x} | np_x \rangle$  span the same subspace of the Hilbert space. For that to be the case, we must prove that the following equation holds

$$\langle \vec{x} | np_x \rangle = \int_{-\infty}^{\infty} dp_y \langle \vec{x} | np_y \rangle \langle np_y | np_x \rangle. \quad (4.2.23)$$

On the other hand

$$\langle np_y | np_x \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \langle np_y | \vec{x} \rangle \langle \vec{x} | np_x \rangle. \quad (4.2.24)$$

Substituting Eq.(4.2.19), Eq.(4.2.21), Eq.(4.2.20), and Eq.(4.2.22) in the above the Eq.(4.2.23), and after using the following identity [33]

$$\int_{-\infty}^{\infty} d\xi \exp(i\xi\eta) \exp\left(-\frac{\xi^2}{2}\right) H_n(\xi) = \sqrt{2\pi} i^n \exp\left(-\frac{\eta^2}{2}\right) H_n(\eta), \quad (4.2.25)$$

we get

$$\langle np_y | np_x \rangle = \exp\left(-i\frac{p_x p_y}{\alpha^2}\right). \quad (4.2.26)$$

Substituting Eq.(4.2.26) in Eq.(4.2.23) and using Eq.(4.2.25) again gives

$$\int_{-\infty}^{\infty} dp_y \langle \vec{x} | np_y \rangle \langle np_y | np_x \rangle = C H_n \left( \alpha \left( y - \frac{p_x}{M\omega} \right) \right) \exp(ip_x x) \exp(-ieBxy), \quad (4.2.27)$$

which completes the proof.

Since all classical orbits are closed and the center of the cyclotron orbit plays the role of a Runge-Lenz vector, it is natural to ask whether the degeneracy is caused by an accidental symmetry. Of course, since the Runge-Lenz vector plays a dual role and is also generating translations (up to gauge transformations), in this case the ‘‘accidental’’ symmetry would just be translation invariance. Indeed, in complete analogy to the classical case, it is easy to convince oneself that  $[H, R_x] = [H, R_y] =$

$[H, L] = 0$ , with the Runge-Lenz vector and the angular momentum operator given by

$$\begin{aligned} R_x &= -\frac{P_y}{eB} = \frac{i\partial_y}{eB}, \quad R_y = \frac{P_x}{eB} = y - \frac{i\partial_x}{eB}, \\ L &= x \left( -i\partial_y + \frac{eBx}{2} \right) - y \left( -i\partial_x + \frac{eBy}{2} \right). \end{aligned} \quad (4.2.28)$$

As in the classical case, the radius of the cyclotron orbit squared is given by

$$r^2 = (x - R_x)^2 + (y - R_y)^2 = \left( x - \frac{i\partial_y}{eB} \right)^2 - \frac{\partial_x^2}{e^2 B^2} = \frac{2H}{M\omega^2}, \quad (4.2.29)$$

and is thus again a conserved quantity. In particular, we can express the Hamiltonian as

$$H = \frac{1}{2} M\omega^2 r^2. \quad (4.2.30)$$

Remarkably, although the two coordinates  $R_x$  and  $R_y$  of the center of the cyclotron circle are not simultaneously measurable, its radius  $r$  has a definite value in an energy eigenstate. As it should, under spatial rotations the Runge-Lenz vector  $(R_x, R_y)$  indeed transforms as a vector, i.e.

$$[L, R_x] = iR_y, \quad [L, R_y] = -iR_x. \quad (4.2.31)$$

These relations suggest to introduce

$$R_{\pm} = R_x \pm iR_y, \quad (4.2.32)$$

which implies

$$[L, R_{\pm}] = \pm R_{\pm}. \quad (4.2.33)$$

Hence,  $R_+$  and  $R_-$  act as raising and lowering operators of angular momentum. Still, it is important to note that  $R_x$ ,  $R_y$ , and  $L$  do not form an  $SU(2)$  algebra. This follows because, in analogy to the classical case

$$[R_x, R_y] = \frac{i}{eB}, \quad (4.2.34)$$

i.e.  $R_x$  and  $R_y$  are generators of a Heisenberg algebra. From the above equation we easily conclude that

$$[R_+, R_-] = \frac{2}{eB}. \quad (4.2.35)$$

#### 4.2.4 Creation and Annihilation Operators

The particle in the magnetic field leads to the (infinitely degenerate) spectrum of a 1-dimensional harmonic oscillator. This suggests that one can construct corresponding creation and annihilation operators such that

$$H = \omega \left( a^\dagger a + \frac{1}{2} \right), \quad [a, a^\dagger] = 1. \quad (4.2.36)$$

Interestingly, the creation and annihilation operators are closely related to the Runge-Lenz vector, i.e. the vector that points to the center of the classical cyclotron orbit. Since we have seen that

$$H = \frac{1}{2} M \omega^2 r^2 = \frac{1}{2} M \omega^2 [(x - R_x)^2 + (y - R_y)^2], \quad (4.2.37)$$

one is led to identify

$$a = \sqrt{\frac{M\omega}{2}} [x - R_x - i(y - R_y)], \quad a^\dagger = \sqrt{\frac{M\omega}{2}} [x - R_x + i(y - R_y)], \quad (4.2.38)$$

which indeed have the desired properties. One also finds that

$$[L, a] = -a, \quad [L, a^\dagger] = a^\dagger, \quad (4.2.39)$$

which implies that  $a^\dagger$  and  $a$  also raise and lower the angular momentum. Interestingly, we have seen before that

$$[L, R_\pm] = \pm R_\pm, \quad [R_+, R_-] = \frac{2}{eB} = \frac{2}{M\omega}. \quad (4.2.40)$$

Therefore,  $R_+$  and  $R_-$  also act as raising and lowering operators of the angular momentum. We can identify another set of creation and annihilation operators, which are

$$b = \sqrt{\frac{M\omega}{2}} R_+, \quad b^\dagger = \sqrt{\frac{M\omega}{2}} R_-, \quad (4.2.41)$$

and obey

$$[L, b] = b, \quad [L, b^\dagger] = -b^\dagger. \quad (4.2.42)$$

As a result,  $b$  raises and  $b^\dagger$  lowers the angular momentum by one unit. Introducing

$$z = x + iy = \sqrt{\frac{2}{M\omega}} (a^\dagger + b), \quad (4.2.43)$$

it is straightforward to derive the commutation relations

$$[a, b] = [a^\dagger, b] = [a, b^\dagger] = [a^\dagger, b^\dagger] = 0, \quad [b, b^\dagger] = 1. \quad (4.2.44)$$

Like in the case of a 2-dimensional harmonic oscillator, the particle in a magnetic field is described by two sets of commuting creation and annihilation operators. However, in contrast to the 2-dimensional harmonic oscillator, the Hamiltonian of the particle in a magnetic field contains only  $a^\dagger a$ , but not  $b^\dagger b$ .



### 4.2.5 Alternative Representation of the Hamiltonian

As we showed in Eq.(2.3.20), the Hamiltonian can be expressed as

$$H = \frac{1}{2}M\omega^2 (R_x^2 + R_y^2) + \omega L. \quad (4.2.45)$$

Using the above equation as well as Eq.(4.2.41) gives

$$H = \omega \left( b^\dagger b + \frac{1}{2} + L \right) = H_0 + \omega L. \quad (4.2.46)$$

Here we have introduced the Hamiltonian of an ordinary 1-dimensional harmonic oscillator

$$H_0 = \omega \left( b^\dagger b + \frac{1}{2} \right), \quad (4.2.47)$$

and the angular momentum operator has been identified as

$$L = a^\dagger a - b^\dagger b. \quad (4.2.48)$$

Interestingly, the creation and annihilation operators  $b^\dagger$  and  $b$  commute with the total energy  $H$  because they raise (lower)  $H_0$  by  $\omega$ , while they lower (raise)  $L$  by 1, such that indeed

$$[H, b] = [H_0, b] + \omega[L, b] = 0, \quad [H, b^\dagger] = [H_0, b^\dagger] + \omega[L, b^\dagger] = 0. \quad (4.2.49)$$

### 4.2.6 Energy Spectrum and Energy Eigenstates

In section (2.3.2) we discussed the explicit solution of the Schrödinger equation for this problem, as well as an alternative method for finding the energy levels and degeneracies from symmetry. Another interesting approach can be is also based also on symmetry, and uses the fact that the algebraic structure of the problem (but not the exact form of the Hamiltonian) is the same as for the 2-dimensional harmonic oscillator. First of all, we construct a state  $|00\rangle$  that is annihilated by both  $a$  and  $b$ , i.e.

$$a|00\rangle = b|00\rangle = 0. \quad (4.2.50)$$

Then we define states

$$|nn'\rangle = \frac{(a^\dagger)^n (b^\dagger)^{n'}}{\sqrt{n!} \sqrt{n'!}} |00\rangle, \quad (4.2.51)$$

which are eigenstates of the total energy

$$H|nn'\rangle = \omega \left( n + \frac{1}{2} \right) |nn'\rangle, \quad (4.2.52)$$

as well as of the angular momentum

$$L|nn'\rangle = (n - n')|nn'\rangle = m|nn'\rangle. \quad (4.2.53)$$

It should be noted that the quantum number  $n \in \{0, 1, 2, \dots\}$  (which determines the energy) is non-negative, while the quantum number  $m = n - n' \in \mathbb{Z}$  (which determines the angular momentum) is an arbitrary integer. The infinite degeneracy of the Landau levels is now obvious because states with the same  $n$  but different values of  $n'$  have the same energy. One may wonder why in subsection (4.2.3) we found an infinite degeneracy labeled by the continuous momentum  $p_y$  and now we only find a countable variety of degenerate states (labeled by the integer  $m$ ). This apparent discrepancy is due to the implicit consideration of two different Hilbert spaces. While the states in the discrete variety labeled by  $m$  are normalizable in the usual sense, the continuous variety of plane wave states labeled by  $p_y$  is normalized to  $\delta$ -functions and thus belongs to an extended Hilbert space.

### 4.2.7 Coherent States

Coherent states are well-known from the harmonic oscillator, and have also been constructed for the Landau level problem [34]. As usual, the coherent states are constructed as eigenstates of the annihilation operators, i.e.

$$a|\lambda\lambda'\rangle = \lambda|\lambda\lambda'\rangle, \quad b|\lambda\lambda'\rangle = \lambda'|\lambda\lambda'\rangle, \quad \lambda, \lambda' \in \mathbb{C}. \quad (4.2.54)$$

Some expectation values in the coherent state  $|\lambda\lambda'\rangle$  are given by

$$\begin{aligned} \langle R_x \rangle &= \sqrt{\frac{2}{M\omega}} \operatorname{Re}\lambda', \quad \Delta R_x = \sqrt{\frac{2}{M\omega}}, \\ \langle R_y \rangle &= \sqrt{\frac{2}{M\omega}} \operatorname{Im}\lambda', \quad \Delta R_y = \sqrt{\frac{2}{M\omega}}, \\ \langle x - R_x \rangle &= \sqrt{\frac{2}{M\omega}} \operatorname{Re}\lambda, \quad \Delta(x - R_x) = \frac{1}{\sqrt{2M\omega}}, \\ \langle y - R_y \rangle &= -\sqrt{\frac{2}{M\omega}} \operatorname{Im}\lambda, \quad \Delta(y - R_y) = \frac{1}{\sqrt{2M\omega}}, \\ \langle Mv_x \rangle &= \langle p_x + eA_x \rangle = \sqrt{2M\omega} \operatorname{Im}\lambda, \quad \Delta(Mv_x) = \sqrt{\frac{M\omega}{2}}, \\ \langle Mv_y \rangle &= \langle p_y + eA_y \rangle = \sqrt{2M\omega} \operatorname{Re}\lambda, \quad \Delta(Mv_y) = \sqrt{\frac{M\omega}{2}}, \\ \langle H \rangle &= \omega \left( |\lambda|^2 + \frac{1}{2} \right), \quad \Delta H = \omega|\lambda|. \end{aligned} \quad (4.2.55)$$

Here  $\Delta O = \sqrt{\langle O^2 \rangle - \langle O \rangle^2}$  describes the quantum uncertainty. In all cases  $\Delta O / \langle O \rangle$  is proportional to  $1/|\lambda|$  or  $1/|\lambda'|$ , which implies that the relative uncertainty goes to zero in the classical limit. Just as for the ordinary harmonic oscillator, the time-dependent Schrödinger equation  $i\partial_t|\Psi(t)\rangle = H|\Psi(t)\rangle$  with an initial coherent state  $|\Psi(0)\rangle = |\lambda(0)\lambda'\rangle$  is solved by

$$|\Psi(t)\rangle = \exp\left(-\frac{i\omega t}{2}\right) |\lambda(t)\lambda'\rangle, \quad \lambda(t) = \lambda(0) \exp(-i\omega t), \quad (4.2.56)$$

i.e. the state remains coherent during its time-evolution. In particular, this implies

$$\begin{aligned} \langle x - R_x \rangle(t) &= \frac{|\lambda|}{\sqrt{2M\omega}} \cos(\omega t), \quad \langle Mv_x \rangle(t) = -\sqrt{2M\omega} |\lambda| \sin(\omega t), \\ \langle y - R_y \rangle(t) &= \frac{|\lambda|}{\sqrt{2M\omega}} \sin(\omega t), \quad \langle Mv_y \rangle(t) = \sqrt{2M\omega} |\lambda| \cos(\omega t). \end{aligned} \quad (4.2.57)$$

Hence, the coherent state represents a wave packet moving around a circular cyclotron orbit just like a classical particle. This can be realized by solving the eigenvalue problem Eqs.(4.2.54). We denote  $\langle \vec{x} | \lambda(t)\lambda' \rangle = \Psi_{\lambda(t)\lambda'}(\vec{x})$ . Substituting for  $a$  and  $b$  from Eq.(4.2.38) and Eq.(4.2.41) in Eqs.(4.2.54), then we will get the following first order partial differential equations

$$\begin{aligned} \sqrt{\frac{M\omega}{2}} \left( \frac{i}{M\omega} \partial_y + \frac{1}{M\omega} \partial_x + iy \right) \Psi_{\lambda(t)\lambda'}(\vec{x}) &= \lambda' \Psi_{\lambda(t)\lambda'}(\vec{x}), \\ \sqrt{\frac{M\omega}{2}} \left( \frac{1}{M\omega} \partial_x - \frac{i}{M\omega} \partial_y + x \right) \Psi_{\lambda(t)\lambda'}(\vec{x}) &= \lambda \Psi_{\lambda(t)\lambda'}(\vec{x}). \end{aligned} \quad (4.2.58)$$

A solution of the above two differential equations is

$$\begin{aligned} \Psi_{\lambda(t)\lambda'}(\vec{x}) &= A(t) \exp \left( \sqrt{\frac{M\omega}{2}} \left( (x - iy)\lambda' + (x + iy)\lambda(t) \right) \right. \\ &\quad \left. - \frac{M\omega}{4} (x^2 + y^2 + 2ixy) \right) \exp \left( -i\frac{\omega}{2}t \right) \end{aligned} \quad (4.2.59)$$

The normalization constant is particularly important in this case because it is time-dependent. From the normalization condition

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \Psi_{\lambda(t)\lambda'}(\vec{x}) \Psi_{\lambda(t)\lambda'}^*(\vec{x}) = 1, \quad (4.2.60)$$

we obtain the value of  $A(t)$ . Accordingly the coherent state takes the following form

$$\begin{aligned} \Psi_{\lambda(t)\lambda'}(\vec{x}) &= \sqrt{\frac{M\omega}{2\pi}} \exp\left(\sqrt{\frac{M\omega}{2}}\left((x-iy)\lambda' + (x+iy)\lambda(t)\right)\right. \\ &\quad \left.- \frac{M\omega}{4}\left(x^2 + y^2 + ixy + (\operatorname{Re}\lambda' - \operatorname{Re}\lambda(t))^2 - (\operatorname{Im}\lambda' - \operatorname{Im}\lambda(t))^2\right)\right) \\ &\quad \times \exp(-i\frac{\omega}{2}t). \end{aligned} \quad (4.2.61)$$

From the above equation we get the probability density given by

$$\begin{aligned} |\langle \vec{x} | \Psi(t) \rangle|^2 &= A \exp\left(-\frac{M\omega}{2}(x - \langle R_x \rangle - d \cos(\omega t - \alpha))^2\right) \\ &\quad \times \exp\left(-\frac{M\omega}{2}(y - \langle R_y \rangle - d \sin(\omega t - \alpha))^2\right), \\ d &= \sqrt{\frac{2}{M\omega}}|\lambda|. \end{aligned} \quad (4.2.62)$$

As it is obvious from the above equation, the coherent state is, in fact, a Gaussian wave packet moving around a circular orbit.

## 4.3 Particle on a Torus

In this section we put the problem in a finite periodic volume. This breaks rotation invariance, but leaves translation invariance intact (at least at the classical level), and leads to an energy spectrum with finite degeneracy. In order to clarify some subtle symmetry properties, we investigate issues of Hermiticity versus self-adjointness of various operators.

### 4.3.1 Constant Magnetic Field on a Torus

In this subsection we impose a torus boundary condition over a rectangular region of size  $L_x \times L_y$ . As a result of this, there is a quantization condition for the magnetic flux. Since the magnetic field is constant, it obviously is periodic. On the other hand, the vector potential of the infinite volume theory  $A_x(x, y) = 0$ ,  $A_y(x, y) = Bx$

obeys the conditions

$$\begin{aligned}
A_x(x + L_x, y) &= A_x(x, y), \\
A_y(x + L_x, y) &= A_y(x, y) + BL_x = A_y(x, y) + \partial_y(BL_x y), \\
A_x(x, y + L_y) &= A_x(x, y), \\
A_y(x, y + L_y) &= A_y(x, y).
\end{aligned} \tag{4.3.1}$$

As a gauge-dependent quantity, the vector potential is periodic only up to gauge transformations, i.e.

$$A_i(x + L_x, y) = A_i(x, y) - \partial_i \varphi_x(y), \quad A_i(x, y + L_y) = A_i(x, y) - \partial_i \varphi_y(x). \tag{4.3.2}$$

The gauge transformations  $\varphi_x(y)$  and  $\varphi_y(x)$  are transition functions in a fiber bundle which specify the boundary condition. In our case the transition functions are given by

$$\varphi_x(y) = \frac{\theta_x}{e} - BL_x y, \quad \varphi_y(x) = \frac{\theta_y}{e}. \tag{4.3.3}$$

Besides the field strength, gauge theories on a periodic volume possess additional gauge invariant quantities — the so-called Polyakov loops — which arise due to the non-trivial holonomies of the torus. For an Abelian gauge theory the Polyakov loops are defined as

$$\Phi_x(y) = \int_0^{L_x} dx A_x(x, y) - \varphi_x(y), \quad \Phi_y(x) = \int_0^{L_y} dy A_y(x, y) - \varphi_y(x). \tag{4.3.4}$$

In our case, they are given by

$$\Phi_x(y) = BL_x y - \frac{\theta_x}{e}, \quad \Phi_y(x) = BL_y x - \frac{\theta_y}{e}. \tag{4.3.5}$$

In order to respect gauge invariance of the theory on the torus, under shifts the wave function must also be gauge transformed accordingly

$$\begin{aligned}
\Psi(x + L_x, y) &= \exp(i e \varphi_x(y)) \Psi(x, y) = \exp(i \theta_x - i e BL_x y) \Psi(x, y), \\
\Psi(x, y + L_y) &= \exp(i e \varphi_y(x)) \Psi(x, y) = \exp(i \theta_y) \Psi(x, y).
\end{aligned} \tag{4.3.6}$$

The angles  $\theta_x$  and  $\theta_y$  parameterize a family of self-adjoint extensions of the Hamiltonian on the torus. Applying the boundary conditions from above in two different orders one obtains

$$\begin{aligned}
\Psi(x + L_x, y + L_y) &= \exp(i \theta_x - i e BL_x(y + L_y)) \Psi(x, y + L_y) \\
&= \exp(i \theta_x + i \theta_y - i e BL_x(y + L_y)) \Psi(x, y), \\
\Psi(x + L_x, y + L_y) &= \exp(i \theta_y) \Psi(x + L_x, y) \\
&= \exp(i \theta_x + i \theta_y - i e BL_x y) \Psi(x, y).
\end{aligned} \tag{4.3.7}$$

Hence, consistency of the boundary condition requires

$$\exp(-ieBL_xL_y) = 1 \Rightarrow B = \frac{2\pi n_\Phi}{eL_xL_y}, \quad n_\Phi \in \mathbb{Z}. \quad (4.3.8)$$

The total magnetic flux through the torus

$$\Phi = BL_xL_y = \frac{2\pi n_\Phi}{e}, \quad (4.3.9)$$

is hence quantized in integer units of the elementary magnetic flux quantum  $2\pi/e$ . Interestingly, the spectrum of a charged particle in a constant magnetic field is discrete (but infinitely degenerate) already in the infinite volume. As we will see, in the finite periodic volume it has only a finite  $|n_\Phi|$ -fold degeneracy determined by the number of flux quanta. A quantum mechanical charged particle is sensitive to the complex phases defined by the Polyakov loops

$$\exp(ie\Phi_x(y)) = \exp(ieBL_xy - i\theta_x), \quad \exp(ie\Phi_y(x)) = \exp(ieBL_yx - i\theta_y), \quad (4.3.10)$$

which are measurable in Aharonov-Bohm type experiments. Remarkably, the Polyakov loops explicitly break the translation invariance of the torus at the quantum level. This is reminiscent of the quantum mechanical breaking of CP invariance due to the  $\theta$ -vacuum angle in non-Abelian gauge theories. The complex phases from above are invariant under shifts by integer multiples of

$$a_x = \frac{2\pi}{eBL_y} = \frac{L_x}{n_\Phi}, \quad a_y = \frac{2\pi}{eBL_x} = \frac{L_y}{n_\Phi}, \quad (4.3.11)$$

in the  $x$ - and  $y$ -directions, respectively. Hence, at the quantum level the continuous translation group of the torus is reduced to a discrete subgroup which plays the role of the accidental symmetry group. In this thesis, we treat the gauge field as a classical background field, while only the charged particle is treated quantum mechanically. It is interesting to note that, once the gauge field is also quantized, the transition functions  $\varphi_x(y)$  and  $\varphi_y(x)$  become fluctuating physical degrees of freedom of the gauge field. Still, as a consequence of

$$\begin{aligned} A_i(x + L_x, y + L_y) &= A_i(x, y + L_y) - \partial_i\varphi_x(y + L_y) \\ &= A_i(x, y) - \partial_i\varphi_x(y + L_y) - \partial_i\varphi_y(x), \\ A_i(x + L_x, y + L_y) &= A_i(x + L_x, y) - \partial_i\varphi_y(x + L_x) \\ &= A_i(x, y) - \partial_i\varphi_y(x + L_x) - \partial_i\varphi_x(y), \end{aligned} \quad (4.3.12)$$

and of

$$\begin{aligned} \Psi(x + L_x, y + L_y) &= \exp(ie\varphi_x(y + L_y))\Psi(x, y + L_y) \\ &= \exp(ie\varphi_x(y + L_y) + ie\varphi_y(x))\Psi(x, y), \\ \Psi(x + L_x, y + L_y) &= \exp(ie\varphi_y(x + L_x))\Psi(x + L_x, y) \\ &= \exp(ie\varphi_y(x + L_x) + ie\varphi_x(y))\Psi(x, y), \end{aligned} \quad (4.3.13)$$

the transition functions must obey the cocycle consistency condition

$$\varphi_y(x + L_x) + \varphi_x(y) - \varphi_x(y + L_y) - \varphi_y(x) = \frac{2\pi n_\Phi}{e}. \quad (4.3.14)$$

In this case, the magnetic flux  $n_\Phi$  specifies a super-selection sector of the theory. Analogous to the  $\mathbb{Z}(N)^d$  center symmetry of non-Abelian  $SU(N)$  gauge theories on a  $d$ -dimensional torus [35], Abelian gauge theories coupled to charged matter have a global  $\mathbb{Z}^d$  center symmetry. The self-adjoint extension parameters  $\theta_x$  and  $\theta_y$  then turn into conserved quantities (analogous to Bloch momenta) of the global  $\mathbb{Z}^2$  symmetry on the 2-dimensional torus. In this sense,  $\theta_x$  and  $\theta_y$  are analogous to the  $\theta$ -vacuum angle of non-Abelian gauge theories, which also distinguishes different super-selection sectors of the theory. The  $\theta$ -vacuum angle is a quantum mechanical source of explicit CP violation. At the classical level, on the other hand, CP invariance remains intact because  $\theta$  does not affect the classical equations of motion. Similarly, for a charged particle on the torus the angles  $\theta_x$  and  $\theta_y$  characterize the explicit breaking of continuous translation invariance down to a discrete subgroup. Just like CP invariance for a non-Abelian gauge theory, for a charged particle on the torus the full continuous translation symmetry remains intact at the classical level, because  $\theta_x$  and  $\theta_y$  do not appear in the classical equations of motion. In this thesis, we treat the charged particle as a test charge which does not surround itself with its own Coulomb field. This would change, once one would derive the charged particle from its own quantum field. For example, if one considers full-fledged QED, a single electron cannot even exist on the torus because the Coulomb field that surrounds it is incompatible with periodic boundary conditions. Indeed, as a consequence of the Gauss law, the total charge on a torus always vanishes. To cure this problem, one could compensate the charge of the electron by a classical background charge homogeneously spread out over the torus. In the present calculation this is not necessary, because the charged particle is treated as a test charge without its own surrounding Coulomb field.

### 4.3.2 Discrete Magnetic Translation Group

As we have seen, in order to respect gauge invariance, on the torus the wave function must obey Eq.(4.3.6), which can be re-expressed as

$$\Psi(x + L_x, y) = \exp\left(i\theta_x - \frac{2\pi i n_\Phi y}{L_y}\right) \Psi(x, y), \quad \Psi(x, y + L_y) = \exp(i\theta_y) \Psi(x, y). \quad (4.3.15)$$

It is interesting to note that a factorization ansatz for the wave function as in Eq.(4.2.16) is inconsistent with the boundary condition. Let us consider the unitary shift operator generating translations by a distance  $a_y$  in the  $y$ -direction

$$T_y = \exp(ia_y P_y), \quad (4.3.16)$$

which acts as

$$T_y \Psi(x, y) = \Psi(x, y + a_y). \quad (4.3.17)$$

It is clear that  $T_y$  commutes with the Hamiltonian because  $P_y$  does. Indeed, the shifted wave function does obey the boundary condition Eq.(4.3.15), i.e.

$$\begin{aligned} T_y \Psi(x + L_x, y) &= \Psi(x + L_x, y + a_y) = \exp\left(i\theta_x - \frac{2\pi i n_\Phi (y + a_y)}{L_y}\right) \Psi(x, y + a_y) \\ &= \exp\left(i\theta_x - \frac{2\pi i n_\Phi y}{L_y}\right) T_y \Psi(x, y), \end{aligned} \quad (4.3.18)$$

which is the case only because

$$a_y = \frac{L_y}{n_\Phi} \Rightarrow \exp\left(-\frac{2\pi i n_\Phi a_y}{L_y}\right) = 1. \quad (4.3.19)$$

Furthermore, we also have

$$\begin{aligned} T_y \Psi(x, y + L_y) &= \Psi(x, y + a_y + L_y) = \exp(i\theta_x) \Psi(x, y + a_y) \\ &= \exp(i\theta_x) T_y \Psi(x, y). \end{aligned} \quad (4.3.20)$$

Hence, as we argued before, the translations in the  $y$ -direction are reduced to the discrete group  $\mathbb{Z}(n_\Phi)$ . In particular, all translations compatible with the boundary conditions  $T_y^{n_y}$  can be expressed as the  $n_y$ -th power of the elementary translation  $T_y$ . According to Eq.(4.2.8),  $P_y = -eBR_x$ , such that

$$T_y = \exp\left(\frac{iL_y P_y}{n_\Phi}\right) = \exp\left(-\frac{ieBL_y R_x}{n_\Phi}\right) = \exp\left(-\frac{2\pi i R_x}{L_x}\right). \quad (4.3.21)$$

Since on the torus the Runge-Lenz vector component  $R_x$ , which determines the  $x$ -coordinate of the center of the cyclotron orbit, is defined only modulo  $L_x$ , it is indeed natural to consider the above translation operator. In fact, although it formally commutes with the Hamiltonian, the operator  $R_x$  itself is no longer self-adjoint in the Hilbert space of wave functions on the torus. The operator  $T_y$ , on the other hand, does act as a unitary operator in the Hilbert space. Similarly, the operator is given by

$$T_x = \exp(ia_x P_x) = \exp(ia_x eBR_y) = \exp\left(\frac{2\pi i n_\Phi a_x R_y}{L_x L_y}\right) = \exp\left(\frac{2\pi i R_y}{L_y}\right). \quad (4.3.22)$$

As a consequence of the commutation relation  $[R_x, R_y] = i/eB$ , one obtains

$$T_y T_x = \exp\left(\frac{2\pi i}{n_\Phi}\right) T_x T_y. \quad (4.3.23)$$



This implies that

$$T_x \Psi(x, y) = \exp\left(\frac{2\pi i y}{L_y}\right) \Psi\left(x + \frac{L_x}{n_\Phi}, y\right), \quad (4.3.24)$$

i.e., up to a periodic gauge transformation  $\exp(2\pi i y/L_y)$ ,  $T_x$  translates the wave function by a distance  $L_x/n_\Phi$ . Remarkably, although at the classical level the torus has two continuous translation symmetries, the corresponding infinitesimal generators  $P_x$  and  $P_y$  are not self-adjoint in the Hilbert space of wave functions on the torus. Only the finite translations  $T_x$  and  $T_y$  are represented by unitary operators, which, however, do not commute with each other. The two operators  $T_x$  and  $T_y$  generate a discrete translation group  $\mathcal{G}$  consisting of the elements

$$\begin{aligned} g(n_x, n_y, m) &= \exp\left(\frac{2\pi i m}{n_\Phi}\right) T_y^{n_y} T_x^{n_x}, \\ n_x, n_y, m &\in \{0, 1, 2, \dots, n_\Phi - 1\}. \end{aligned} \quad (4.3.25)$$

The group multiplication rule takes the form

$$g(n_x, n_y, m)g(n'_x, n'_y, m') = g(n_x + n'_x, n_y + n'_y, m + m' - n_x n'_y), \quad (4.3.26)$$

with all summations being understood modulo  $n_\Phi$ . Obviously, the unit-element is represented by

$$\mathbb{1} = g(0, 0, 0), \quad (4.3.27)$$

while the elements

$$z_m = g(0, 0, m) = \exp\left(\frac{2\pi i m}{n_\Phi}\right), \quad (4.3.28)$$

form the cyclic Abelian subgroup  $\mathbb{Z}(n_\Phi) \subset \mathcal{G}$ . The inverse of a general group element  $g(n_x, n_y, m)$  is given by

$$g(n_x, n_y, m)^{-1} = g(-n_x, -n_y, -m - n_x n_y), \quad (4.3.29)$$

because

$$g(n_x, n_y, m)g(-n_x, -n_y, -m - n_x n_y) = g(0, 0, -n_x n_y + n_x n_y) = g(0, 0, 0) = \mathbb{1}. \quad (4.3.30)$$

It is interesting to consider the conjugacy class of a group element  $g(n_x, n_y, m)$  which consists of the elements

$$\begin{aligned} &g(n'_x, n'_y, m')g(n_x, n_y, m)g(n'_x, n'_y, m')^{-1} = \\ &g(n'_x + n_x, n'_y + n_y, m' + m - n'_x n_y)g(-n'_x, -n'_y, -m' - n'_x n_y) = \\ &g(n_x, n_y, m - n'_x(n_y + n'_y) + (n'_x + n_x)n'_y) = g(n_x, n_y, m + n_x n'_y - n'_x n_y). \end{aligned} \quad (4.3.31)$$

Hence, as one would expect, the elements  $g(0, 0, m) = z_m \in \mathbb{Z}(n_\Phi)$  are conjugate only to themselves and thus form  $n_\Phi$  single-element conjugacy classes. In the case of odd  $n_\Phi$ , the  $n_\Phi$  elements  $z_m g(n_x, n_y, 0)$  with  $(n_x, n_y) \neq (0, 0)$  form  $n_\Phi^2 - 1$  additional conjugacy classes. When  $n_\Phi$  is even, this is still true as long as not both  $n_x$  and  $n_y$  are even. On the other hand, when  $n_x$ ,  $n_y$ , and  $n_\Phi$  are all even, the elements  $z_m g(n_x, n_y, 0)$  with  $m$  even and with  $m$  odd form two distinct conjugacy classes, each consisting of  $n_\Phi/2$  elements. Obviously, multiplication by a phase  $z_m$  is just a global gauge transformation and thus leaves the physical state invariant. Hence, the conjugacy classes correspond to gauge equivalence classes. The elements  $g(0, 0, m) = z_m$  commute with all other elements and thus form the center  $\mathbb{Z}(n_\Phi)$  of the group  $\mathcal{G}$ . Since the individual elements of the center form separate conjugacy classes, the center is a normal subgroup and can hence be factored out. The center itself represents global phase transformations of the wave function, and hence factoring it out corresponds to identifying gauge equivalence classes. Physically speaking, the quotient space  $\mathcal{G}/\mathbb{Z}(n_\Phi) = \mathbb{Z}(n_\Phi) \otimes \mathbb{Z}(n_\Phi)$  corresponds to discrete translations up to gauge transformations. It should be pointed out that  $\mathcal{G}$  is not simply given by the direct product  $\mathbb{Z}(n_\Phi) \otimes \mathbb{Z}(n_\Phi) \otimes \mathbb{Z}(n_\Phi)$ . In fact, the quotient space  $\mathbb{Z}(n_\Phi) \otimes \mathbb{Z}(n_\Phi)$  is not a subgroup of  $\mathcal{G}$ , and hence  $\mathcal{G}$  is also not the semi-direct product  $\mathbb{Z}(n_\Phi) \otimes \mathbb{Z}(n_\Phi) \times \mathbb{Z}(n_\Phi)$ . All we can say is that  $\mathcal{G}$  is a particular central extension of  $\mathbb{Z}(n_\Phi) \otimes \mathbb{Z}(n_\Phi)$  by the center subgroup  $\mathbb{Z}(n_\Phi)$ .

### 4.3.3 Spectrum and Degeneracy on the Torus

It should be pointed out that on the torus the Hamiltonian is identically the same as in the infinite volume. It now just acts on the restricted set of wave functions obeying the boundary condition Eq.(4.3.15). In particular, the finite volume wave functions are appropriate linear combinations of the infinitely many degenerate states of a given Landau level. As a result, the energy spectrum remains unchanged, but the degeneracy is substantially reduced. Let us use the fact that  $T_y$  commutes with the Hamiltonian to construct simultaneous eigenstates of both  $H$  and  $T_y$ . Since  $T_y^{n_\Phi} = \exp(i\theta_y)\mathbb{1}$ , the eigenvalues of  $T_y$  are given by  $\exp(i(2\pi l_y + \theta_y)/n_\Phi)$  with  $l_y \in \{0, 1, \dots, n_\Phi - 1\}$ , while the eigenvalues of  $H$  are still given by  $E_n = \omega(n + \frac{1}{2})$ . Hence, we can construct simultaneous eigenstates  $|nl_y\rangle$  such that

$$H|nl_y\rangle = \omega\left(n + \frac{1}{2}\right)|nl_y\rangle, \quad T_y|nl_y\rangle = \exp\left(\frac{2\pi i l_y + i\theta_y}{n_\Phi}\right)|nl_y\rangle. \quad (4.3.32)$$

The states  $|nl_y\rangle$  are the finite-volume analog of the states  $|np_y\rangle$  of Eq.(4.2.19) with  $p_y = (2\pi l_y + \theta_y)/L_y$ . In coordinate representation these states are given by the wave

functions

$$\begin{aligned} \langle \vec{x} | n l_y \rangle &= \sum_{n_x \in \mathbb{Z}} \psi_n \left( x + \left( n_\Phi n_x + l_y + \frac{\theta_y}{2\pi} \right) \frac{L_x}{n_\Phi} \right) \\ &\times \exp \left( \frac{2\pi i y}{L_y} \left( n_\Phi n_x + l_y + \frac{\theta_y}{2\pi} \right) - i \theta_x n_x \right). \end{aligned} \quad (4.3.33)$$

As a consequence of Eq.(4.3.23) one obtains

$$T_y T_x |n l_y\rangle = \exp \left( \frac{2\pi i}{n_\Phi} \right) T_x T_y |n l_y\rangle = \exp \left( \frac{2\pi i (l_y + 1) + i \theta_y}{n_\Phi} \right) T_x |n l_y\rangle, \quad (4.3.34)$$

from which we conclude that

$$T_x |n l_y\rangle = |n (l_y + 1)\rangle. \quad (4.3.35)$$

Since  $[T_x, H] = 0$ , the  $n_\Phi$  states  $|n l_y\rangle$  with  $l_y \in 0, 1, \dots, n_\Phi - 1$  thus form an irreducible representation of the magnetic translation group. Using  $n_\Phi = 4$  as a concrete example, a matrix representation of the two generators of  $\mathcal{G}$  is given by

$$T_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad T_y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \exp \left( \frac{i \theta_y}{4} \right). \quad (4.3.36)$$

Similarly, one can construct simultaneous eigenstates  $|n l_x\rangle$  of  $H$  and  $T_x$

$$H |n l_x\rangle = \omega \left( n + \frac{1}{2} \right) |n l_x\rangle, \quad T_x |n l_x\rangle = \exp \left( \frac{2\pi i l_x + i \theta_x}{n_\Phi} \right) |n l_x\rangle. \quad (4.3.37)$$

The states  $|n l_x\rangle$  are the finite-volume analog of the states  $|n p_x\rangle$  of Eq.(4.2.21) with  $p_x = (2\pi l_x + \theta_x)/L_x$ . In coordinate representation these states are given by the wave functions

$$\begin{aligned} \langle \vec{x} | n l_x \rangle &= \sum_{n_y \in \mathbb{Z}} \psi_n \left( y - \left( n_\Phi n_y + l_x + \frac{\theta_x}{2\pi} \right) \frac{L_y}{n_\Phi} \right) \\ &\times \exp \left( \frac{2\pi i x}{L_x} \left( n_\Phi n_y + l_x + \frac{\theta_x}{2\pi} - \frac{n_\Phi y}{L_y} \right) + i \theta_y n_y \right). \end{aligned} \quad (4.3.38)$$

It is worth noting that

$$T_y |n l_x\rangle = |n (l_x - 1)\rangle. \quad (4.3.39)$$

Just as in the infinite volume, it is straightforward to show that the two sets of eigenstates  $\langle \vec{x} | n l_y\rangle$  and  $\langle \vec{x} | n l_x\rangle$  span the same subspace of the Hilbert space.

### 4.3.4 Coherent States on the Torus

The coherent state should also satisfy the boundary condition in Eq.(4.3.13). The proposed form for the coherent state is

$$\begin{aligned} \tilde{\Psi}_{\lambda(t)\lambda'}(x, y) &= C \sum_{n_x=-\infty}^{\infty} \sum_{n_y=-\infty}^{\infty} \Psi_{\lambda(t)\lambda'}(x + n_x L_x, y + n_y L_y) \\ &\times \exp\left(\frac{2\pi i y}{L_y} n_\Phi n_x\right) \exp(-i\theta_x n_x) \exp(-i\theta_y n_y), \end{aligned} \quad (4.3.40)$$

where  $C$  is a normalization constant that will be obtained in the next subsection.

#### Normalization of the Wave Function

The normalization condition can be obtained from the following equation

$$\int_0^{L_x} \int_0^{L_y} dx dy \tilde{\Psi}_{\lambda(t)\lambda'}(x, y) \tilde{\Psi}_{\lambda(t)\lambda'}^*(x, y) = 1. \quad (4.3.41)$$

The coherent state for motion on the torus given by Eq.(4.3.40) can be written in terms of the coherent state in the infinite volume with the help of the shift operators  $(T_x)^{n_\Phi}$  and  $(T_y)^{n_\Phi}$ , and we have

$$|\widetilde{\lambda\lambda'}\rangle = C \sum_{n_x=-\infty}^{\infty} \sum_{n_y=-\infty}^{\infty} e^{-i\theta_x n_x} e^{-i\theta_y n_y} \tilde{T}_y^{n_y} \tilde{T}_x^{n_x} |\lambda\lambda'\rangle, \quad (4.3.42)$$

where  $\tilde{T}_x = (T_x)^{n_\Phi}$ , and  $\tilde{T}_y = (T_y)^{n_\Phi}$ .

In order to find the normalization constant, we write

$$\begin{aligned} \langle \widetilde{\lambda'\lambda} | \widetilde{\lambda\lambda'} \rangle &= |C|^2 \sum_{n_x=-\infty}^{\infty} \sum_{n_y=-\infty}^{\infty} \sum_{n'_x=-\infty}^{\infty} \sum_{n'_y=-\infty}^{\infty} \langle \lambda'\lambda | (\tilde{T}_x^\dagger)^{n'_x} (\tilde{T}_y^\dagger)^{n'_y} \tilde{T}_y^{n_y} \tilde{T}_x^{n_x} | \lambda\lambda' \rangle \\ &\times e^{-i\theta_x(n_x - n'_x)} e^{-i\theta_y(n_y - n'_y)}, \end{aligned} \quad (4.3.43)$$

where

$$\langle \widetilde{\lambda'\lambda} | \widetilde{\lambda\lambda'} \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \tilde{\Psi}_{\lambda(t)\lambda'}(x, y) \tilde{\Psi}_{\lambda(t)\lambda'}^*(x, y). \quad (4.3.44)$$

In the above equation, the intervals of the double integral in Eq.(4.3.41) were extended to  $(-\infty, \infty)$ . This will prove to be useful later in calculating the normalization constant  $C$  from Eq.(4.3.43). This term diverge, but we should not worry

about that because it can be cancelled from both sides of Eq.(4.3.43) as we will see later. Now using Eq.(4.3.41), Eq.(4.3.44) can be written as

$$\begin{aligned}
\langle \widetilde{\lambda}' \lambda | \widetilde{\lambda} \lambda' \rangle &= \sum_{n_x=-\infty}^{\infty} \sum_{n_y=-\infty}^{\infty} \int_0^{L_x} \int_0^{L_y} dx dy \\
&\times \widetilde{\Psi}_{\lambda(t)\lambda'}(x + n_x L_x, y + n_y L_y) \widetilde{\Psi}_{\lambda(t)\lambda'}^*(x + n_x L_x, y + n_y L_y) \\
&= \sum_{n_x=-\infty}^{\infty} \sum_{n_y=-\infty}^{\infty} 1.
\end{aligned} \tag{4.3.45}$$

Using the definition of the shift operators in Eq.(4.3.22), and Eq.(4.3.21) it is easy to show

$$\widetilde{T}_x^\dagger = \widetilde{T}_x^{-1}, \quad \widetilde{T}_y^\dagger = \widetilde{T}_y^{-1}. \tag{4.3.46}$$

Substituting for  $\widetilde{T}_x^{-1}$  and  $\widetilde{T}_y^{-1}$  from Eq.(4.3.46) into Eq.(4.3.43) then, the result is an expression inside the summations that depends only on  $n_x - n'_x$  and  $n_y - n'_y$ . This leads to

$$\begin{aligned}
\langle \widetilde{\lambda}' \lambda | \widetilde{\lambda} \lambda' \rangle &= |C|^2 \sum_{n_x - n'_x = m_x} \sum_{n_y - n'_y = m_y} \sum_{m_x = -\infty}^{\infty} \sum_{m_y = -\infty}^{\infty} \langle \lambda' \lambda | \widetilde{T}_y^{m_y} \widetilde{T}_x^{m_x} | \lambda \lambda' \rangle \\
&\times e^{-i\theta_x m_x} e^{-i\theta_y m_y},
\end{aligned} \tag{4.3.47}$$

From the above equation and Eq.(4.3.45) we get

$$|C|^{-2} = \sum_{m_x = -\infty}^{\infty} \sum_{m_y = -\infty}^{\infty} \langle \lambda' \lambda | \widetilde{T}_y^{m_y} \widetilde{T}_x^{m_x} | \lambda \lambda' \rangle e^{-i\theta_x m_x} e^{-i\theta_y m_y}, \tag{4.3.48}$$

In the following we will obtain the value of  $C$ . From the definition of the shift operators in Eq.(4.3.22) and Eq.(4.3.21), and using Eqs.(4.2.41), we can write

$$\begin{aligned}
\widetilde{T}_x &= \exp(2\pi i n_\Phi \frac{R_y}{L_y}) = \exp(c_x (b^\dagger - b)), \\
\widetilde{T}_y &= \exp(-2\pi i n_\Phi \frac{R_x}{L_y}) = \exp(-i c_y (b^\dagger + b)),
\end{aligned} \tag{4.3.49}$$

where

$$c_x = \sqrt{\frac{2}{M\omega}} \frac{\pi}{L_y} n_\Phi, \quad c_y = \sqrt{\frac{2}{M\omega}} \frac{\pi}{L_x} n_\Phi. \tag{4.3.50}$$

Substituting Eqs.(4.3.49) in Eq.(4.3.48) gives

$$|C|^{-2} = \sum_{m_x = -\infty}^{\infty} \sum_{m_y = -\infty}^{\infty} \langle \lambda' \lambda | e^{-m_x c_x (b^\dagger - b)} e^{-i m_y c_y (b^\dagger + b)} | \lambda \lambda' \rangle e^{-i\theta_x m_x} e^{-i\theta_y m_y}. \tag{4.3.51}$$

A well-known theorem [34] can be applied to find the value of  $C$ . If  $A$  and  $B$  are two operators such that  $[A, B]$  commutes with both  $A$  and  $B$  then

$$e^{A+B} = e^{-\frac{1}{2}[A,B]}e^Ae^B, \quad e^Ae^B = e^{[A,B]}e^Be^A. \quad (4.3.52)$$

Using the identities of Eqs.(4.3.52) one can shift all the exponentials with  $b^\dagger$  to the left, and exponentials with  $b$  to the right. After some calculations, and using the fact that the state  $|\lambda\lambda'\rangle$  is an eigenstate of the operator  $b$  with eigenvalue  $\lambda'$ , Eq.(4.3.51) can be written as

$$\begin{aligned} |C|^{-2} &= \langle \lambda'\lambda | \lambda\lambda' \rangle \sum_{m_x, m_y} \exp\left(-\frac{1}{2}c_x^2 m_x^2 + i(-2c_x \text{Im}\lambda' - \theta_x)m_x\right) \\ &\times \exp\left(-\frac{1}{2}c_y^2 m_y^2 - i(2c_y \text{Re}\lambda' + \theta_y)m_y\right) \exp(ic_x c_y m_x m_y). \end{aligned} \quad (4.3.53)$$

The summation over  $m_x$  converges to a special function called the  $\vartheta_3$  function. The elliptic function plays an important role in the following discussions, therefore it is useful to me define these functions and discuss some of their properties. It is defined by the following equation

$$\vartheta_2(z, q) = \sqrt[4]{q} \sum_{m=-\infty}^{m=\infty} q^{m(m+1)} e^{(2m+1)iz}, \quad (4.3.54)$$

$$\vartheta_3(z, q) = \sum_{m=-\infty}^{m=\infty} q^{m^2} e^{2miz}, \quad (4.3.55)$$

$$\begin{aligned} \vartheta_4(z, q) &= \sum_{m=-\infty}^{m=\infty} (-1)^m q^{m^2} e^{2miz}, \\ q &= i\pi\tau, \quad |q| < 1. \end{aligned} \quad (4.3.56)$$

Accordingly, Eq.(4.3.55) can be written as

$$\begin{aligned} |C|^{-2} &= \sum_{m_y} \vartheta_3\left(\frac{1}{2}(c_x c_y m_y - 2c_x \text{Im}\lambda' - \theta_x), e^{-\frac{1}{2}c_x^2}\right) \\ &\times \exp\left(-\frac{1}{2}c_y^2 m_y^2 - 2ic_y m_y \text{Re}\lambda' - i\theta_y m_y\right). \end{aligned} \quad (4.3.57)$$

The following properties of the  $\vartheta_3$  function are useful to find the value of the above summation

$$\begin{aligned} \vartheta_3(z + n\pi, q) &= \vartheta_3(z, q), \\ \vartheta_3\left(z + (2n+1)\frac{\pi}{2}, q\right) &= \begin{cases} \vartheta_3(z, q), & \text{for even } k, \\ \vartheta_4(z, q), & \text{for odd } k, \end{cases} \\ q &= i\pi\tau, \quad |q| < 1, \quad n \in \mathbb{Z}. \end{aligned} \quad (4.3.58)$$

It is easy to see from Eq.(4.3.50) that

$$c_x c_y = \pi n_\Phi. \quad (4.3.59)$$

From the above equation and Eqs.(4.3.57), for even  $n_\Phi$  one obtains

$$\begin{aligned} |C|^{-2} &= \vartheta_3 \left( -c_x \text{Im} \lambda' - \frac{1}{2} \theta_x, e^{-\frac{1}{2} c_x^2} \right) \\ &\times \left( \sum_{m_y} \exp \left( -2c_y^2 m_y^2 - 4ic_y m_y \text{Re} \lambda' - 2i\theta_y m_y \right) \right. \\ &\left. + \sum_{m_y} \exp \left( -\frac{1}{2} c_y^2 (2m_y + 1)^2 - 2ic_y (2m_y + 1) \text{Re} \lambda' - i\theta_y (2m_y + 1) \right) \right). \end{aligned} \quad (4.3.60)$$

From the above equation and Eqs.(4.3.58), for odd  $n_\Phi$  one finds

$$\begin{aligned} |C|^{-2} &= \vartheta_3 \left( -c_x \text{Im} \lambda' - \frac{1}{2} \theta_x, e^{-\frac{1}{2} c_x^2} \right) \sum_{m_y} \exp \left( -2c_y^2 m_y^2 - 4ic_y m_y \text{Re} \lambda' - 2i\theta_y m_y \right) \\ &+ \vartheta_4 \left( -c_x \text{Im} \lambda' - \frac{1}{2} \theta_x, e^{-\frac{1}{2} c_x^2} \right) \\ &\times \sum_{m_y} \exp \left( -\frac{1}{2} c_y^2 (2m_y + 1)^2 - 2ic_y (2m_y + 1) \text{Re} \lambda' - i\theta_y (2m_y + 1) \right). \end{aligned} \quad (4.3.61)$$

Using the definition of  $\vartheta_3$  in Eq.(4.3.55) we can write Eq.(4.3.60) for even  $n_\Phi$

$$\begin{aligned} |C|^{-2} &= \vartheta_3 \left( -c_x \text{Im} \lambda' - \frac{1}{2} \theta_x, e^{-\frac{1}{2} c_x^2} \right) \left( \vartheta_3 \left( -2c_y \text{Re} \lambda' - \theta_y, e^{-2c_y^2} \right) \right. \\ &\left. + \exp \left( -2ic_y \text{Re} \lambda' - i\theta_y - \frac{1}{2} c_y^2 \right) \vartheta_3 \left( ic_y^2 - 2c_y \text{Re} \lambda' - \theta_y, e^{-2c_y^2} \right) \right). \end{aligned} \quad (4.3.62)$$

Using the following identity [31]

$$\begin{aligned} \vartheta_3 \left( z + (2n + 1) i \frac{\ln(q)}{2}, q \right) &= q^{-n^2 - n - \frac{1}{4}} e^{(2n+1)iz} \vartheta_2(z, q), \\ q &= i\pi\tau, \quad |q| < 1, \end{aligned} \quad (4.3.63)$$

for  $n = -1$ , Eq.(4.3.62) can be written as

$$|C|^{-2} = \vartheta_3 \left( -c_x \text{Im}\lambda' - \frac{1}{2}\theta_x, e^{-\frac{1}{2}c_x^2} \right) \left( \vartheta_3 \left( -2c_y \text{Re}\lambda' - \theta_y, e^{-2c_y^2} \right) + \exp(-c_y^2) \vartheta_2 \left( -2c_y \text{Re}\lambda' - \theta_y, e^{-2c_y^2} \right) \right). \quad (4.3.64)$$

Finally, for even  $n_\Phi$  the normalization constant can be written as

$$C = e^{i\alpha} \left( \vartheta_3 \left( -c_x \text{Im}\lambda' - \frac{1}{2}\theta_x, e^{-\frac{1}{2}c_x^2} \right) \left( \vartheta_3 \left( -2c_y \text{Re}\lambda' - \theta_y, e^{-2c_y^2} \right) + \exp(-c_y^2) \vartheta_2 \left( -2c_y \text{Re}\lambda' - \theta_y, e^{-2c_y^2} \right) \right) \right)^{-\frac{1}{2}}, \quad (4.3.65)$$

where  $\alpha$  is a constant arbitrary phase. For odd  $n_\Phi$  the result is

$$C = e^{i\alpha} \left( \vartheta_3 \left( -c_x \text{Im}\lambda' - \frac{1}{2}\theta_x, e^{-\frac{1}{2}c_x^2} \right) \vartheta_3 \left( -2c_y \text{Re}\lambda' - \theta_y, e^{-2c_y^2} \right) + \exp(-c_y^2) \vartheta_4 \left( -c_x \text{Im}\lambda' - \frac{1}{2}\theta_x, e^{-\frac{1}{2}c_x^2} \right) \vartheta_2 \left( -2c_y \text{Re}\lambda' - \theta_y, e^{-2c_y^2} \right) \right)^{-\frac{1}{2}}. \quad (4.3.66)$$

As it is obvious from Eq.(4.3.65) and Eq.(4.3.66), the first argument of  $\vartheta_i, i = 2, 3, 4$  either depend on  $c_x \text{Im}\lambda'$  shifted by  $\theta_x$ , or on  $2c_y \text{Re}\lambda'$  shifted by  $\theta_y$ . The theta functions are periodic functions with  $\vartheta_i(z + n\pi, q) = \vartheta_i(z, q), i = 3, 4$ , and  $\vartheta_2(z + n\pi, q) = (-1)^n \vartheta_2(z, q)$ , with  $n \in \mathbb{Z}$ . This means that the normalization constant is a  $\pi$ -periodic function. That is because when we shift  $c_x \text{Im}\lambda'$  or  $c_y \text{Re}\lambda'$  or both by  $\pi$ , then  $C$  maintains the same value.

### 4.3.5 Calculating the Expectation Value of $T_x^l$ and $T_y^l$

The expectation value for the shift operator  $T_x$  raised to the power  $l$  for a coherent state can be evaluated from the following equation

$$\begin{aligned} \langle \widetilde{\lambda}' \lambda | T_k^l | \widetilde{\lambda} \lambda' \rangle &= \sum_{n_x=-\infty}^{\infty} \sum_{n_y=-\infty}^{\infty} \sum_{n'_x=-\infty}^{\infty} \sum_{n'_y=-\infty}^{\infty} e^{-i\theta_x(n_x - n'_x)} e^{-i\theta_y(n_y - n'_y)} \\ &\times \langle \lambda' \lambda | (\widetilde{T}_x^\dagger)^{n'_x} (\widetilde{T}_y^\dagger)^{n'_y} T_k^l \widetilde{T}_y^{n_y} \widetilde{T}_x^{n_x} | \lambda \lambda' \rangle, \end{aligned} \quad (4.3.67)$$



where  $k = x, y$ . The shift operators raised to the power  $l$  can be written as

$$\begin{aligned} T_x^l &= \exp(2in_\Phi\pi\frac{R_y}{L_y}) = \exp(d_x(b^\dagger - b)), \\ T_y^l &= \exp(-2in_\Phi\pi\frac{R_x}{L_x}) = \exp(-id_y(b^\dagger + b)), \end{aligned} \quad (4.3.68)$$

where,

$$d_x = \sqrt{\frac{2}{M\omega}} \frac{\pi}{L_y} l, \quad d_y = \sqrt{\frac{2}{M\omega}} \frac{\pi}{L_x} l \quad (4.3.69)$$

We apply the same technique that was used for calculating the normalization constant by shifting all the exponentials with  $b^\dagger$  to the left, and exponentials with  $b$  to the right. Accordingly, for odd  $l$  the operator  $T_x^l$  takes the following form

$$\begin{aligned} \langle \widetilde{\lambda'}\lambda | T_x^l | \widetilde{\lambda}\lambda' \rangle &= |C|^2 \langle \lambda'\lambda | \lambda\lambda' \rangle e^{-\frac{1}{2}d_x^2} \exp(-2id_x \text{Im}\lambda') \sum_{m_x, m_y} (-1)^{m_y} \\ &\times \exp(-d_x c_x m_x - \frac{1}{2}c_x^2 m_x^2 + i(c_x c_y m_y - 2c_x \text{Im}\lambda' - \theta_x) m_x) \\ &\times \exp(-\frac{1}{2}c_y^2 m_y^2 - i(2c_y \text{Re}\lambda' + \theta_y) m_y), \end{aligned} \quad (4.3.70)$$

while for even  $l$

$$\begin{aligned} \langle \widetilde{\lambda'}\lambda | T_x^l | \widetilde{\lambda}\lambda' \rangle &= |C|^2 \langle \lambda'\lambda | \lambda\lambda' \rangle e^{-\frac{1}{2}d_x^2} \exp(-2id_x \text{Im}\lambda') \\ &\times \sum_{m_x, m_y} \exp(-d_x c_x m_x - \frac{1}{2}c_x^2 m_x^2 + i(c_x c_y m_y - 2c_x \text{Im}\lambda' - \theta_x) m_x) \\ &\times \exp(-\frac{1}{2}c_y^2 m_y^2 - i(2c_y \text{Re}\lambda' + \theta_y) m_y). \end{aligned} \quad (4.3.71)$$

We can write Eq.(4.3.71) and Eq.(4.3.70) in terms of the  $\vartheta$ -functions. For even  $n_\Phi$  we get

$$\begin{aligned} \langle \widetilde{\lambda'}\lambda | T_x^l | \widetilde{\lambda}\lambda' \rangle &= |C|^2 e^{-\frac{1}{2}d_x^2} \exp(-2id_x \text{Im}\lambda') \\ &\times \left( \vartheta_3 \left( i\frac{d_x c_x}{2} - c_x \text{Im}\lambda' - \frac{1}{2}\theta_x, e^{-\frac{1}{2}c_x^2} \right) \left( \vartheta_3 \left( -2c_y \text{Re}\lambda' - \theta_y, e^{-2c_y^2} \right) \right. \right. \\ &\left. \left. + (-1)^l \exp(-c_y^2) \vartheta_2 \left( -2c_y \text{Re}\lambda' - \theta_y, e^{-2c_y^2} \right) \right) \right), \end{aligned} \quad (4.3.72)$$

and for odd  $n_\Phi$  we get

$$\begin{aligned}
\langle \widetilde{\lambda'}\lambda | T_x^l | \widetilde{\lambda}\lambda' \rangle &= |C|^2 e^{-\frac{1}{2}d_x^2} \exp(-2id_x \text{Im}\lambda') \left( \vartheta_3 \left( i\frac{d_x c_x}{2} - c_x \text{Im}\lambda' - \frac{1}{2}\theta_x, e^{-\frac{1}{2}c_x^2} \right) \right. \\
&\times \vartheta_3 \left( -2c_y \text{Re}\lambda' - \theta_y, e^{-2c_y^2} \right) \\
&+ (-1)^l \exp(-c_y^2) \vartheta_2 \left( -2c_y \text{Re}\lambda' - \theta_y, e^{-2c_y^2} \right) \\
&\left. \times \vartheta_4 \left( i\frac{d_x c_x}{2} - c_x \text{Im}\lambda' - \frac{1}{2}\theta_x, e^{-\frac{1}{2}c_x^2} \right) \right). \tag{4.3.73}
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
\langle \widetilde{\lambda'}\lambda | T_y^l | \widetilde{\lambda}\lambda' \rangle &= |C|^2 \langle \lambda'\lambda | \lambda\lambda' \rangle e^{-\frac{1}{2}d_y^2} \exp(-2id_y \text{Re}\lambda') \sum_{m_x, m_y} (-1)^{m_x} \\
&\times \exp(-d_y c_y m_y - \frac{1}{2}c_x^2 m_x^2 + i(c_x c_y m_y - 2c_x \text{Im}\lambda' - \theta_x) m_x) \\
&\times \exp(-\frac{1}{2}c_y^2 m_y^2 - i(2c_y \text{Re}\lambda' + \theta_y) m_y), \tag{4.3.74}
\end{aligned}$$

and for even  $l$

$$\begin{aligned}
\langle \widetilde{\lambda'}\lambda | T_y^l | \widetilde{\lambda}\lambda' \rangle &= |C|^2 \langle \lambda'\lambda | \lambda\lambda' \rangle e^{-\frac{1}{2}d_y^2} \exp(-2id_y \text{Re}\lambda') \\
&\times \sum_{m_x, m_y} \exp(-d_y c_y m_y - \frac{1}{2}c_x^2 m_x^2 + i(c_x c_y m_y - 2c_x \text{Im}\lambda' - \theta_x) m_x) \\
&\times \exp(-\frac{1}{2}c_y^2 m_y^2 - i(2c_y \text{Re}\lambda' + \theta_y) m_y). \tag{4.3.75}
\end{aligned}$$

We can write Eq.(4.3.75) in terms of the  $\vartheta$ -functions. For even  $n_\Phi$  we get

$$\begin{aligned}
\langle \widetilde{\lambda'}\lambda | T_y^l | \widetilde{\lambda}\lambda' \rangle &= |C|^2 e^{-\frac{1}{2}d_y^2} \exp(-2id_y \text{Re}\lambda') \\
&\times \vartheta_3 \left( -c_x \text{Im}\lambda' - \frac{1}{2}\theta_x, e^{-\frac{1}{2}c_x^2} \right) \left( \vartheta_3 \left( id_y c_y - 2c_y \text{Re}\lambda' - \theta_y, e^{-2c_y^2} \right) \right. \\
&+ \exp(-c_y^2) \vartheta_2 \left( id_y c_y - 2c_y \text{Re}\lambda' - \theta_y, e^{-2c_y^2} \right) \left. \right), \tag{4.3.76}
\end{aligned}$$

and for odd  $n_\Phi$  we get

$$\begin{aligned}
\langle \widetilde{\lambda}' \lambda | T_y^l | \widetilde{\lambda} \lambda' \rangle &= |C|^2 e^{-\frac{1}{2}d_y^2} \exp(-2id_y \text{Re} \lambda') \left( \vartheta_3 \left( -c_x \text{Im} \lambda' - \frac{1}{2} \theta_x, e^{-\frac{1}{2}c_x^2} \right) \right. \\
&\times \vartheta_3 \left( id_y c_y - 2c_y \text{Re} \lambda' - \theta_y, e^{-2c_y^2} \right) \\
&+ \exp(-c_y^2) \vartheta_4 \left( -c_x \text{Im} \lambda' - \frac{1}{2} \theta_x, e^{-\frac{1}{2}c_x^2} \right) \\
&\left. \times \vartheta_2 \left( id_y c_y - 2c_y \text{Re} \lambda' - \theta_y, e^{-2c_y^2} \right) \right). \tag{4.3.77}
\end{aligned}$$

For odd  $l$  and even  $n_\Phi$  Eq.(4.3.70) gives

$$\begin{aligned}
\langle \widetilde{\lambda}' \lambda | T_y^l | \widetilde{\lambda} \lambda' \rangle &= |C|^2 e^{-\frac{1}{2}d_y^2} \exp(-2id_y \text{Re} \lambda') \\
&\times \vartheta_4 \left( -c_x \text{Im} \lambda' - \frac{1}{2} \theta_x, e^{-\frac{1}{2}c_x^2} \right) \left( \vartheta_3 \left( id_y c_y - 2c_y \text{Re} \lambda' - \theta_y, e^{-2c_y^2} \right) \right. \\
&\left. - \exp(-c_y^2) \vartheta_2 \left( id_y c_y - 2c_y \text{Re} \lambda' - \theta_y, e^{-2c_y^2} \right) \right), \tag{4.3.78}
\end{aligned}$$

while for odd  $n_\Phi$ , we get

$$\begin{aligned}
\langle \widetilde{\lambda}' \lambda | T_y^l | \widetilde{\lambda} \lambda' \rangle &= |C|^2 e^{-\frac{1}{2}d_y^2} \exp(-2id_y \text{Re} \lambda') \\
&\times \left( \vartheta_4 \left( -c_x \text{Im} \lambda' - \frac{1}{2} \theta_x, e^{-\frac{1}{2}c_x^2} \right) \vartheta_3 \left( id_y c_y - 2c_y \text{Re} \lambda' - \theta_y, e^{-2c_y^2} \right) \right. \\
&- \exp(-c_y^2) \vartheta_3 \left( -c_x \text{Im} \lambda' - \frac{1}{2} \theta_x, e^{-\frac{1}{2}c_x^2} \right) \\
&\left. \times \vartheta_2 \left( id_y c_y - 2c_y \text{Re} \lambda' - \theta_y, e^{-2c_y^2} \right) \right). \tag{4.3.79}
\end{aligned}$$

Here we must keep in mind that the value of  $C$  depends on whether  $n_\Phi$  is odd or even. This is obvious from Eq.(4.3.65) and Eq.(4.3.66).



# Chapter 5

## Conclusions

We have considered the physics of a particle confined to the surface of a cone with deficit angle  $\delta$  and bound to its tip by a  $1/r$  or an  $r^2$  potential. In both cases, for rational  $s = 1 - \delta/2\pi$ , all bound classical orbits are closed and there are accidental degeneracies in the discrete energy spectrum of the quantum system. There is an accidental  $SU(2)$  symmetry generated by the Runge-Lenz vector and by the angular momentum. However, the Runge-Lenz vector is not necessarily a physical operator. For example, by acting with the Runge-Lenz vector on a physical state one may generate an unphysical wave function outside the domain of the Hamiltonian. As a result, the representations of the accidental  $SU(2)$  symmetry are larger than the multiplets of degenerate physical states. In particular, some physical states are contained in multiplets with an unusual value of the Casimir spin  $S$  which is neither an integer nor a half-integer. Still, the fractional value of the spin yields the correct value of the quantized energy.

In the study of the problem of a particle moving on a cone, a procedure was developed to reduce the problem of finding the conserved components of the Runge-Lenz vector to a problem of solving a system of differential equations (see section (3.4) and subsection (3.5.5)). The ansatz was made with the help of the classical expression of the Runge-Lenz vector. This procedure will be an important tool to solve more complicated problems like the cone problem in higher dimensions.

The particle on a cone provides us with an interesting physical system in which symmetries manifest themselves in a very unusual manner. Although the Hamiltonian commutes with the generators of an  $SU(2)$  symmetry, the multiplets of degenerate states do not always correspond to integer or half-integer Casimir spin. This is because the application of the symmetry generators may lead us out of the domain of the Hamiltonian. Only the states with square-integrable single-valued  $2\pi$ -periodic wave functions belong to the physical spectrum, and all other members of the corre-

sponding “ $SU(2)$ ” representation must be discarded. Mathematically speaking, the symmetry generators — although Hermitean in their respective domain — do not act as Hermitean operators in the domain of the Hamiltonian.

It contrast to many other quantum mechanics problems, in order to understand motion on a cone it was necessary to address mathematical issues such as the domains of operators as well as Hermiticity versus self-adjointness. Still, we have not elaborated on some questions related to different possible self-adjoint extensions of the Hamiltonian. For the particle on the cone, such issues seem worth investigating. In this work, we have limited ourselves to the standard Friedrichs extension of the Hamiltonian. Alternative self-adjoint extensions correspond to an additional  $\delta$ -function potential located at the tip of the cone. This will modify the problem in an interesting way. In particular, we expect that, in the presence of an additional  $\delta$ -function potential, the accidental degeneracy will be partly lifted. However, since the  $\delta$ -function only affects states with  $m = 0$ , some accidental degeneracy will remain. The particle on the cone provides us with another example for the deep connection between the closedness of all bound classical orbits and accidental degeneracies in the discrete spectrum of the Hamiltonian. Even if the classical system has various quantum analogues (because there are different possible self-adjoint extensions) some accidental degeneracy still persists. It is also remarkable that, like in other cases with accidental symmetries, for the particle on the cone semi-classical Bohr-Sommerfeld quantization provides the exact quantum energy spectrum.

We are unaware of another system for which a similarly unusual symmetry behavior has been observed. It is interesting to ask if symmetry can manifest itself in this unusual manner also in other quantum systems. For example, cones of graphene may provide a motivation to study accidental degeneracies of the Dirac equation on a cone. Also higher-dimensional spaces with conical singularities may be worth investigating. In any case, motion on a cone provides an illuminating example for a rather unusual manifestation of symmetry in quantum mechanics.

We have also re-investigated an old and rather well-studied problem in quantum mechanics — a charged particle in a constant magnetic field — from an unconventional accidental symmetry perspective. While many aspects of this problem are well-known, and some results of this thesis can be found in various places in the literature, we believe that we have painted a picture of cyclotron motion that reveals new aspects of this fascinating system, which behaves in a unique and sometimes counter-intuitive manner. The fact that all classical cyclotron orbits are closed circles identifies the center of the circle as a conserved quantity analogous to the Runge-Lenz vector of the Kepler problem. Remarkably, the corresponding “accidental” symmetry is just translation invariance (up to gauge transformations). In particular, the coordinates  $(R_x, R_y) = (-P_y, P_x)/eB$  of the center of the cyclotron circle simultaneously generate infinitesimal translations  $-P_y$  and  $P_x$  (up to gauge

transformations) in the  $y$ - and  $x$ -directions, respectively. As is well-known, in a constant magnetic field, translations in the  $x$ - and  $y$ -directions do not commute, i.e.  $[P_x, P_y] = ieB$ , and thus the two coordinates  $R_x$  and  $R_y$  of the center of the cyclotron circle are also not simultaneously measurable at the quantum level. In contrast, the radius of the cyclotron circle has a sharp value in an energy eigenstate. The accidental symmetry leads to the infinite degeneracy of the Landau levels.

In order to further investigate the nature of the accidental symmetry, we have put the system in a finite rectangular periodic volume. Obviously, this breaks rotation invariance, but leaves translation invariance (and thus the accidental symmetry) intact — at least at the classical level. Interestingly, at the quantum level continuous translation invariance is explicitly broken down to a discrete magnetic translation group, due to the existence of two angles  $\theta_x$  and  $\theta_y$  which parametrize a family of self-adjoint extensions of the Hamiltonian on the torus. In a field theoretical context, in which the gauge field is dynamical (and not just treated as a classical background field), the parameters  $\theta_x$  and  $\theta_y$  characterize super-selection sectors. In this sense, they are analogous to the vacuum angle  $\theta$  of non-Abelian gauge theories. Just like the  $\theta$ -vacuum angle explicitly breaks CP invariance at the quantum level but is classically invisible, the angles  $\theta_x$  and  $\theta_y$  lead to quantum mechanical explicit breaking of continuous translation invariance down to the discrete magnetic translation group. The magnetic translation group  $\mathcal{G}$  itself, which plays the role of the accidental symmetry in the periodic volume, is a particular central extension of  $\mathbb{Z}(n_\Phi) \otimes \mathbb{Z}(n_\Phi)$  by the center subgroup  $\mathbb{Z}(n_\Phi)$ , where  $n_\Phi$  is the number of magnetic flux quanta trapped in the torus. We find it remarkable that the simple fact that all classical cyclotron orbits are closed circles has such intricate effects at the quantum level.

One of the most important results of studying the cyclotron motion on a torus concerns the angles  $\theta_x$  and  $\theta_y$  which are parameters of self-adjoint extensions. Here we must stress that not taking these parameters into account leads to a conceptual difficulty. To understand this, let us ignore these parameters, and let us consider the case when the flux quantum number  $n_\Phi = 1$ . Then there is no degeneracy. The wave function in Eq.(4.3.33) then is a coherent state. Since there is no preferred point on the torus, we expect that the probability density is constant. However, this is not the case. Instead, the probability density is coordinate-dependent and has a maximum at the origin. This means that we would have a preferred point at the origin. The introduction of  $\theta_x$  and  $\theta_y$  solves this problem. For  $n_\Phi = 1$  the probability density is centered at the point  $(\theta_x L_x/2\pi, \theta_y L_y/2\pi)$ . This is what one expects since these parameters break the translation symmetry of the system.

The treatment that has been used in the cyclotron problem can be generalized to include the effect of spin. Then we can repeat the whole treatment using the Pauli equation or the Dirac equation. This will be a possible subject for future work.

The observation that there is a deep relation between accidental symmetry and bound classical orbits being closed has been supported by the construction of the conserved quantities for systems with closed bound orbits. However, there is no rigorous proof of this relation based on fundamental laws of classical mechanics. The work in this thesis gives a motivation to find such a proof, and derive a general procedure to find conserved quantities generated by accidental symmetry, or for any other classical system.

It is worth mentioning that it is possible that a system with continuous fields may have an accidental symmetry. In this case a closed orbit can not necessarily be defined although the system possesses an accidental symmetry. For example, Fröhlich and Studer proved [36] expanded the Dirac equation for a particle in an electromagnetic field in powers of the inverse mass  $1/M$ . If we assume that  $M$  is sufficiently large, then we can limit ourselves to terms up to the power  $1/M^3$ . The result is an equation with an accidental  $SU(2)$  gauge symmetry in addition to the usual  $U(1)$  gauge symmetry of electromagnetism. The derivation of this result is not based on a standard theorem applied to a Lagrangian of classical fields, but based on observation. This indicates the potential of a very important problem in classical mechanics of finding a general theorem that is applicable to continuous as well as discrete systems, illuminating the relation between closed bound orbits and accidental symmetry. In addition to that, such a theorem may provide a valuable tool to extract conserved quantities from a complicated Lagrangian when mere intuition fails to do so.

We find it remarkable that a well-studied subject such as quantum mechanics still confronts us with interesting puzzles. In particular, accidental symmetries hold the promise to further deepen our understanding of the essence of quantum physics.



# Acknowledgements

I like to thank U.-J. Wiese for supervising this thesis. I am indebted to him for his valuable explanation of important facts regarding the foundations of quantum physics and for his endless patience in addressing concerns and answering questions related to this thesis and beyond. My discussions with him improved my understanding of quantum physics significantly.

Further thanks go to Prof. Shailesh Chandrasekharan for several interesting discussions and, in particular, for his willingness to act as a co-referee of this thesis.

I am also indebted to F. Niedermayer for illuminating discussions.

Deep thanks go to Ottilia Haenni for her very valuable support in finding accommodation, dealing with administrative issues, and for her warm and kind treatment.



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