Investigation of $\mathbb{C}\mathrm{P}(N-1)$ models in the large N limit

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Faculty of Science, University of Bern

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Marco Brühlmann

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Supervisor:

Prof. Dr. Uwe-Jens Wiese

Abstract

We explore the two-dimensional $\mathbb{C}\mathrm{P}(N-1)$ model in the limit $N \to \infty$. In order to do this, the D-theory formalism is employed. D-theory allows us to formulate the two-dimensional $\mathbb{C}\mathrm{P}(N-1)$ model at zero vacuum angle through a ferromagnetic $\mathrm{SU}(N)$ quantum Heisenberg model on a $(2+1)$-dimensional lattice with a Euclidean extra dimension. We confirm, that this formalism works in the large N limit. We work towards an analytic description of the $\mathbb{C}\mathrm{P}(N-1)$ model at large N by using clusters of spins in the $\mathrm{SU}(N)$ lattice model. On a lattice with two sites a phase transition that originates form an infinite number of spin-flavors is discovered.

Contents

1. Introduction

The $(1+1)$-dimensional $\mathbb{C}\mathrm{P}(N-1)$ model, like other two-dimensional non-linear σ models, displays a series of interesting similarities to the Yang-Mills theories in $(3+1)$ dimensions. This makes them very attractive toy models for quantum chromodynamics (QCD). As part of the standard model, QCD, an $\mathrm{SU}(3)$ non-abelian gauge theory, describes the strong interaction between quarks mediated by gluons. While we are unable to solve QCD analytically, there are a number of methods that are used to produce accurate predictions. Due to the asymptotic freedom of QCD we can apply perturbation theory only at high energies. At low energies the theory becomes non-perturbative. In this low-energy regime, chiral perturbation theory (ChPT) can be used to describe QCD using hadrons as the degrees of freedom. Non-perturbative effects are often investigated using lattice QCD. Another approach is the large N expansion, also known as the $1/N$-expansion, in which properties of the theory are expressed perturbatively in powers of the small parameter $1/N$. Despite the number of colors in QCD being $N_C = 3$, the method has been successfully used to produce new insights, such as the Witten-Veneziano mechanism for explaining why the η'-meson is not a massless Goldstone boson. Unfortunately, even at an infinite number of colors currently an analytic solution of QCD is not known.

For theories as complex as QCD it is common practice to test new methods on toy models. Such toy models need to be more simple than the original and share key characteristics that are relevant to the method or problem at hand. As mentioned above, two-dimensional non-linear σ models are a popular choice of toy models for QCD. A comprehensive definition of a general non-linear σ model is given in a paper by D'Adda, Lüscher and Di Vecchia [5]. What makes $\mathbb{C}\mathrm{P}(N-1)$ models stand out is that they seem to be the simplest series of topologically non-trivial, non-linear σ models. Indeed, while in the $\mathbb{C}\mathrm{P}(N-1)$ models topologically non-trivial configuration have been found for all values of N, the $\mathrm{O}(N)$ σ models are topologically trivial for $N \geq 4$ [5]. Topological non-triviality gives rise to a topological charge term Q and a vacuum angle θ. This characteristic is essential to making $\mathbb{C}\mathrm{P}(N-1)$ models such powerful toy models for QCD. There are a number of problems directly related to the topology of QCD, such as the "strong CP problem" and the "$\mathrm{U}(1)$ problem" which was solved by 't Hooft [9]. Other characteristics the $(1+1)$-dimensional $\mathbb{C}\mathrm{P}(N-1)$ model shares with QCD are a dynamically generated mass gap and asymptotic freedom. The $(N-1)$ dimensional complex projective space $\mathbb{C}\mathrm{P}(N-1)$ can be identified with the coset space $\mathrm{SU}(N)/U(N-1)$. At higher space-time dimensions d the particles in this model are therefore massless Goldstone bosons generated in the symmetry breaking $\mathrm{SU}(N) \to U(N-1)$. In $d = 2$ however, according to the Mermin-Wagner theorem [1], continuous symmetries cannot be broken spontaneously; therefore, no massless modes exist and the particles pick up a mass non-perturbatively. In the classical formulation, the model contains no dimensionful parameters. However, upon quantization, dimensional transmutation takes place. The regularization of divergences implies that the bare coupling is characterized by a typical energy scale. This is essential for the generation of the mass-gap.

D-theory provides an alternative to the standard quantum field theory (QFT) approach of quantizing physical systems, which uses path integrals of classical fields. First proposed in 1996 by Chandrasekharan and Wiese [18], it aims to provide a resource efficient way of quantizing field theories via the dimensional reduction of discrete variables. A paper by Brower, Chandrasekharan, Riederer, and Wiese [22] provides a thorough introduction with a number of applications. In this method the classical fields of a d-dimensional field theory are found to be the low-energy degrees of freedom of a system of discrete quantized variables. It is imperative for the $(d+1)$-dimensional low-energy degrees of freedom to be massless. The quantized variables live on a d-dimensional lattice and evolve in an extra dimension. When the extent of the extra dimension becomes small compared to the correlation length, dimensional reduction takes place and the original d-dimensional QFT emerges.

The aim of this thesis is to explore the large N limit of the two-dimensional $\mathbb{C}\mathrm{P}(N-1)$ nonlinear σ model at zero vacuum angle and work towards an analytic solution of the theory in this limit. In Chapter 2, we consider the standard field theory formulation of $\mathbb{C}\mathrm{P}(N-1)$ models. We derive the self-duality equation which defines instanton solutions that arise as a consequence of the non-trivial topology, and we confirm the existence of a mass-gap. In Chapter 3, we confirm that the ferromagnetic $\mathrm{SU}(N)$ quantum Heisenberg model can indeed be used as a D-theory regularization of the $\mathbb{C}\mathrm{P}(N-1)$ model at zero vacuum angle and we discuss the constraints under which this is applicable. We also show that D-theory is applicable even in the large N limit. In Chapter 4, we discuss the potential of clusters, originating from the concept of cluster algorithms in Monte Carlo calculations, to be used in the formulation of an analytic solution of the $\mathbb{C}\mathrm{P}(N-1)$ model. In Chapter 5, we describe how in the calculation of the $\mathrm{SU}(N)$ quantum ferromagnet on a two-site lattice as a test case for potential cluster descriptions, we encountered an interesting phase transition. Finally, we draw conclusions and point out potential for future research in Chapter 6.

2. Field theory formulation of $\mathbb{C}\mathrm{P}(N-1)$ non-linear σ models

The $\mathbb{C}\mathrm{P}(N-1)$ non-linear σ model in two space-time dimensions is a topologically non-trivial toy model for QCD in four space-time dimensions. In the following we introduce two representations of the $\mathbb{C}\mathrm{P}(N-1)$ model in $(1+1)$-dimensional space-time with a Euclidean time dimension using classical field variables. More prevalent in the literature are complex N-vector field variables, this will be referred to as the z-notation. An alternative representation was introduced by Lüscher [8] that uses NxN-matrices as field variables for the Goldestone bosons. Both are subject to constraints. This chapter is mainly based on the paper by D'Adda, Lüscher, and Vecchia [5]. We introduce the classical formulation of the $\mathbb{C}\mathrm{P}(N-1)$ model and investigate its topology. The self-duality equation defining instantons and the leading order of the mass-gap in a large N expansion are derived.

2.1. Definition of the two-dimensional $\mathbb{C}\mathrm{P}(N-1)$ model

In a first step we identify the building blocks used to construct the action. We remind ourselves that the field variables are elements of the coset space $\mathbb{C}\mathrm{P}(N-1) = \mathrm{SU}(N)/U(N-1)$. To fully capture the action's symmetries we make use of complex N component scalar-fields normalized to 1,

$$z^T = (z_1, z_2, ..., z_N), \qquad \overline{z_i} z_i = 1, \tag{2.1}$$

upon which we impose an equivalence relation. Two vectors z and z' are considered to be equivalent if

$$z'_i(x) = e^{i\Lambda(x)} z_i(x) \tag{2.2}$$

for all $i \in \{1, ..., N\}$, $\Lambda \in \mathbb{C}$ and space-time positions $x \in \mathbb{R}^2$. We have imposed a $U(1)$ gauge symmetry to deal with the artificial degree of freedom introduced by using vector-fields z instead of the equivalence classes $[z]$ that make up $\mathrm{SU}(N)/U(N-1)$. In order to build the gauge invariant action

$$S[z, A_\mu] = \int d^2x \frac{2}{g^2} \overline{D_\mu z_i} D_\mu z_i, \quad D_\mu = \partial_\mu + iA_\mu \tag{2.3}$$

the gauge field $A_\mu(x)$ which transforms as

$$A'_\mu(x) = A_\mu(x) - \partial_\mu \Lambda(x). \tag{2.4}$$

is introduced. This action is invariant under global $\mathrm{SU}(N)$ transformations $z'(x) = \Omega z(x)$, where $\Omega \in \mathrm{SU}(N)$. Note that A_μ is a dimensionful field variable. Consequently, a dynamical term for the gauge field would break the theory's classical scale invariance. Therefore, it cannot

be included in the action. Since (2.3) is quadratic in A_μ, the field can be integrated out. This results in

$$A_\mu = \frac{1}{2} i \left(\overline{z_i} \partial_\mu z_i - (\partial_\mu \overline{z_i}) z_i \right), \tag{2.5}$$

which is also the classical equation of motion. We have now successfully built a classical action that is invariant under global $\mathrm{SU}(N)$ transformation and has a $U(N-1)$ ambiguity. Before we discuss the model's topology, we introduce an alternative formalism that allows us to work without gauge fields. It works by using inherently $U(1)$-invariant field variables $P(x)$ as defined by

$$P = |z\rangle \langle z| = z z^\dagger, \qquad P_{ij} = z_i \overline{z_j}. \tag{2.6}$$

From the construction of P and (2.1) follow directly the characteristics

$$P^2 = P, \qquad \mathrm{Tr}\, P = 1, \qquad P^\dagger = P. \tag{2.7}$$

The local $U(1)$ symmetry also follows directly from the construction of P,

$$P'_{ij}(x) = z'_i(x)\overline{z'_j(x)} = e^{i\Lambda(x)} z_i(x) \overline{e^{i\Lambda(x)} z_j(x)} = e^{i\Lambda(x)} e^{-i\Lambda(x)} z_i(x) \overline{z_j(x)} = P_{ij}(x). \tag{2.8}$$

Under global $\mathrm{SU}(N)$ transformations, $P(x)$ transforms in the adjoint representation

$$P'(x) = \Omega P(x) \Omega^\dagger, \quad \Omega \in \mathrm{SU}(N). \tag{2.9}$$

We now have everything to build the classical action

$$S[P] = \int d^2x \frac{1}{g^2} \mathrm{Tr} \left(\partial_\mu P \partial_\mu P \right). \tag{2.10}$$

In the following we work with the formulation that is more suited to the task at hand.
We have mentioned in the introduction that the $\mathbb{C}\mathrm{P}(N-1)$ models exhibit an interesting topological structure. The fields map the two-dimensional space-time onto a coset space G/H. At space-time infinity they cannot necessarily be continuously deformed into each other. The fields are classified by their behavior at infinity. The set of these so-called homotopy classes is the second homotopy group $\pi_2(G/H)$ of G/H. Homotopy theory provides us with a very useful theorem [6] to identify models with an interesting topology. Under the assumption, that G is both simply connected as well as connected, meaning

$$\pi_0(G) = 0, \qquad \pi_1(G) = 0, \tag{2.11}$$

there exists an isomorphism between the second homotopy group of G/H and the first homotopy group of H

$$\pi_2(G/H) \simeq \pi_1(H). \tag{2.12}$$

Let us consider the manifolds that locally correspond to the group and subgroup of the coset that is the complex projective space,

$$\begin{aligned} \mathrm{SU}(N) &\sim S^3 \otimes S^5 \otimes S^7 \otimes ... \otimes S^{2N-1}, \\ U(N-1) = SU(N-1) \otimes U(1) &\sim S^1 \otimes S^3 \otimes ... \otimes S^{2N-3}, \\ CP(N-1) = \mathrm{SU}(N)/U(N-1) &\sim S^{2N-1}/S^1. \end{aligned} \tag{2.13}$$

We combine this with the above mentioned theorem to find for $N \geq 2$

$$\pi_2(\mathbb{C}\mathrm{P}(N-1)) \simeq \pi_1(S^1) = \mathbb{Z}. \tag{2.14}$$

Therefore $\mathbb{C}\mathrm{P}(N-1)$ indeed has non-trivial topology. The valid field configurations are sorted into classes of fields that can be continuously deformed into each other. Each class is labeled by its topological charge Q; an integer. The topological charge expressions for both notations are

$$Q[z] = \frac{1}{2\pi} \int d^2x \epsilon_{\mu\nu} \partial_\mu A_\nu, \tag{2.15}$$

and

$$Q[P] = \frac{i}{2\pi} \int d^2x \epsilon_{\mu\nu} \operatorname{Tr}(P \partial_\mu P \partial_\nu P) \tag{2.16}$$

respectively. We check, that Q is a real valued quantity by showing that the integrand is completely imaginary.

$$\begin{aligned} \epsilon_{\mu\nu} \operatorname{Tr}(P \partial_\mu P \partial_\nu P)^* &\overset{P^\dagger = P}{=} \epsilon_{\mu\nu} \operatorname{Tr}(P^\mathsf{T} \partial_\mu P^\mathsf{T} \partial_\nu P^\mathsf{T}) \\ &= \epsilon_{\mu\nu} \operatorname{Tr}((\partial_\nu P \partial_\mu P P)^\mathsf{T}) \\ &= -\epsilon_{\mu\nu} \operatorname{Tr}(P \partial_\mu P \partial_\nu P). \end{aligned} \tag{2.17}$$

A proof that $Q[z]$ is indeed an integer can be found in [25]. In Appendix A.1 we prove the equivalence of the two expressions (2.15) and (2.16).

The full expression for the partition function for the two-dimensional $\mathbb{C}\mathrm{P}(N-1)$ non-linear σ model is finally given by

$$Z = \int \mathcal{D}\bar{z} \mathcal{D}z \, \exp(-S[z] + i\theta Q[z]) \tag{2.18}$$

or

$$Z = \int \mathcal{D}P \, \exp(-S[P] + i\theta Q[P]) \tag{2.19}$$

respectively.

2.2. Self-duality equations for instantons

It is found that the topological charge puts a lower bound on the action of a field configuration. Consider the inequalities

$$0 \leq \int d^2x \operatorname{Tr}\left(|P(a\partial_\mu P \pm ib\epsilon_{\mu\nu}\partial_\nu P)(a\partial_\mu P \mp ib\epsilon_{\mu\nu}\partial_\nu P)|\right) \tag{2.20}$$

$$0 \leq \int d^2x \; |aD_\mu z_i \pm ib\epsilon_{\mu\nu} D_\nu z_i|^2 \tag{2.21}$$

to find that

$$S \geq \frac{8\pi}{g^2} \left| \frac{ab}{a^2+b^2} Q \right|. \tag{2.22}$$

For non-zero values of a and b the inequality is strongest when $a = b$. Field configurations that minimize the action within a sector of definite topological charge are called instantons. They are defined as solutions of the self-duality equations

$$\partial_\mu P = \pm i\epsilon_{\mu\nu}\partial_\nu P \quad \text{and} \quad D_\mu z_i = \pm i\epsilon_{\mu\nu} D_\nu z_i, \tag{2.23}$$

respectively. The minus signs define anti-instantons. Instantons describe the tunnelling process between classical vacuum states. Any solution of the first-order self-duality equation is automatically also a solution of the second-order Euler-Lagrange field equation

$$D_\mu D_\mu z_i + \left(\overline{D_\mu z_i} D_\mu z_i\right) z = 0. \tag{2.24}$$

Deriving the Euler-Lagrange equation is non-trivial, due to the constrains acting on the fields z and P. For the z-formalism this is done in Appendix A.2, in the P-formalism the constraints become more difficult to implement explicitly, therefore this is left out here. A special property of $\mathbb{C}\mathrm{P}(N-1)$ models is that the self-duality equations (2.23) are completely solvable. This is done explicitly in the paper by D'Adda, Lüscher, and Vecchia [5].

2.3. Leading order mass-gap equation

In this section we derive the leading order expression of the mass-gap equation for the quantum $\mathbb{C}\mathrm{P}(N-1)$ model using the $1/N$-expansion. For this task the z-formalism is chosen, since, as mentioned in the previous section, in the P-formalism the explicit formulation of the constraints on the field variables by means of Lagrange multipliers is difficult. The following section is based on the work by D'Adda et al. [5]. To take the limit of $N \to \infty$ is conceptually more complicated than to take a limit of other parameters of the model. After all, to change N is to change the symmetry of system. It is therefore necessary to ask if any special conditions apply. One can argue that the so called 't Hooft coupling [3]

$$f = \frac{Ng^2}{4} \tag{2.25}$$

must be kept fixed as N goes to infinity for the limit to be well defined. The rough idea behind this can be outlined by considering diagrams in perturbation theory. Each time a loop is added without crossing over any existing lines in the diagram, the amplitude gains a factor proportional to Ng^2; the N arises from the possible flavors of the new virtual particle. In order for the amplitudes not to diverge, the 't Hooft coupling must remain fixed. It should be noted that f is a bare coupling. Our goal is to express the mass-gap via the saddle-point equation that arises, when taking the limit $N \to \infty$. This concept only holds for the quantum $\mathbb{C}\mathrm{P}(N-1)$ model since the coupling only gains a scale through dimensional transmutation. We begin by writing down the partition function and integrating out the gauge fields which appear quadratically in the action. Note that we do not keep track of overall factors in front of Z,

$$\begin{aligned} Z &= \int \mathcal{D}z\mathcal{D}\overline{z} \int \mathcal{D}A_\mu \exp\left\{-N\int d^2x \frac{1}{2f}\overline{D_\mu z_i} D_\mu z_i\right\} \\ &= \int \mathcal{D}z\mathcal{D}\overline{z} \int \mathcal{D}A_\mu \exp\left\{-N\int d^2x \frac{1}{2f}(\partial_\mu \overline{z}_i \partial_\mu z_i + A_\mu A_\mu \overline{z}_i z_i - A_\mu(i\overline{z}_i \overleftrightarrow{\partial}_\mu z_i))\right\} \\ &= \int \mathcal{D}z\mathcal{D}\overline{z} \int \mathcal{D}A_\mu \exp\left\{-N\int d^2x \frac{1}{2f}(\partial_\mu \overline{z}_i \partial_\mu z_i + \frac{1}{4}(\overline{z}_i \overleftrightarrow{\partial}_\mu z_i)^2)\right\}. \end{aligned} \tag{2.26}$$

Integrating out the gauge field has the same effect as plugging in the solution of the classical equation of motion. We made use of the notation

$$\overline{z}_i \overleftrightarrow{\partial}_\mu z_i = \overline{z}_i \partial_\mu z_i - (\partial_\mu \overline{z}_i) z_i. \tag{2.27}$$

For later convenience we rescale the field variable $\tilde{z}_i = \sqrt{\frac{N}{2f}} z_i$,

$$Z = \int \mathcal{D}z \mathcal{D}\overline{z} \int \mathcal{D}A_\mu \exp\left\{ -N \int d^2x (\partial_\mu \overline{z}_i \partial_\mu z_i + \frac{f}{2N} (\overline{z}_i \overleftrightarrow{\partial}_\mu z_i)^2) \right\}. \tag{2.28}$$

In a next step we introduce the auxiliary field variables λ_μ with the identity

$$\begin{aligned} &\int \mathcal{D}\lambda_\mu \exp\left\{ -\int d^2x \frac{1}{N} |z|^2 \lambda_\mu \lambda_\mu \pm \frac{i}{\sqrt{N}} (\overline{z}_i \overleftrightarrow{\partial}_\mu z_i) \lambda_\mu \right\} \\ &= \sqrt{\frac{\pi^2 N}{|z|^2}} \exp\left\{ \int d^2x \frac{-(\overline{z}_i \overleftrightarrow{\partial}_\mu z_i)^2}{4|z|^2} \right\} \\ &= \sqrt{2\pi f} \exp\left\{ -\int d^2x \frac{f}{2N} (\overline{z}_i \overleftrightarrow{\partial}_\mu z_i)^2 \right\} \end{aligned} \tag{2.29}$$

in order to make the action quadratic in z. We realize the constraint on z by introducing a Lagrange multiplier field $\alpha(x)$.

$$\int \mathcal{D}\alpha \exp\left\{ \pm \int d^2x \frac{i\alpha}{\sqrt{N}} (\overline{z}_i z_i - \frac{N}{2f}) \right\} = \delta\left(|z|^2 - \frac{N}{2f} \right) \tag{2.30}$$

Demanding $U(1)$ gauge invariance of the action, we find the transformation behavior of the new field variables to be

$$z'(x) = e^{i\Lambda(x)} z(x), \quad \lambda'_\mu(x) = \lambda_\mu + \partial_\mu \Lambda(x), \quad \alpha'(x) = \alpha(x). \tag{2.31}$$

Considering that $|z|^2 = \overline{z}_i z_i$ is constant, we are free to rescale Z by a constant factor

$$\exp\left\{ -\int d^2x \; m^2 |z|^2 \right\} \tag{2.32}$$

that carries the meaning of a mass term in the action. Plugging in all the above relations, the partition function takes the form

$$Z = \int \mathcal{D}\lambda_\mu \mathcal{D}\alpha \; \left[\int \mathcal{D}z_i \mathcal{D}\overline{z}_i \exp\left\{ -\int d^2x \, \overline{z}_i \Delta z_i \right\} \right]^N \exp\left\{ \int d^2x \; \frac{i\sqrt{N}}{2f} \alpha \right\}, \tag{2.33}$$

where

$$\begin{aligned} \Delta :&= -\partial_\mu \partial_\mu + \frac{1}{N} \lambda_\mu \lambda_\mu + \frac{i}{\sqrt{N}} \partial_\mu \cdot \lambda_\mu + \frac{i}{\sqrt{N}} \lambda_\mu \partial_\mu + \frac{i\alpha}{\sqrt{N}} + m^2 \\ &= -\underbrace{\left(\partial_\mu - \frac{i}{\sqrt{N}} \lambda_\mu \right)^2}_{D_\mu D_\mu} + \frac{i\alpha}{\sqrt{N}} + m^2. \end{aligned} \tag{2.34}$$

The dot in the third term of the operator Δ signifies that the derivative acts on both λ_μ and the field z_i that the operator is applied to. Note that in (2.33) the index i of z is no longer summed over. Also, we find that $D_\mu D_\mu$ is Hermitian, while $i\alpha(x)$ is anti-Hermitian. However, we find that, at leading order, α is constant; therefore the integral over $\overline{z}_i$ and z_i is Gaussian. At higher orders it is not obvious that the integral should remain Gaussian and we recommend consulting [5]. Performing the Gaussian integral reduces the partition function to the highly symbolic expression

$$Z = \int \mathcal{D}\lambda_\mu \mathcal{D}\alpha \ \exp \underbrace{\left\{ -N \operatorname{Tr}(\log \Delta) + \frac{i\sqrt{N}}{2f} \int d^2x \alpha(x) \right\}}_{-S_{\text{eff}}}. \tag{2.35}$$

The effective action may now be expanded in a power series of $1/\sqrt{N}$:

$$S_{\text{eff}} = N S_{\text{eff}}^{(0)} + \sqrt{N} S_{\text{eff}}^{(1)} + S_{\text{eff}}^{(2)} + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right). \tag{2.36}$$

It turns out that $S_{\text{eff}}^{(0)}$ is a constant with respect to the fields α and λ_μ; consequently, it only contributes to the partition function as an irrelevant overall factor. At large N the exponent is therefore dominated by

$$\begin{aligned} S_{\text{eff}}^{(1)} &= \frac{i}{2f} \int d^2x \ \alpha(x) - \operatorname{Tr}\left(\frac{i\partial_\mu \cdot \lambda_\mu + i\lambda_\mu \partial_\mu + i\alpha}{-\Box + m^2} \right) \\ &= i\tilde{\alpha}(0) \left(\frac{1}{2f} - \int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2 + m^2} \right). \end{aligned} \tag{2.37}$$

In Appendix A.3 the derivation of the explicit form of $S_{\text{eff}}^{(1)}$ and $S_{\text{eff}}^{(0)}$ can be found. $\tilde{\alpha}(p)$ is the Fourier transform of $\alpha(x)$ using the convention

$$\tilde{\alpha}(p) = \int d^2x \ \exp(-ipx)\alpha(x). \tag{2.38}$$

The momentum integral in the second line of (2.37) displays a logarithmic ultraviolet divergence. We employ a Pauli-Villars cutoff to regularize the integral. This results in

$$\int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2 + m^2} = 1/(4\pi) \log(\Lambda^2/m^2). \tag{2.39}$$

Let the bare coupling run with the cutoff such that the divergence at $\Lambda \to \infty$ cancels:

$$\frac{1}{2f} = \frac{1}{4\pi} \log\left(\frac{\Lambda^2}{\mu^2}\right) + \frac{1}{2f_R(\mu)}. \tag{2.40}$$

The quantity $f_R(\mu)$ is the renormalized coupling and μ is the so-called normalization point. Unless $\sqrt{N} S_{\text{eff}}^{(1)}$ vanishes, it will introduce rapid oscillations to the partition function Z in the limit $N \to \infty$. However, since m is a free parameter, we impose on it the mass-gap equation

$$m^2 = \mu^2 \exp\left(-\frac{2\pi}{f_R(\mu)}\right). \tag{2.41}$$

The physical particles of the two-dimensional $\mathbb{C}\mathrm{P}(N-1)$ non-linear σ model indeed gain a mass dynamically. D'Adda et al. go on to discuss the quadratic part $S^{(2)}_{\text{eff}}$ of the effective action. In 1990 the exact mass-gap was derived in the $\overline{\text{MS}}$ renormalization scheme for some two-dimensional asymptotically free field theories by Hasenfratz and Niedermayer [13]. One such theory is the $\mathrm{O}(3)$ model, which is equivalent to the $\mathbb{C}\mathrm{P}(1)$ model. For any other $\mathbb{C}\mathrm{P}(N-1)$ models with $2 < N < \infty$ the mass gap cannot be determined using their technique. In order to compare the two results, one would have to match the different renormalization schemes.

3. D-theory regularization

In D-theory we recover the classical fields of a d-dimensional QFT as the low-energy degrees of freedom of quantized variables on a $(d+1)$-dimensional lattice after dimensional reduction. The $(d+1)$-dimensional theory must produce massless Goldstone bosons at low energies. If the extent of the extra Euclidean dimension becomes small with respect to the correlation length, dimensional reduction takes place. In the following we go step-by-step through the D-theory regularization of the 2-dimensional $\mathbb{C}\mathrm{P}(N-1)$ model with vacuum angle $\theta = 0$. There are a multitude of possible D-theory formulations of this model. We will use the ferromagnetic $\mathrm{SU}(N)$ quantum Heisenberg model as the underlying lattice model. Quantization of the $\mathbb{C}\mathrm{P}(N-1)$ model at vacuum angle $\theta = 0, \pi$ using $\mathrm{SU}(N)$ quantum spin-ladders has also been explored in [23].

3.1. $\mathrm{SU}(N)$ quantum ferromagnet

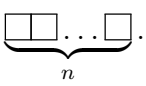
The $\mathrm{SU}(N)$ quantum Heisenberg models are built from a set of quantum spin operators on a lattice, where each spin typically interacts with its nearest neighbors. Their main purpose is the description of ferromagnetism and anti-ferromagnetism in solids. In this first section we give an introduction to said models and investigate the characteristics necessary for retrieving the $\mathbb{C}\mathrm{P}(N-1)$ models. Explicitly this includes the symmetries of the system and its ground states. It is found that, in order to achieve the desired symmetry breaking, one is restricted to symmetric representations of $\mathrm{SU}(N)$. Furthermore, magnon states are investigated in preparation for formulating the low-energy effective field theory (EFT) in the next section.

Let us introduce the ferromagnetic Heisenberg model on a d-dimensional hypercubic lattice with a lattice spacing a and with sides of length L with periodic boundary conditions. It is customary to set $a = 1$. For important results, however, we will reinsert a for completeness and to ensure that the dimensions match. For the application to a regularization of the $(1+1)$-dimensional $\mathbb{C}\mathrm{P}(N-1)$ model we only need to consider the case of $d = 2$. Nonetheless, in this section it is not yet necessary to restrict d, therefore we leave it arbitrary for now. The Hamiltonian associated with the ferromagnetic quantum Heisenberg model is given by

$$H = -J \sum_{\langle xy \rangle} T_x^a T_y^a, \tag{3.1}$$

where $J > 0$ is the coupling constant that controls the strength of the interaction. The indices x and y are d-dimensional position vectors of the lattice sites. The quantum spins located on the lattice sites x are represented by operators T_x^a. They are generators of the $\mathfrak{su}(N)$ Lie algebra with the normalization condition $\mathrm{Tr}(T^a T^b) = \delta_{ab}/2$. The representation of the generators plays a significant role later, since only for certain representations the desired symmetry breaking

arises. As mentioned above, our main interest lies with symmetric representations that are described by Young tableaux of n boxes in a row



(3.2)

We therefore use the value of n to identify the representations. As generators of $\mathfrak{su}(N)$, the matrices T^a_x are Hermitian and traceless with the structure constants f and the d-coefficients defining the commutation and anti-commutation relations, respectively. Therefore, in the fundamental representation $n = 1$, the matrices T^a are generalized Gell-Mann matrices rescaled by a factor of $1/2$. The summation in (3.1) denotes the sum over all pairs of nearest neighbors on the lattice. By the Einstein summation convention the index a, which is not to be confused with the lattice spacing, is summed over $a \in \{1, \ldots, N^2-1\}$, where N^2-1 is the dimension of the $\mathrm{SU}(N)$ group. Note that J has the dimension of an energy $[J] = 1$. The partition function for such a canonical ensemble is given by

$$Z = \mathrm{Tr}\left(e^{-\beta H}\right). \tag{3.3}$$

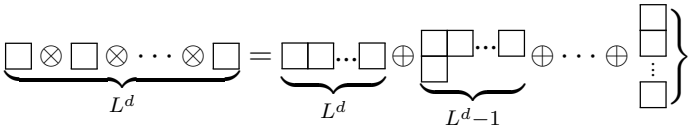
Since H is time-independent, the time evolution operator in imaginary time takes the form $U = \exp(-tH)$. As a consequence, the inverse temperature $\beta = 1/T$ ($k_B = 1$) can also be interpreted as the extent of a Euclidean time dimension in which the system evolves. The cyclicity of the trace demands periodic boundary conditions in this direction. This extra dimension will later disappear via the dimensional reduction step of the D-theory formalism. Then one of the space dimensions will be reinterpreted as the Euclidean time. The trace extends over the physical Hilbert space. The system is invariant under global $\mathrm{SU}(N)$ transformations

$$[H, T^a] = 0, \tag{3.4}$$

where $T^a = \sum_x T^a_x$ is the total spin. This is shown explicitly in Appendix B.1.

3.1.1. Spontaneous symmetry breaking

It turns out that the ground states do not respect the full symmetry of the system. As mentioned above, the representation of spin operators on the lattice sites is crucial for determining the symmetry breaking pattern of the ground state. For simplicity's sake let us first investigate the fundamental representation $\{N\}$ with $n = 1$. The coupling of L^d spins via a direct product can be decomposed into irreducible representations



(3.5)

The sum extends over all Young tableaux consisting of L^d boxes. $\mathrm{SU}(N)$ in the fundamental representation has N flavor states $|q\rangle$, $q \in \{u, d, s, ...\}$. In the ferromagnetic system the energy is minimized for completely symmetric states, i.e. states that belong to the first Young tableau on the right hand side of (3.5). Such a ground state breaks the $\mathrm{SU}(N)$ symmetry, but still retains a $U(N-1)$ symmetry. For example, we may choose the ground state $|uu\ldots u\rangle$. It is invariant

under $SU(N-1)$ transformations that mix the remaining $N-1$ flavors. The vacuum is also left unchanged by a global phase factor. This implies an additional $U(1)$ symmetry. We conclude that completely symmetric vacua display the desired symmetry breaking

$$\mathrm{SU}(N) \to SU(N-1) \otimes U(1) = U(N-1), \tag{3.6}$$

where the low-energy degrees of freedom live in the coset space

$$\mathbb{C}\mathrm{P}(N-1) = \mathrm{SU}(N)/U(N-1). \tag{3.7}$$

Let us now generalize these observations to arbitrary irreducible spin representations. Any irreducible representation has a corresponding Young tableau, all of which can be generated by coupling together fundamental representations. In order to experience the desired symmetry breaking the ground state of the system must be a state of the completely symmetric Young tableau. By drawing the coupling of L^d spins in terms of their Young tableaux, it becomes evident that we may only find such a ground state if the spins on each lattice site are themselves in a symmetric representation. A symmetric ground state is an element of the multiplet in

$$\underbrace{\square\square\ldots\square}_{nL^d} = \left\{ \frac{(N+nL^d-1)!}{(N-1)!(nL^d)!} \right\}. \tag{3.8}$$

On each site there is now a choice of $\frac{(N+n-1)!}{(N-1)!(n)!}$ flavors $q \in \{u, d, s, \ldots\}$. Each of these flavors can be built as a symmetric combination of n fundamental spin states. The vacuum energy of the system is a quantity of interest needed later in order to construct the low-energy effective field theory and match it to the underlying $\mathrm{SU}(N)$ model. Let $|0\rangle$ be the completely symmetric ground state with maximum spin projection $|0\rangle = |uu\ldots u\rangle$ just like above. Using the identity

$$T_x^a T_y^a = \frac{1}{2}(T_x^a + T_y^a)^2 - \frac{1}{2}(T_x^a)^2 - \frac{1}{2}(T_y^a)^2, \tag{3.9}$$

the Hamiltonian can be written terms of quadratic Casimir operators

$$C_2(\mathcal{R}) = T^a T^a, \qquad T^a = \sum_x T_x^a, \tag{3.10}$$

where $\mathcal{R}$ determines the representation. Let us use the following notation for symmetric and antisymmetric representations.

$$\begin{aligned} C_2(\underbrace{\square\square\ldots\square}_{m_S}) &= C_2(m_S) \\ C_2\left({}_{m_A}\left\{ \begin{array}{c}\square\\ \square\\ \vdots\\ \square\end{array} \right. \right) &= C_2(m_A) \end{aligned} \tag{3.11}$$

This implies the expression

$$\begin{aligned} H\,|0\rangle &= -J \sum_{\langle xy \rangle} \left(\frac{C_2(2n_S)}{2} - C_2(n_S) \right) |0\rangle \\ &= -Jd \left(\frac{L}{a} \right)^d \frac{N-1}{2N} n^2\,|0\rangle \\ &= E_0\,|0\rangle \end{aligned} \tag{3.12}$$

for the Hamiltonian acting on a completely symmetric ground state of $\mathrm{SU}(N)$ spins in the symmetric representation of n boxes in a row. The value of the Casimir operator in symmetric and antisymmetric representations was derived in [20].

$$C_2(m_S) = C_f \frac{m(N+m)}{N+1} \tag{3.13}$$

$$C_2(m_A) = C_f \frac{m(N-m)}{N-1} \tag{3.14}$$

$$C_f = \frac{N^2-1}{2N} \tag{3.15}$$

The expression for antisymmetric representations will come in handy in a later chapter. The factor C_f is the quadratic Casimir of the fundamental representation. The summands in (3.12) are independent of the choice of x and y for this ground state. Thus the sum simply yields a factor of the number of nearest-neighbors on the lattice with periodic boundary conditions dL^d. Moreover, the lattice spacing a was reinserted $L \to L/a$.

3.1.2. Magnon states

Now that we have determined the vacuum energy let us derive the dispersion relation for magnons (also known as spin-waves). To this end, we first introduce an alternative basis for $\mathfrak{su}(N)$ in the fundamental representation. It consists of $N(N-1)/2$ shift operators

$$\begin{aligned} T^{\pm,1} &= T^1 \pm iT^2 \\ T^{\pm,2} &= T^4 \pm iT^5 \\ T^{\pm,3} &= T^6 \pm iT^7 \\ &\dots \end{aligned} \tag{3.16}$$

and $N-1$ diagonal operators

$$\tilde{T}^a = \frac{1}{2}\sqrt{\frac{2}{a(a+1)}}\,\mathrm{diag}(\underbrace{1,\dots,1}_{a},-a,0,\dots,0), \quad a \in \{1,\dots,N-1\}. \tag{3.17}$$

The diagonal operators are in fact the same as in the generalized Gell-Mann matrices, only with different indices. To avoid confusion they are denoted with a tilde. The shift operators $T^{\pm,b}$ with $b \in \{1,...,N(N-1)/2\}$ are built from pairs of off-diagonal generalized Gell-Mann matrices with non-zero entries at the same positions just like in (3.16). Each pair of shift operators causes a shift between two specific spin flavors in the multiplet. Let us consider the example of $SU(3)$ with $n=1$,

$$\begin{aligned} T^{+,1}|u\rangle &= 0, & T^{-,1}|u\rangle &= |d\rangle, \\ T^{+,1}|d\rangle &= |u\rangle, & T^{-,1}|d\rangle &= 0, \\ T^{+,1}|s\rangle &= 0, & T^{-,1}|s\rangle &= 0, \end{aligned} \tag{3.18}$$

$$\begin{aligned} T^{+,2}|u\rangle &= 0, & T^{-,2}|u\rangle &= |s\rangle, \\ T^{+,2}|d\rangle &= 0, & T^{-,2}|d\rangle &= 0, \\ T^{+,2}|s\rangle &= 0\,|u\rangle, & T^{-,2}|s\rangle &= 0, \end{aligned} \tag{3.19}$$

$$\begin{aligned} T^{+,3}\left|u\right\rangle &= 0, & T^{-,3}\left|u\right\rangle &= 0, \\ T^{+,3}\left|d\right\rangle &= 0, & T^{-,3}\left|d\right\rangle &= \left|s\right\rangle, \\ T^{+,3}\left|s\right\rangle &= \left|d\right\rangle, & T^{-,3}\left|s\right\rangle &= 0. \end{aligned} \tag{3.20}$$

Furthermore, for $n \neq 1$ the generators can be constructed via the symmetrization of the direct product of generators in the fundamental representation. For $n = 2$ this looks as follows:

$$\begin{aligned} T^a_{\square\otimes\square} &= T^a_{\square} \otimes 1 + 1 \otimes T^a_{\square}, \\ T^a_{\square\square} &= T^a_{\square\otimes\square}|_{\text{symm}}. \end{aligned} \tag{3.21}$$

After building the tensor product, one can bring the generators into block-diagonal form by a change of basis. For $n = 2$ only the block corresponding to the symmetric representation survives the symmetrization of the matrices. For higher n other blocks will partially survive as well. The change of basis introduces factors of $1/\sqrt{n}$ to the generators. The shift operators connect all states in a multiplet. Figure 3.1 illustrates the example of $N = 3$ and $n = 2$.

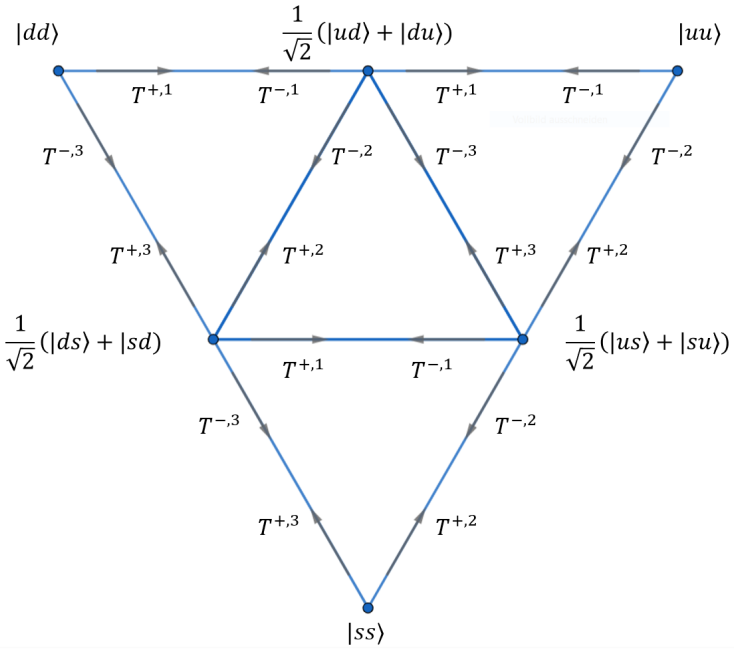


Figure 3.1.: States of the sextet representation of $SU(3)$.

This formulation of the $\mathfrak{su}(N)$ generators is very convenient in the definition of magnon states

$$\left|p\right\rangle = \sum_x \exp\left(ip_i x_i\right) T_x^{\pm,b} \left|0\right\rangle, \tag{3.22}$$

where, in accordance with the previous notation, x is a d-dimensional position vector. The following treatment of magnons follows [25]. Let $|0\rangle$ again be the completely symmetric ground state of maximum spin projection and $T_x^{\pm,b}$ the shift operator that acts on the spin at position x; here $T_x^{-,1}$. We proceed by investigating how the Hamiltonian acts on $T_x^{-,1}\left|0\right\rangle$. Let H_x contain all summands of H that act on the site x and H_{rest} the rest,

$$H = H_x + H_{\text{rest}}. \tag{3.23}$$

The local shift operator $T_x^{-,1}$ commutes with H_{rest}, thus we find in analogy to (3.12) that

$$H_{\text{rest}} T_x^{-,1} |0\rangle = -J \left(d \left(\frac{L}{a} \right)^d - 2d \right) \frac{N-1}{2N} n^2 \, T_x^{-,1} |0\rangle \,, \tag{3.24}$$

where the term $-2d$ comes from the missing number of terms in H_{rest} compared to H. Furthermore we split H_x into a diagonal part H_x^d and an off-diagonal part H_x^s that can be written in terms of shift operators, hence the index s,

$$\begin{aligned} H_x^d &= -J \sum_{y|<x,y>} \sum_{a=1}^{N-1} \tilde{T}_x^a \tilde{T}_y^a, \\ H_x^s &= -J \sum_{y|<x,y>} \sum_{b=1}^{N(N-1)/2} (T_x^{+,b} T_y^{-,b} + T_x^{-,b} T_y^{+,b}). \end{aligned} \tag{3.25}$$

The relation

$$T_x^1 T_y^1 + T_x^2 T_y^2 = T_x^{+,1} T_y^{-,1} + T_x^{-,1} T_y^{+,1}, \tag{3.26}$$

and its equivalents for higher indices, were used to rewrite H_x^s in (3.25). The expression $y|\langle xy\rangle$ stands for "y such that y is a neighbor of x". The eigenvalue of $|0'_x\rangle = T_x^- |0\rangle$ acted on by H_x^d is straightforward to calculate,

$$\begin{aligned} H_x^d |0'_x\rangle &= -J2d \left(\tilde{T}_x^1 \tilde{T}_y^1 |0'_x\rangle + \sum_{a=2}^{N-1} \tilde{T}_x^a \tilde{T}_y^a |0'_x\rangle \right) \\ &= -J2d \left(\left(\frac{n}{2} - 1 \right) \frac{n}{2} |0'_x\rangle + \sum_{a=2}^{N-1} \frac{n^2}{2a(a+1)} |0'_x\rangle \right) \\ &= -J2d \left(\left(\frac{n}{2} - 1 \right) \frac{n}{2} + \frac{n^2}{2} \frac{N-2}{2N} \right) |0'_x\rangle \,. \end{aligned} \tag{3.27}$$

The effect of the shift-operator part H_x^s is derived in Appendix B.2. We find

$$H_x^s |p\rangle = -nJ \sum_{\mu=1}^{d} \cos(p_\mu) |p\rangle \,. \tag{3.28}$$

Therefore we now have all the necessary information in order to find the dispersion relation of

magnon states

$$\begin{aligned} E_p - E_0 &= -J\left[\left(d\left(\frac{L}{a}\right)^d - 2d\right)\frac{N-1}{2N}n^2 + 2d\left(\left(\frac{n}{2}-1\right)\frac{n}{2} + \frac{n^2}{2}\frac{N-2}{2N}\right) + n\sum_{\mu=1}^{d}\cos(p_\mu)\right] \\ &\quad + J\left[d\left(\frac{L}{a}\right)^d \frac{N-1}{2N}n^2\right] \\ &= -J\left[-d\frac{n^2}{2} + 2d\left(\frac{n}{2}-1\right)\frac{n}{2} + n\sum_{\mu=1}^{d}\cos(p_\mu)\right] \\ &= -J\left[-dn + n\sum_{\mu=1}^{d}\cos(p_\mu)\right] \\ &= Jn\sum_{\mu=1}^{d}(1-\cos(p_\mu)). \end{aligned} \tag{3.29}$$

For small values of p we may expand the expression to second order and explicitly reintroduce the lattice spacing a. We find that the difference between the magnon energy and the ground state energy is

$$E_p - E_0 \simeq \frac{Jn}{2}a^2p^2. \tag{3.30}$$

The energy is not proportional to the magnitude of the momentum, thus the magnons are of a non-relativistic nature.

3.2. Chiral perturbation theory

We now move toward recovering the $\mathbb{C}\mathrm{P}(N-1)$ model in the low-energy degrees of freedom by building an effective field theory for the symmetry breaking $\mathrm{SU}(N) \to U(N-1)$. To this end chiral perturbation theory is utilized. The fields of an effective field theory for the symmetry breaking $G \to H$ reside in the coset space G/H [2]. Here $\mathrm{SU}(N)/U(N-1) = \mathbb{C}\mathrm{P}(N-1)$, thus we use the matrices P introduced in Section 2.1 to express the effective action. In this section the lattice dimension is set to $d = 2$ and the extent of all dimensions L and β are set to infinity. Let us construct the leading order effective Lagrangian that respects the symmetry relations of the microscopic theory. All terms need to respect global $\mathrm{SU}(N)$ invariance, this can be achieved by working with traces and determinants. It turns out, however, that the determinant of P and its derivatives only lead to trivial Lagrangian terms. The projector property $P^2 = P$ in combination with $\mathrm{Tr}(P) = 1$ implies

$$\det(P) = 0 \tag{3.31}$$

and

$$\partial_\mu P = (\partial_\mu P)P + P\partial_\mu P. \tag{3.32}$$

Therefore, the determinant of derivatives of P vanishes,

$$\det(\partial_\mu P) = 0, \tag{3.33}$$

and all terms of the effective Lagrangian need to be traces. The Hamiltonian (3.1) is invariant under reflection along either axis, thus space derivatives $\partial_i P$, $i = 1, 2$ must only appear in pairs. Note also that, since we are working with a non-relativistic theory, the Lagrangian doesn't have to be Lorentz-invariant. Since P is a projector it is naturally dimensionless; therefore, the terms contributing to the Lagrangian are ordered by the number of derivatives. Terms without any derivatives are trivial due to the characteristics $\mathrm{Tr}(P) = 1$ and $P^2 = P$. The leading order terms containing spatial derivatives are of the forms

$$\int d^2x \int dt \, \mathrm{Tr}\left(f_1(P)\partial_i P \partial_i P\right) \tag{3.34}$$

and

$$\int d^2x \int dt \, \mathrm{Tr}\left(f_2(P)\partial_i \partial_i P\right). \tag{3.35}$$

The projector property of P means that $f_{1,2}$ are of the form $f_{1,2}(P) = a + bP$, where a and b are real constants. It is also cause for the following three identities:

$$\begin{aligned} \mathrm{Tr}(\partial_i \partial_i P) &= 2\,\mathrm{Tr}(\partial_i P \partial_i P) + 2\,\mathrm{Tr}(P \partial_i \partial_i P), \\ \mathrm{Tr}(P \partial_i \partial_i P) &= -2\,\mathrm{Tr}(P \partial_i P \partial_i P), \\ \mathrm{Tr}(\partial_i P \partial_i P) &= 2\,\mathrm{Tr}(P \partial_i P \partial_i P). \end{aligned} \tag{3.36}$$

Using the first two identities we can show that terms of the second form (3.35) can be absorbed into $f_1(P)$. With the help of the third identity, the term in (3.34) coming from bP can be absorbed into a. For spatial derivative terms we are thus left with a single term $\mathrm{Tr}(\partial_i P \partial_i P)$ in the Lagrangian. Since the extra time dimension is not restricted by a reflection symmetry, terms with a single time derivative are allowed. Naively all such terms seem to be trivial,

$$\mathrm{Tr}(\partial_t P) \tag{3.37}$$

is a total derivative and

$$\mathrm{Tr}(P \partial_t P) \tag{3.38}$$

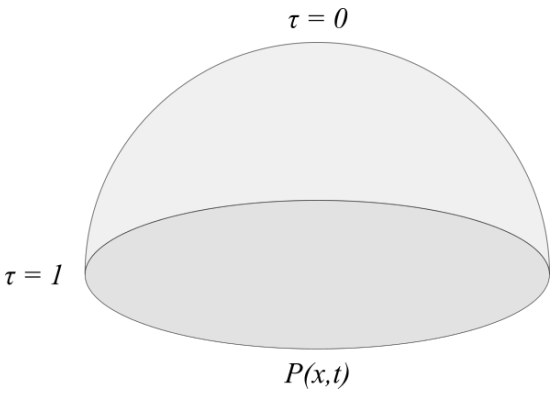
vanishes due to the projector property of P. However, for $\mathbb{C}\mathrm{P}(1)$ H. Leutwyler [17] found that by introducing an additional, purely mathematical dimension $\tau \in [0, 1]$ one can build a so-called Wess-Zumino-Witten term

$$\int d^2x \int dt \int d\tau \, \mathrm{Tr}\left(P \partial_t P \partial_\tau P - P \partial_\tau P \partial_t P\right). \tag{3.39}$$

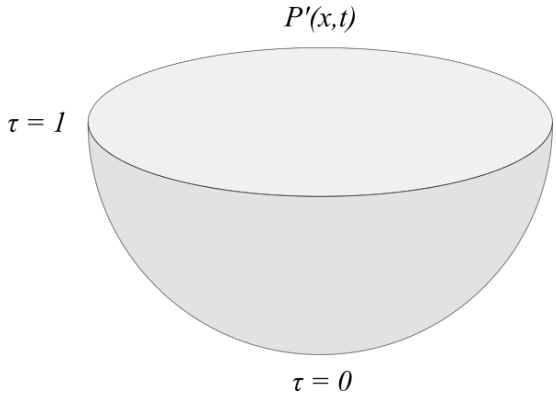
This is also possible for higher values of N. Where $P(x, t)$ was previously defined on $\mathbb{R}^d \times S^1$, S^1 being the compactified time dimension, it is now extended to $P(x, t, \tau)$ defined on $\mathbb{R}^d \times H^2$. H^2 is the 2-dimensional hemisphere in Figure 3.2a. We impose the boundary condition that, at $\tau = 1$, the interpolated field matches the physical field $P(x, t, \tau = 1) = P(x, t)$.

Since τ parametrizes just a mathematical dimension, the physics must be invariant under the choice of interpolation. Let us consider two different choices P and P' interpolated on H^2. We then change the orientation of the hemisphere on which P' is defined. The difference in the action between the two interpolations with a normalization factor results in an integer ambiguity

$$\begin{aligned} &\frac{1}{i2\pi}\left(\int_{H^2} dt d\tau \, \mathrm{Tr}\left(P \partial_t P \partial_\tau P - P \partial_\tau P \partial_t P\right) - \int_{H^{2\prime}} dt d\tau \, \mathrm{Tr}\left(P \partial_t P \partial_\tau P - P \partial_\tau P \partial_t P\right)\right) \\ &= \frac{1}{i2\pi} \int_{S^2} dt d\tau \, \mathrm{Tr}\left(P \partial_t P \partial_\tau P - P \partial_\tau P \partial_t P\right) = k \in \mathbb{Z}. \end{aligned} \tag{3.40}$$



(a) Interpolation of $P(x,t,\tau)$ on hemisphere H^2



(b) Alternative interpolation $P'(x,t,\tau)$ on hemisphere $H^{2\prime}$

Figure 3.2.: Extension of the P-field into the extra dimension parametrized by τ. The interpolated field at $\tau = 1$ corresponds to the physical field.

We used that the union of H^2 and $H^{2\prime}$ with the direction of the normal vector flipped by the negative sign, is a 2-sphere S^2. In the last step of (3.40) we used that the second homotopy group of $\mathbb{C}\mathrm{P}(N-1)$ is $\pi_2[\mathbb{C}\mathrm{P}(N-1)] = \mathbb{Z}$. The invariance of the physics against this ambiguity demands the quantization of the prefactor m of the Wess-Zumino-Witten term

$$\begin{aligned} S_{\mathrm{WZW}}[P] &= -2m \int d^2x \int_{H^2} dt d\tau \, \mathrm{Tr}\left(P\partial_t P \partial_\tau P - P\partial_\tau P \partial_t P\right), \\ S_{\mathrm{WZW}}[P] - S_{\mathrm{WZW}}[P'] &= i2\pi(-2m)k \underbrace{\int d^2x}_{V} \overset{!}{\in} 2\pi\mathbb{Z} \quad \Rightarrow \quad Vm \in \frac{\mathbb{Z}}{2}. \end{aligned} \tag{3.41}$$

We now put together the full leading order effective action.

$$S_{\mathrm{eff}}[P] = \int d^2x \left(\int_{S^1} dt \rho_s \, \mathrm{Tr}(\partial_i P \partial_i P) - 2m \int_{H^2} dt d\tau \, \mathrm{Tr}\left(P\partial_t P \partial_\tau P - P\partial_\tau P \partial_t P\right) \right) \tag{3.42}$$

The choice of parameters in the microscopic system controls the parameters of the EFT, hence we want to express the spin stiffness ρ_s and the magnetization density m in terms of J and n. The results are derived in Appendix B.3. We find

$$m = \frac{n}{2a^2}, \qquad \rho_s = \frac{Jn^2}{4}. \tag{3.43}$$

3.3. Dimensional reduction

The EFT derived in (3.42) already bears a lot of similarity to the $\mathbb{C}\mathrm{P}(N-1)$ model. However, to truly recover the desired degrees of freedom, the EFT has to be reduced in its time dimension. This section is based on [22]. Thus far we have worked with a lattice of infinite extent $L, \beta = \infty$, the spontaneous symmetry breaking in this case gives rise to massless Goldstone bosons. It follows that the EFT possesses an infinite correlation length ξ. If the extent of the

time dimension β is taken to be finite, then the existence of massless Goldstone bosons is prohibited by the Mermin-Wagner-theorem. A finite β therefore gives rise to a non-perturbative, dynamically generated mass m that is the inverse of the correlation length $\xi = 1/m$. The dispersion relation is used to find the space-equivalent a' to β

$$\frac{1}{\beta} = \frac{Jn}{2}\frac{1}{(a')^2} \quad \Rightarrow \quad a' = \sqrt{\frac{\beta Jn}{2}}a. \tag{3.44}$$

We assume that the finite correlation length respects the relation

$$\xi \gg a'. \tag{3.45}$$

This will prove to be correct a-posteriori. This condition implies, that the field $P(x)$ doesn't change significantly within a cube of dimensions $a' \times a' \times \beta$. One can use block spin renormalization group transformations to map the $(2+1)$-dimensional EFT onto a 2-dimensional lattice as depicted in Figure 3.3. This was done in the two-dimensional $O(3)$ model, which is equivalent to the $\mathbb{C}\mathrm{P}(1)$ model, by Hasenfratz and Niedermayer [14]. A different, more intuitive approach was chosen by [11]. They argue, that the partition function is dominated by the t-independent field configurations with

$$\partial_t P(t,x) = 0. \tag{3.46}$$

This can also be derived via a Fourier analysis in the time dimension, where the zero-momentum mode is found to be dominant. We obtain the effective action of a two-dimensional system

$$S_{\text{eff}}[P] = \int d^2x \beta\rho_s \,\mathrm{Tr}(\partial_i P \partial_i P), \tag{3.47}$$

which takes the form of the $\mathbb{C}\mathrm{P}(N-1)$ action

$$S[P] = \int d^2x \frac{1}{g^2}\,\mathrm{Tr}(\partial_\mu P \partial_\mu P) \tag{3.48}$$

with the bare coupling at the scale a' being $1/g(a')^2 = \beta\rho_s$. The two expressions differ in that the effective theory lives in two space dimensions, while the $\mathbb{C}\mathrm{P}(N-1)$ model lives in $(1+1)$ dimensions. However, in Euclidean time we may simply reinterpret one of the lattice dimensions as the imaginary time. What was previously seen as the time dimension in the microscopic model now becomes some unphysical extra dimension. Let us now check that (3.45) is indeed satisfied. The $\mathbb{C}\mathrm{P}(N-1)$ model in two dimensions is asymptotically free, consequently one may use the universal asymptotic formula for the correlation length [19]

$$\frac{a'}{\xi} = C\left(\beta_1 g^2\right)^{-\frac{\beta_2}{\beta_1^2}} \exp\left(-\frac{1}{\beta_1 g^2}\right)(1+\mathcal{O}(g^2)). \tag{3.49}$$

The 1- and 2-loop coefficients for this model are $\beta_1 = \frac{N}{4\pi}$ and $\beta_2 = \frac{8\pi}{N^2}$ [26]. Therefore at leading order, ξ is given by

$$\frac{\xi}{a'} = C_N \left(\frac{N}{4\pi\beta\rho_s}\right)^{\frac{2}{N}} \exp\left(\frac{4\pi\beta\rho_s}{N}\right). \tag{3.50}$$

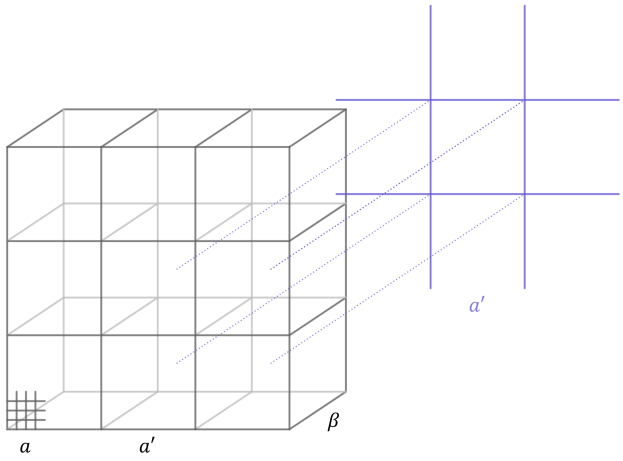


Figure 3.3.: Reduction of the (2+1)-dimensional EFT in the extra dimension β leads to a 2-dimensional lattice field theory with spacing a'.

Calculations by B. Beard [24] and experience with $O(N)$ models suggest that the model specific constant C_N doesn't change significantly with N. Though it may seem counterintuitive, $\xi \gg a'$ is satisfied for sufficiently large values of β. The continuum limit is taken by letting $\beta \to \infty$. We have mentioned in the previous chapter, that the so-called 't Hooft coupling

$$\frac{1}{g'^2_{\mathrm{tH}}} = \frac{1}{Ng^2} = \frac{\beta\rho_s}{N} = \frac{\beta J n^2}{4N} \tag{3.51}$$

must be kept fixed in the large N limit. Since this is a bare coupling, it is not constant when taking the continuum limit. To check whether dimensional reduction is still possible at large N, we express ξ/a' in terms of g'_{tH}

$$\frac{\xi}{a'} = C_N \underbrace{\left(\frac{g'^2_{\mathrm{tH}}}{4\pi}\right)^{\frac{2}{N}}}_{\to 1} \exp\left(\frac{4\pi}{g'^2_{\mathrm{tH}}}\right). \tag{3.52}$$

Since the exponent is invariant against a change of N and the remaining factors are well behaved, we find that $\xi \gg a'$ indeed holds for large N. However, we can only retrieve information about ξ from a finite-size $\mathrm{SU}(N)$ lattice model if a' does not diverge. It follows, that the scaling of βJ and n must be $\beta J \sim 1/N$ and $n \sim N$. D-Theory relies on ensembles of spins to generate the continuous values of the field variable P. This naturally works only for $L > a'$. Finite size lattices also give rise to infrared (IR) effects and we can only hope to recover accurate values of observables like the correlation length if $L \gg \xi$. If all these conditions are met, we have shown that the $d = (1+1)$ $\mathbb{C}\mathrm{P}(N-1)$ non-linear σ model is indeed recovered as a low-energy effective theory of the $d = (2+1)$ ferromagnetic $\mathrm{SU}(N)$ quantum Heisenberg model.

3.3.1. C, P, and T symmetries

We should check whether the $\mathbb{C}\mathrm{P}(N-1)$ model and the $\mathrm{SU}(N)$ ferromagnet share the same behavior under charge conjugation (C), parity transformation (P), and time reversal (T). In order

to derive the transformation behavior of the fields, it is more convenient to work in the z-notation. Under all three transformations the expression $\int d^2x A_\mu j_\mu$ must remain invariant. Here $j_\mu = (\rho, \vec{j})$ is a conserved current consisting of some charge density ρ and a current $\vec{j}$. The known transformation behavior of j_μ implies the behavior of A_μ

$$\begin{aligned}
{}^C j_\mu(t,x) &= \begin{pmatrix} -\rho(t,x) \\ -\vec{j}(t,x) \end{pmatrix}, & {}^C A_\mu(t,x) &= \begin{pmatrix} -A_t(t,x) \\ -A_x(t,x) \end{pmatrix}, \\
{}^P j_\mu(t,x) &= \begin{pmatrix} \rho(t,-x) \\ -\vec{j}(t,-x) \end{pmatrix}, & {}^P A_\mu(t,x) &= \begin{pmatrix} A_t(t,-x) \\ -A_x(t,-x) \end{pmatrix}, \\
{}^T j_\mu(t,x) &= \begin{pmatrix} \rho(-t,x) \\ -\vec{j}(-t,x) \end{pmatrix}, & {}^T A_\mu(t,x) &= \begin{pmatrix} A_t(-t,x) \\ -A_x(-t,x) \end{pmatrix}.
\end{aligned} \tag{3.53}$$

This, combined with the previously derived identity

$$A_\mu = \frac{i}{2}\left(\overline{z}_\alpha \partial_\mu z_\alpha - (\partial_\mu \overline{z}_\alpha) z_\alpha\right), \tag{3.54}$$

provides the transformation behavior of the complex vectors z and consequently also P

$$\begin{aligned}
{}^C z_\alpha(t,x) &= \overline{z}_\alpha(t,x), & {}^C P(t,x) &= \overline{P}(t,x) = P(t,x)^\mathsf{T}, \\
{}^P z_\alpha(t,x) &= z_\alpha(t,-x), & {}^P P(t,x) &= P(t,-x), \\
{}^T z_\alpha(t,x) &= \overline{z}_\alpha(-t,x), & {}^T P(t,x) &= \overline{P}(-t,x) = P(-t,x)^\mathsf{T}.
\end{aligned} \tag{3.55}$$

The action

$$S[P] = \int d^2x \frac{1}{g^2} \operatorname{Tr}\left(\partial_\mu P \partial_\mu P\right), \tag{3.56}$$

is clearly invariant under all three transformations. The topological charge term

$$Q[P] = \frac{i}{4\pi} \int d^2x \epsilon_{\mu\nu} \operatorname{Tr}\left(P \partial_\mu P \partial_\nu P\right), \tag{3.57}$$

however breaks both the C- and P-symmetry

$${}^C Q = -Q, \qquad {}^P Q = -Q, \qquad {}^T Q = Q. \tag{3.58}$$

Because the ferromagnetic $\mathrm{SU}(N)$ quantum Heisenberg model provides a quantization of the $\mathbb{C}\mathrm{P}(N-1)$ model at vacuum angle $\theta = 0$ we expect it to respect all three symmetries. The P and T transformations correspond to a reflection along one of the axes due to the reinterpretation of the lattice dimensions in the dimensional reduction step. The lattice neighbors $\langle xy \rangle$ in the Hamiltonian do not change under such transformations, consequently the system is indeed invariant under parity and time reversal. Under charge conjugation the representation of the generators in the Hamiltonian is changed to its conjugate $\tilde{T}^a = -(T^a)^*$. For $N > 2$ the $\mathrm{SU}(N)$ models have complex representations. Therefore, contrary to our initial expectation, the C-symmetry is explicitly broken. Note also that the symmetry breaking pattern changes under charge conjugation. This behavior also carries over to the EFT. There the symmetry breaking is controlled by the Wess-Zumino-Witten term

$$S_{\mathrm{WZW}}([P]) = \frac{1}{i2\pi} \int d^2x\, dt\, d\tau\ \operatorname{Tr}\left(P \partial_t P \partial_\tau P - P \partial_\tau P \partial_t\right). \tag{3.59}$$

Again it is important to remember that not only τ but also t denotes not the time but an extra dimension. The C-symmetry is thus restored in the process of dimensional reduction. However, in the EFT there are still subleading terms that break C-invariance. One example of such a term is

$$\frac{1}{\beta^2}\epsilon_{\mu\nu}\,\mathrm{Tr}\left(P\partial_\mu P\partial_\nu P\partial_\sigma P\partial_\sigma P + P\partial_\sigma P\partial_\sigma P\partial_\mu P\partial_\nu P\right). \tag{3.60}$$

One might find C-violating effects at high energies due to the underlying microscopic model. Such terms in the dimensionally reduced two-dimensional $\mathbb{C}\mathrm{P}(N-1)$ model are irrelevant in the renormalization group sense.

4. Cluster representation of $\mathbb{C}\mathrm{P}(N-1)$ models

In an $\mathrm{SU}(N)$ quantum ferromagnet with a magnetization density of $n/2$, the number of possible flavor states on each lattice site at large N presents a problem in computational physics. A possible approach to creating efficient Monte Carlo algorithms is through the implementation of clusters. The first cluster algorithm was introduced by Swendsen and Wang [10] in 1987. In 1989 Wolff [12] developed an improved algorithm that decreases the autocorrelation time by updating multiple clusters of spins at once. Applications of cluster algorithms to the $\mathbb{C}\mathrm{P}(N-1)$ models via D-Theory can be found in [23, 24]. The goal of this chapter is not to perform Monte Carlo calculations, hence there will be no detailed introduction to this topic. We merely use concepts from the Monte Carlo method in order to convince ourselves of the correctness of the cluster formulation of the $\mathrm{SU}(N)$ ferromagnet. This chapter covers the implementation of clusters with the goal of introducing an alternate, large N friendly way to describe spin configurations of the lattice. With this cluster description we work towards a new way of directly solving $(2+1)$-dimensional ferromagnetic $\mathrm{SU}(N)$ quantum Heisenberg models, and consequently $(1+1)$-dimensional $\mathbb{C}\mathrm{P}(N-1)$ models. The introduction initially follows the PhD thesis of Riederer [25] and then implements the concept used by Kawashima and Gubernatis [16] for the generalization to higher spins.

4.1. Cluster representation of $\mathrm{SU}(N)$ quantum ferromagnets

The Monte Carlo method can be used as a way of calculating expectation values by importance sampling of configuration space. The partition function in the path integral formalism is an extremely high-dimensional integral. In the infinite volume or number of flavors the integral even becomes infinite-dimensional. Configurations in the path integral of the partition function are generated numerically, their Boltzmann weight factor acting as a quantification of their importance. The series of configurations that contribute to the expectation value in question is called a Markov chain. An algorithm has to generate the Markov chain in a way that is ergodic, i.e. any field configuration can be reached in a finite number of steps from any arbitrary initial configuration. For the distribution of configurations in the chain to converge uniquely to the desired Boltzmann distribution it is sufficient but not necessary that the algorithm respects detailed balance

$$\exp\left(-S[q]\right) w[q,q'] = \exp\left(-S[q']\right) w[q',q]. \tag{4.1}$$

Here $w[q,q']$ is the probability density for the system in configuration $[q]$ to transition to a configuration $[q']$, and $\exp(-S[q])$ is the Boltzmann weight factor of the configuration $[q]$.

Let us construct said path integral representation of

$$Z = \mathrm{Tr} \exp\left(-\beta H\right), \qquad H = -J \sum_{\langle xy \rangle} T_x^a T_y^a, \tag{4.2}$$

by discretizing the system in the Euclidean time dimension of extent β. For simplicity's sake we return to interpreting this as a time dimension as opposed to some extra dimension. The transition to the path integral representation is performed by insertion of a full basis after discrete time-steps of size ϵ, where $\beta = M\epsilon$. Directly working in arbitrary symmetric representations of T_x^a would greatly complicate the computation of the transfer matrix elements. We start by introducing clusters in a system of fundamental spins $n = 1$ and arbitrary N, ignoring the relation between the two. Only later will we make the generalization to arbitrary n. The discretization in time produces the partition function

$$Z = \left(\prod_{i=1}^{M} \sum_{q_i} \right) \langle q_1 | \exp(-\epsilon H) | q_2 \rangle \langle q_2 | \exp(-\epsilon H) | q_3 \rangle \ldots \langle q_M | \exp(-\epsilon H) | q_1 \rangle , \tag{4.3}$$

where

$$\sum_{q_i} |q_i\rangle \langle q_i| = \mathbb{1}. \tag{4.4}$$

Each $|q_i\rangle$ denotes a spin state of the complete lattice at a fixed point in time. The goal now is to manipulate the Hamiltonian in such a way that each discrete time step is decomposed into successive sub-steps of plaquette interactions that do not interfere with each other. This is done by application of the Suzuki-Trotter expansion [15]. We prepare the Hamiltonian by splitting it into subsets of terms that commute with each other

$$H = \sum_{i=1}^{2d} H_i, \tag{4.5}$$

where H_i is defined by

$$H_i = - \sum_{x | x_i \text{ even}} J T_x^a T_{x+\hat{i}}^a \qquad H_{i+d} = - \sum_{x | x_i \text{ odd}} J T_x^a T_{x+\hat{i}}^a \tag{4.6}$$

for $i \in \{1, \ldots, d\}$ with the notation $\hat{i} = \vec{e_i}$. This decomposition requires L to be even. We separate the exponential and insert full basis sets in between.

$$\begin{aligned} Z = \lim_{\epsilon \to 0} \sum_{[q]} & \langle q_1 | \exp\left(-\epsilon H_1\right) | q_2 \rangle \langle q_2 | \exp\left(-\epsilon H_2\right) | q_3 \rangle \ldots \langle q_{2d} | \exp\left(-\epsilon H_{2d}\right) | q_{2d+1} \rangle \\ & \cdot \langle q_{2d+1} | \exp\left(-\epsilon H_1\right) | q_{2d+2} \rangle \ldots \end{aligned} \tag{4.7}$$

The sum extends over all lattice configurations $[q] = (|q_1\rangle, \ldots, |q_{M2d}\rangle)$. This leads to an error of order ϵ^2 that only disappears in the continuum time limit $\epsilon \to 0$, $M \to \infty$. The Suzuki-Trotter expansion splits each time step into $2d$ sub-steps where interactions only take place on what we call the active plaquettes. Figure 4.1 illustrates this for $d = 1$. It follows, that the partition function is simply the product of the Boltzmann-weight factors of active plaquettes.

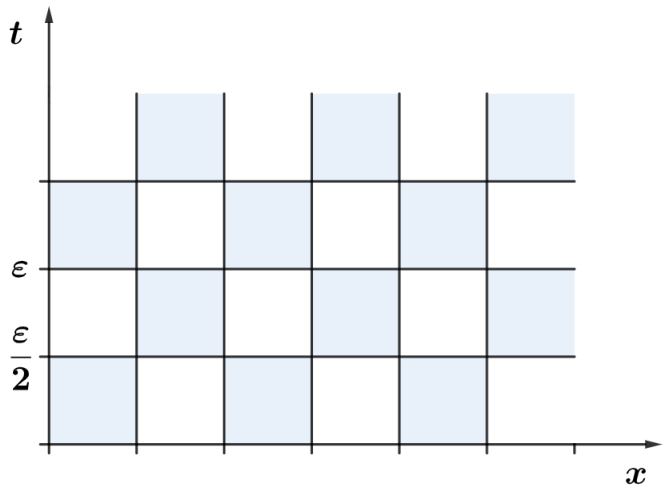


Figure 4.1.: The Suzuki-Trotter expansion of a $(d+1)$-dimensional lattice separates each discrete time step of $\Delta t = \epsilon$ into $2d$ sub-steps, in which interactions occur on the active plaquettes colored white in this example for $d = 1$.

We introduce the notation $q_{(x,t)}$ for the spin on site x at time $\beta t/(M2d)$ and rewrite the partition function as

$$Z = \sum_{[q]} \exp(-S[q]), \tag{4.8}$$

where the Boltzmann weight factor is defined as

$$\begin{aligned}\exp(S[q]) = \prod_{j=0}^{M-1}\prod_{i=1}^{d} &\left(\prod_{\substack{x|x_i \text{ even} \\ t=2dj+i-1}} \exp(-S(q_{(x,t)}, q_{(x+\hat{i},t)}, q_{(x,t+1)}, q_{(x+\hat{i},t+1)})) \right) \\ &\times \left(\prod_{\substack{x|x_i \text{ odd} \\ t=2dj+d+i-1}} \exp(-S(q_{(x,t)}, q_{(x+\hat{i},t)}, q_{(x,t+1)}, q_{(x+\hat{i},t+1)})) \right).\end{aligned} \tag{4.9}$$

The weight of a single active plaquette is an element of the transfer matrix

$$\exp(-S(q^a_{(x,t)}, q^b_{(x+\hat{i},t)}, q^c_{(x,t+1)}, q^d_{(x+\hat{i},t+1)})) = \langle q^a_{(x,t)}, q^b_{(x+\hat{i},t)}| \exp(-\epsilon T^a_x T^a_{x+\hat{i}}) \, |q^c_{(x,t+1)}, q^d_{(x+\hat{i},t+1)}\rangle \,. \tag{4.10}$$

In this notation the index $a \in \{1, \ldots, N\}$ defines the spin flavor. Let us calculate the transfer matrix elements. Two neighboring spins at equal time $q^a_{(x,t)}$ and $q^b_{(x+\hat{i},t)}$ both transform in the fundamental representation $\{N\}$. The coupling of the two can be decomposed into $N(N+1)/2$ symmetric and $N(N-1)/2$ antisymmetric states.

$$\square \otimes \square = \square\square \oplus \begin{array}{l}\square \\ \square\end{array} \tag{4.11}$$

The symmetrized and anti-symmetrized states take the form

$$\begin{aligned}|aa\rangle &= |q^a_{(x,t)}, q^a_{(x+\hat{i},t)}\rangle \,, \\ |ab\rangle_s &= \frac{1}{\sqrt{2}}(|q^a_{(x,t)}, q^b_{(x+\hat{i},t)}\rangle + |q^b_{(x,t)}, q^a_{(x+\hat{i},t)}\rangle)\end{aligned} \tag{4.12}$$

and

$$|ab\rangle_a = \frac{1}{\sqrt{2}}(|q^a_{(x,t)}, q^b_{(x+\hat{i},t)}\rangle - |q^b_{(x,t)}, q^a_{(x+\hat{i},t)}\rangle) \tag{4.13}$$

respectively. Consequently we can decompose the basis states into their symmetric and anti-symmetric parts

$$\begin{aligned} |q^a_{(x,t)}, q^a_{(x+\hat{i},t)}\rangle &= |aa\rangle_s\,, \\ |q^a_{(x,t)}, q^b_{(x+\hat{i},t)}\rangle &= \frac{1}{\sqrt{2}}(|ab\rangle_s + |ab\rangle_a), \qquad a \neq b, \\ |q^b_{(x,t)}, q^a_{(x+\hat{i},t)}\rangle &= \frac{1}{\sqrt{2}}(|ab\rangle_s - |ab\rangle_a), \qquad a \neq b. \end{aligned} \tag{4.14}$$

We recognize the quadratic Casimir operator in the transfer matrix

$$\tau_{x,x+\hat{i}} = \exp\left(\epsilon J T^a_x T^a_{x+\hat{i}}\right) = \exp\left(\frac{\epsilon J}{2}\left((T^a_x + T^a_{x+\hat{i}})^2 - (T^a_x)^2 - (T^a_{x+\hat{i}})^2\right)\right), \tag{4.15}$$

and use the relations (3.13) and (3.14) to find

$$\begin{aligned} \langle q^a_{(x,t)}, q^a_{(x+\hat{i},t)}|\,\tau_{(x,x+\hat{i})}\,|q^a_{(x,t+1)}, q^a_{(x+\hat{i},t+1)}\rangle &= \exp\left(\epsilon J\frac{N-1}{2N}\right), \\ \langle q^a_{(x,t)}, q^b_{(x+\hat{i},t)}|\,\tau_{(x,x+\hat{i})}\,|q^a_{(x,t+1)}, q^b_{(x+\hat{i},t+1)}\rangle &= \frac{1}{2}\left(\exp\left(\epsilon J\frac{N-1}{2N}\right) + \exp\left(-\epsilon J\frac{N+1}{2N}\right)\right), \\ \langle q^b_{(x,t)}, q^a_{(x+\hat{i},t)}|\,\tau_{(x,x+\hat{i})}\,|q^b_{(x,t+1)}, q^a_{(x+\hat{i},t+1)}\rangle &= \frac{1}{2}\left(\exp\left(\epsilon J\frac{N-1}{2N}\right) - \exp\left(-\epsilon J\frac{N+1}{2N}\right)\right). \end{aligned} \tag{4.16}$$

All other transfer matrix elements reduce to zero. The physics of the system are invariant under the change of an overall factor of the Boltzmann weights. We multiply the transfer matrix elements by the factor

$$\exp\left(-\epsilon J\left(\frac{N-1}{2N} - \frac{1}{2}\right)\right)\frac{1}{\cosh\left(\frac{\epsilon J}{2}\right)}. \tag{4.17}$$

The resulting Boltzmann weight factors of plaquettes are depicted in Table 4.1. In order to move between valid configurations in the Markov chain, whole clusters of spins have to be flipped at a time. In a single-cluster algorithm step one starts with an initial spin configuration and constructs probabilistically a cluster of spins of the same flavor in accordance with detailed balance. This works by picking an initial spin at $t = 0$ and probabilistically choosing a bond on each plaquette that is visited such that detailed balance is respected. The bonds define the cluster flow through the lattice in space and time. The possible bond types on an active plaquette and their probabilities of being chosen in the cluster building process are shown in Table 4.2. We find that a cluster can never move backwards in time. The boundary conditions imply, that a cluster can loop around the edges of the lattice if it reaches $t = \beta$, $x_i = 1$ or $x_i = L$. The cluster is completed once it connects back to the initial site at $t = 0$. In order to build the next spin configuration in the Markov-chain the whole cluster is then assigned a random flavor. Figure 4.2 shows an example of a cluster for $d = 1$. The concept can be taken a step further in the form

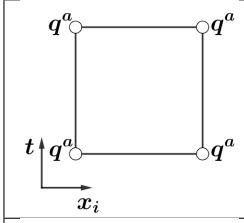
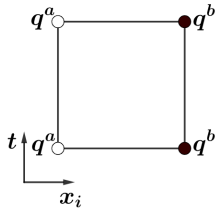
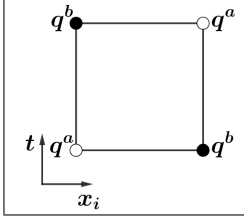
	$\exp\left(-S(q^a_{(x,t)}, q^a_{(x+\hat{i},t)}, q^a_{(x,t+1)}, q^a_{(x+\hat{i},t+1)})\right) = \exp\left(\frac{\epsilon J}{2}\right)\cosh\left(\frac{\epsilon J}{2}\right)^{-1}$
	$\exp\left(-S(q^a_{(x,t)}, q^b_{(x+\hat{i},t)}, q^a_{(x,t+1)}, q^b_{(x+\hat{i},t+1)})\right) = 1$
	$\exp\left(-S(q^a_{(x,t)}, q^b_{(x+\hat{i},t)}, q^b_{(x,t+1)}, q^a_{(x+\hat{-i},t+1)})\right) = \tanh\left(\frac{\epsilon J}{2}\right)$

Table 4.1.: All plaquette configurations with non-zero weights are listed here. The fundamental spins q^a and q^b take different flavors, $q^a \neq q^b$.

of multi-cluster algorithms, in which the complete $d+1$ dimensional lattice is covered with clusters. At this point it has to be noted that the Boltzmann factor of the plaquette $q^a q^a \to q^a q^a$, that can take both bond types, is exactly the sum of the other two Boltzmann factors. It is possible to split up the spin configurations into spin-bond configurations by attributing to the $q^a q^a \to q^a q^a$ plaquette the weight

$$\exp\left(-S(q^a_{(x,t)}, q^a_{(x+\hat{i},t)}, q^a_{(x,t+1)}, q^a_{(x+\hat{i},t+1)}; \mathrm{X})\right) = \tanh(\epsilon J/2) \tag{4.18}$$

if it carries a cross-bond, and

$$\exp\left(-S(q^a_{(x,t)}, q^a_{(x+\hat{i},t)}, q^a_{(x,t+1)}, q^a_{(x+\hat{i},t+1)}; |\ |)\right) = 1 \tag{4.19}$$

if it carries a parallel-bond. The path integral in the partition function then sums over all valid combinations of spin and bond configurations

$$Z = \sum_{[q,b]} \exp(-S[q,b]) = \sum_{[q,b]} \prod_{\square} \exp(-S(q_\square, b_\square)). \tag{4.20}$$

Here the product ranges over all active plaquettes denoted by the $\square$-symbol. The spins and bond types on the plaquette are $q_\square$ and $b_\square$, respectively. By considering that all spins on a cluster need to be of the same flavor, we find that each bond configuration $[b]$ appears in $N^{\#\mathcal{C}}$ spin configurations, where $\#\mathcal{C}[b]$ denotes the number of clusters in the bond configuration. However, the Boltzmann factor of these configurations does not depend on the flavor of the spins; therefore, we are able to integrate out the spins from the partition function

$$Z = \sum_{[b]} \exp(-S[b]) N^{\#\mathcal{C}[b]}, \qquad \exp(-S[b]) = \prod_{\square} \exp(-S(b_\square)). \tag{4.21}$$

In the large N limit this provides a great advantage since the dimensionality of the path integral now only depends on the lattice volume and its segmentation in the Euclidean dimension.

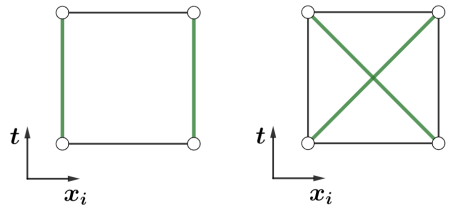
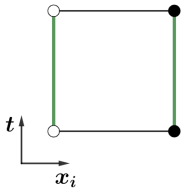
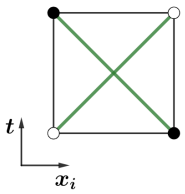
	$p(\vert\ \vert) = \frac{1}{1+\tanh(\epsilon J/2)}$ $p(\mathrm{X}) = \frac{\tanh(\epsilon J/2)}{1+\tanh(\epsilon J/2)}$
	$p(\vert\ \vert) = 1$ $p(\mathrm{X}) = 0$
	$p(\vert\ \vert) = 0$ $p(\mathrm{X}) = 1$

Table 4.2.: Bonds connect two spins of the same flavor in order to build clusters of spins. Changing the flavor of two connected spins must yield another plaquette configuration with non-zero weight. The bond types are denoted as cross-bond and parallel-bond.

4.1.1. Generalization to arbitrary n

In Section 3.3 we established that our interest lies with systems in which $n \sim N$ and $\beta J \sim 1/N$. We thus generalize the previous introduction of clusters to systems with arbitrary values of n. To do so we take inspiration from Kawashima and Gubernatis [16]. Instead of explicitly repeating the calculations for such representations, it is simpler to make use of the fact that spin states of a symmetric Young diagram of n boxes are simply the symmetrization of n fundamental spins. Accordingly, we are able to work with n fundamental spins on each site, that are symmetrized by a projector $\mathcal{P}_x$. This is done by introducing an extra lattice dimension, in which n layers of fundamental spin lattices are stacked without any notion of neighborhood. This means, that all n spins on a site interact equally with all n spins of a neighboring site. In the Hamiltonian this manifests as

$$H = -J \sum_{\langle x,y \rangle} \left(\sum_{i=1}^{n} T^a_{x,i} \right) \left(\sum_{j=1}^{n} T^a_{y,j} \right), \tag{4.22}$$

where i and j denote the layer. In this notation we impose a periodic boundary condition on the layer index $T^a_{x,i+n} = T^a_{x,i}$. Now the symmetrization operator $\mathcal{P}$ appears in the partition function

$$Z = \mathrm{Tr}\left(\exp\left(-\beta H\right)\mathcal{P}\right), \qquad \mathcal{P} = \prod_x \mathcal{P}_x. \tag{4.23}$$

The projectors $\mathcal{P}_x$ are given by

$$\mathcal{P}_x = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma. \tag{4.24}$$

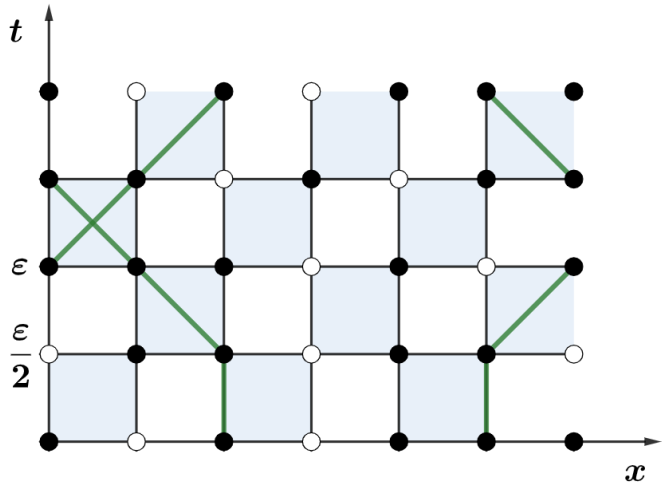


Figure 4.2.: In a single-cluster algorithm a cluster is built probabilistically on a given spin configuration. This shows an example of a cluster on a lattice with $d = 1$, $L = 6$, and $\beta = 2\epsilon$

As an explicit example let us consider $N = 3$ and $n = 2$. We find

$$\mathcal{P}_x = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \text{ in the basis } \begin{pmatrix} uu \\ ud \\ du \\ dd \\ ds \\ sd \\ ss \\ us \\ su \end{pmatrix}. \tag{4.25}$$

We observe, that the symmetrization operator commutes with the Hamiltonian,

$$\begin{aligned}
[H,\mathcal{P}] &= \frac{-J}{n}\sum_{\langle x,y\rangle}\sum_{i,j,k,l=1}^{n}\left[T^a_{x,i}T^a_{y,j},\prod_z T^a_{z,k}T^a_{z,l}\right]\\
&= \frac{-J}{n}\sum_{\langle x,y\rangle}\sum_{i,j=1}^{n}\sum_{k_1,l_1=1}^{n}\sum_{k_2,l_2=1}^{n}\left[T^a_{x,i}T^a_{y,j},T^b_{x,k_1}T^b_{x,l_1}T^c_{y,k_2}T^c_{y,l_2}\right]\sum_{k_3,l_3=1}^{n}\prod_{z\neq x,y}T^d_{z,k_3}T^d_{z,l_3}\\
&= \frac{-J}{n}\sum_{\langle x,y\rangle}\sum_{i,j=1}^{n}\sum_{k_1,l_1=1}^{n}\sum_{k_2,l_2=1}^{n}\Big(\left[T^a_{x,i},T^b_{x,k_1}T^b_{x,l_1}\right]T^a_{y,j}T^c_{y,k_2}T^c_{y,l_2}\\
&\quad + T^a_{x,i}T^b_{x,k_1}T^b_{x,l_1}\left[T^a_{y,j},T^c_{y,k_2}T^c_{y,l_2}\right]\Big)\sum_{k_3,l_3=1}^{n}\prod_{z\neq x,y}T^d_{z,k_3}T^d_{z,l_3}\\
&= \frac{-J}{n}\sum_{\langle x,y\rangle}\sum_{i,j=1}^{n}\sum_{k_1,l_1=1}^{n}\sum_{k_2,l_2=1}^{n}\Big(if^{abe}\left(\delta_{i,k_1}T^e_{x,k_1}T^b_{x,l_1}+\delta_{i,l_1}T^b_{x,k_1}T^e_{x,l_1}\right)T^a_{y,j}T^c_{y,k_2}T^c_{y,l_2}\\
&\quad + T^a_{x,i}T^b_{x,k_1}T^b_{x,l_1}if^{ace}\left(\delta_{j,k_1}T^e_{y,k_2}T^c_{y,l_2}+\delta_{j,l_2}T^c_{y,k_2}T^e_{y,l_2}\right)\Big)\sum_{k_3,l_3=1}^{n}\prod_{z\neq x,y}T^d_{z,k_3}T^d_{z,l_3}\\
&= 0.
\end{aligned}\tag{4.26}$$

In the last step we have used, that f^{abe} is antisymmetric under the exchange of b and e, while the term in brackets, that it is multiplied by, is symmetric. The term containing f^{ace} disappears analogously. Consequently, it is sufficient to only symmetrize the system at $t=\beta$. Let us take a closer look at the partition function. We again start by discretizing the Euclidean dimension and separating the Hamiltonian into $2d$ terms that are defined by (4.6) with $T^a_x=\sum_{i=1}^n T^a_{x,i}$. Each H_i is then decomposed further into n expressions consisting of terms that commute with each other.

$$H_{i,j}=\sum_{x|x_i\text{ even}}-J\sum_{k=0}^{n-1}T^a_{x,k}T^a_{x+\hat{i},k+j},\qquad H_{i+d,j}=\sum_{x|x_i\text{ odd}}-J\sum_{k=0}^{n-1}T^a_{x,k}T^a_{x+\hat{i},k+j},\tag{4.27}$$

where $j\in\{1,\dots,n\}$. We then continue with the Suzuki-Trotter expansion just as we had done before. The Boltzmann factor of a spin configuration is again the product of the weights of the active plaquettes, but now an additional factor $\langle q_{t=\beta}|\,\mathcal{P}\,|q_{t=0}\rangle$ comes from the symmetrization. Each plaquette still connects four fundamental spins, therefore the same plaquette configurations with the same weights appear as in the $n=1$ case. Let us take a look at the Boltzmann weights and cluster flow in the symmetrization step between $t=\beta$ and $t=0$. In Table 4.3 we treat the example with $N=2$ and $n=2$. The weights of the spin configurations are split equally into weights of spin-bond configurations, such that

$$\mathcal{P}_x(q_{(x,1,\beta)},\dots,q_{(x,n,\beta)},q_{(x,1,0)},\dots,q_{(x,n,0)};b)=\frac{1}{n!},\tag{4.28}$$

where b is a bond type that is compatible with the spins involved, and the second index of q denotes the layer. For incompatible combinations, the weight vanishes. This way of splitting

satisfies

$$\sum_b \mathcal{P}_x(q_{(x,1,\beta)},\dots,q_{(x,n,\beta)},q_{(x,1,0)},\dots,q_{(x,n,0)};b) = \mathcal{P}_x(q_{(x,1,\beta)},\dots,q_{(x,n,\beta)},q_{(x,1,0)},\dots,q_{(x,n,0)}). \tag{4.29}$$

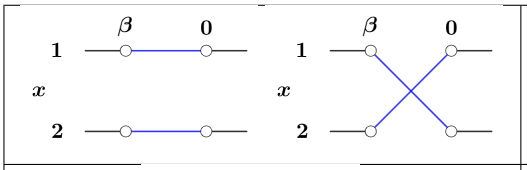
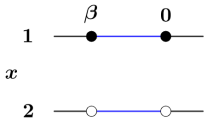
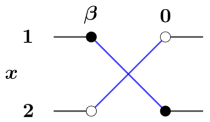
	$\langle q^a q^a \vert \mathcal{P}_x \vert q^a q^a \rangle = 1$
	$\langle q^b q^a \vert \mathcal{P}_x \vert q^b q^a \rangle = \frac{1}{2}$
	$\langle q^b q^a \vert \mathcal{P}_x \vert q^a q^b \rangle = \frac{1}{2}$

Table 4.3.: The symmetrizing projector gives a non-zero contribution to all configurations where the spins on the layers at $(x,0)$ are a permutation of the spins at (x,β). This table shows the possible cluster flows in spin configurations of a system with $N=2$ and $n=2$. The black dots represent a different spin flavor than the white ones.

It is now possible to again express the partition function as a sum over spin-bond configurations.

$$Z = \sum_{[q,b]} \exp(-S[q,b]) \left(\frac{1}{n!}\right)^{L^d} = \sum_{[q,b]} \prod_{\square} \exp(-S(q_\square, b_\square)) \left(\frac{1}{n!}\right)^{L^d} \tag{4.30}$$

The symmetrization adds a factor $1/n!$ for every lattice site that can be ignored as an overall factor of the partition function. More importantly the symmetrization affects the number of configurations in the path integral. Consequently, the spins can be integrated out of the partition function analogously to the $n=1$ system, and we end up with

$$Z = \sum_{[b]} \exp(-S[b]) N^{\#\mathcal{C}[b]}, \qquad \exp(-S[b]) = \prod_{\square} \exp(-S(b_\square)). \tag{4.31}$$

4.1.2. Continuous time limit

The Suzuki-Trotter expansion generates an error of order $\mathcal{O}(\epsilon)$ that only disappears in the continuum time limit $\epsilon \to 0$. In the previous calculations we have chosen the overall factor of plaquette Boltzmann weights such that parallel-bonds carry a weight of 1. A bond configuration is thus completely defined by the number and position of cross-bonds and the permutation

among layers of each site at $t = \beta$

$$Z = \underbrace{\left(\prod_{i=1}^{n^2dL^d}\left(\sum_{n_i=0}^{M}\left(\sum_{t_1^i=1}^{M-n_i+1}\sum_{t_2^i=t_1^i+1}^{M-n_i+2}\cdots\sum_{t_{n_i}^i=t_{n_i-1}^i+1}^{M}\right)\right)\right)\left(\prod_x\sum_{\sigma_x\in S_n}\right)}_{\sum_{[b]}} \times \tanh\left(\frac{\epsilon J}{2}\right)^{\sum_{i=1}^{n^2dL^d} n_i} N^{\#\mathcal{C}[b]}. \tag{4.32}$$

In this notation we index the channels between sites and layers in which cross-bonds can appear with $i \in \{1, \ldots, n^2dL^d\}$. For each channel the sum over the number of cross-bonds n_i on channel i is taken. This number is limited by the number of discrete time steps $M = \beta/\epsilon$. The innermost brackets contain the sum over the positions of the cross-bonds in the time dimension, where t_m^i is the number of the plaquette on which the m-th cross-bond of channel i is situated. Finally, the path integral contains for every lattice site, marked by a d-dimensional vector x, the sum over all possible permutations among the n layers at said site. The weight of a configuration gains a factor $\tanh(\epsilon J/2)$ for every cross-bond and a factor N for every cluster. The number of clusters is a highly complex non-local function for which we have no analytical formulation. In the continuous time limit we encounter

$$\epsilon\sum_{t_m^i=t_{m-1}^i}^{M-n_i+m} \xrightarrow{\epsilon\to 0} \int_{t_{m-1}^i}^{\beta} dt_m^i \tag{4.33}$$

and

$$\frac{1}{\epsilon}\tanh\left(\frac{\epsilon J}{2}\right) \xrightarrow{\epsilon\to 0} \frac{J}{2}. \tag{4.34}$$

The question needs to be asked, whether $\#\mathcal{C}[b]$ is well defined at the limit $\epsilon \to 0$. Problems arise if there is a finite chance of an infinite number of transitions appearing in a channel while β is finite. We remind ourselves, that the 't Hooft coupling demands $\beta J \sim 1/N$. The natural choice is to let β be finite and independent of N and $J \sim 1/N$. The continuous time limit is therefore well defined in the large N limit and yields the partition function

$$\begin{aligned} Z &= \underbrace{\left(\prod_{i=1}^{n^2dL^d}\left(\sum_{n_i=0}^{\infty}\left(\int_0^\beta dt_1^i\int_{t_1^i}^\beta dt_2^i\cdots\int_{t_{n_i-1}}^\beta dt_{n_i}^i\right)\right)\right)\left(\prod_x\sum_{\sigma_x\in S_n}\right)}_{\int\mathcal{D}b}\left(\frac{J}{2}\right)^{\sum_{i=1}^{n^2dL^d} n_i} N^{\#\mathcal{C}[b]} \\ &= \left(\prod_{i=1}^{n^2dL^d}\left(\sum_{n_i=0}^{\infty}\left(\frac{1}{n_i!}\int_0^\beta dt_1^i\int_0^\beta dt_2^i\cdots\int_0^\beta dt_{n_i}^i\right)\right)\right)\left(\prod_x\sum_{\sigma_x\in S_n}\right)\left(\frac{J}{2}\right)^{\sum_{i=1}^{n^2dL^d} n_i} N^{\#\mathcal{C}[b]}. \end{aligned} \tag{4.35}$$

4.1.3. Large N limit

The prime advantage of the cluster formalism emerges at large values of N. It allows us to avoid dealing with an infinite number of spin flavors. In this formalism three things happen

in the large N limit. The number of layers n becomes infinite while the ratio n/N remains constant, the factor $N^{\#\mathcal{C}}$ becomes infinite and the weight of a crossing $J/2$ goes to zero. While $J \to 0$ implies that the chance to find a cross-bond in a specific channel is vanishingly small, one finds that the chance of a cluster, starting on a specific lattice site and layer, transitioning to an arbitrary neighboring site and layer is actually finite. Generally a cluster configuration has an infinite number of crossings due to the number of channels growing with N^2. Consider the probability p_m to encounter m cross-bonds between two specific layers of neighboring sites. Under the assumption that the sum over all configurations with m such cross-bonds will wash out the effect that m has on $\#\mathcal{C}$, we find

$$p_m = \frac{\frac{1}{m!}\left(\frac{\beta J}{2}\right)^m}{\sum_{n_i=0}^{\infty}\frac{1}{n_i!}\left(\frac{\beta J}{2}\right)^{n_i}}. \tag{4.36}$$

On average we expect to find

$$\sum_{m=0}^{\infty} m p_m = \frac{\beta J}{2} \tag{4.37}$$

cross-bonds on a channel. This implies a probability per unit time to encounter a cross-bond on a specific channel of $J/2$. If we define $J := j/n$ and consider that each site and layer is connected via $2n$ channels, then the probability per unit time of a cluster, starting on a specific lattice site and layer, transitioning to any neighboring site and layer is given by j.
Since the lattice at the large N limit grows infinitely large in the extra dimension of the n layers, it becomes impossible to generate or work with full cluster configurations. We need a different approach. Let us consider the system consisting of only a single lattice site. Without any neighbors, no interactions can take place in the time between $t = 0$ and β. A cluster configuration is therefore fully defined by the permutation at β. Consider the single-site model at arbitrary finite N and n. The cluster configurations are the elements of the symmetric group S_n. The elements σ are assigned the Boltzmann factor $N^{\#\mathcal{C}}$, where the number of clusters is equal to the number of partitions of σ and therefore only depends on the conjugacy class. Table 4.4 illustrates this for the example of $n = 3$. We are interested in the average size of

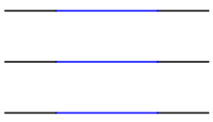
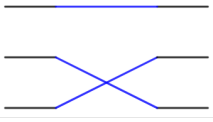
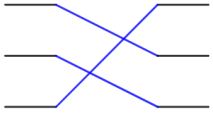
Partition	Size of the conjugacy class	Boltzmann factor	Example
$1+1+1$	1	N^3	
$2+1$	3	N^2	
3	2	N^1	

Table 4.4.: The cluster configurations of the single-site model in the symmetric representation of n boxes are given by the permutations of the symmetric group S_n at $t = \beta$. This table shows the Boltzmann factor of configurations in the example $n = 3$.

a cluster that is found by picking a random lattice site $\langle|\mathcal{C}|\rangle_{SC}$. The subscript SC stands for

"single cluster" and is added to avoid confusion with the average cluster size in a configuration. Explicit calculations of examples with small n suggest the recursive definition of the cluster size probability distribution

$$p(|\mathcal{C}|=m+1)=p(|\mathcal{C}|=m)\frac{n-m}{N+n-m-1}, \qquad p(|\mathcal{C}|=1)=\frac{N}{N+n-1}. \tag{4.38}$$

At large N this reduces to

$$p(|\mathcal{C}|=m)=\frac{N}{n}\left(\frac{n}{N+n}\right)^m, \tag{4.39}$$

which ultimately results in the expectation value

$$\langle|\mathcal{C}|\rangle_{SC}=\sum_{m=1}^{\infty} mp(|\mathcal{C}|=m)=1+\frac{n}{N}. \tag{4.40}$$

The expectation value for the size of a single cluster turns out to be finite despite the infinite number of layers. In chapter 5 we confirm this quantity to be finite in the two-site lattice model as well. This result suggests the possibility of extracting information about the $\mathrm{SU}(N)$ quantum ferromagnet at large N by analyzing the flow of a single cluster through the lattice. To this end we propose treating the propagation from $t=0$ to β as a diffusion process. Each time the cluster reaches β at the original site the probability of closing the cluster is evaluated. The idea of treating the cluster flow as a diffusion process is restricted to the large N limit since it is based on the assumption that a cluster never interacts with itself due to the infinite number of layers. By this we mean that if the cluster were to loop back onto itself at $t\neq 0$ it would force a transition of the cluster flow. If the cluster reaches β at its initial lattice site, there is a probability of the cluster to close with the right permutation of layers. The proposed approach would need to take into account that not closing the cluster leads to a decrease in $\#\mathcal{C}$. This also happens each time the cluster crosses β at another lattice site. The issue is further complicated by the fact that the inspected cluster limits the space for the remaining clusters in the system to propagate. So far our attempts at describing the probability to close the cluster have proven inadequate and further investigation is required.

5. $\mathrm{SU}(N)$ ferromagnet with two sites at $N = \infty$

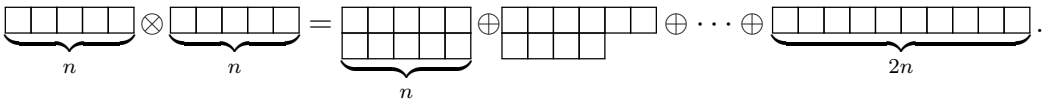
D-Theory has shown that we can use the 2-dimensional ferromagnetic $\mathrm{SU}(N)$ quantum Heisenberg model to solve the $(1+1)$-dimensional $\mathbb{C}\mathrm{P}(N-1)$ at any value of N. In the section on dimensional reduction we have pointed out that $L > \xi$ and a sufficiently large value of β are necessary to recover reasonable measurements through D-Theory. Still, considering the $\mathrm{SU}(N)$ spin model at small volumes has its uses. At low temperatures it turns out that the two-site problem can be directly solved at $N = \infty$ with a reasonable amount of effort. This can be used as a check for procedures that are more naturally extended to larger systems.
Let us consider the Hamiltonian of the $\mathrm{SU}(N)$ ferromagnet with two sites

$$H = -JT_x^a T_y^a. \tag{5.1}$$

We can be rewrite it to accommodate the use of Casimir operators

$$H = -\frac{J}{2}\left((T_x^a + T_y^a)^2 - (T_x^a)^2 - (T_y^a)^2\right). \tag{5.2}$$

We again consider a totally symmetric representation on each site. In order to diagonalize the complete system, these representations are coupled together and decomposed with the help of Young tableaux



(5.3)

This sum extends over all diagrams with $2n$ boxes in at most two rows. A diagram of an irreducible representation in (5.3) with m_i boxes in the i-th row has a degeneracy

$$D^N_{m_1,m_2} = \frac{(N+m_1-1)!}{(N-1)!}\,\frac{(N+m_2-2)!}{(N-1)!}\,\frac{m_1-m_2+1}{(m_1+1)!\,m_2!}. \tag{5.4}$$

From [20] we take the quadratic Casimir for arbitrary irreducible representations

$$C_2(m_1,m_2,\ldots,m_k) = \frac{nN}{2} + \sum_{i=1}^{k}\frac{m_i(m_i+1-2i)}{2} - \frac{n^2}{2N}, \tag{5.5}$$

where k denotes the number of rows in the diagram. Let us replace m_1 and m_2 by $n_1 = m_1 - m_2$ and $n_2 = m_2$. We find

$$D^N_{n_1,n_2} = \frac{(N+n_1+n_2-1)!}{(N-1)!\,(n_1+n_2)!}\,\frac{(N+n_2-2)!}{(N-2)!\,n_2!}\,\frac{n_1+1}{n_1+n_2+1} \tag{5.6}$$

and

$$\begin{aligned} C_2(n_1, n_2) = \frac{1}{2N} \big(&(N-1)n_1^2 + 2(N-2)n_2^2 + 2(N-2)n_1 n_2 \\ &+ N(N-1)n_1 + 2N(N-2)n_2 \big) . \end{aligned} \tag{5.7}$$

If we take into consideration that each diagram consists of $2n$ boxes, therefore $n_1 + 2n_2 = 2n$, these expressions reduce to

$$\begin{aligned} D^N_{2n-2n_2,n_2} &= \frac{(N+2n-n_2-1)!}{(N-1)!\,(2n-n_2)!} \frac{(N+n_2-2)!}{(N-2)!\,n_2!} \frac{2n-2n_2+1}{2n-n_2+1}, \\ C_2(2n-2n_2, n_2) &= \frac{1}{2N} \big(4(N-1)n^2 - 4Nnn_2 + 2Nn_2^2 + 2(N-1)n - 2Nn_2 \big) . \end{aligned} \tag{5.8}$$

The eigenvalues of the Hamiltonian take the form

$$E_{n,n_2} = -\frac{J}{2}(C_2(2n-2n_2, n_2) - 2C_2(n,0)) = -\frac{J}{2}\left(\frac{N-1}{N} n^2 - 2nn_2 + n_2^2 - n_2 \right) . \tag{5.9}$$

In Section 3.3 we have argued that βJ must scale with $1/N$ and $n \sim N$. We therefore let $J = j/n$ and leave β independent of N. The variable n_2 can take integer values from 0 to n. In the limit $N \to \infty$ we keep n/N and n_2/N fixed and obtain

$$E_{n,n_2} = -\frac{j}{2n} (n - n_2)^2 . \tag{5.10}$$

We define an effective action through the relation

$$\exp\left(-S_{\text{eff}}(n_2)\right) = D^N_{n,n_2} \exp\left(-\beta E_{n,n_2}\right) \tag{5.11}$$

in order to write the partition function as

$$Z = \sum_{n_2=0}^{\infty} \exp\left(-S_{\text{eff}}(n_2)\right). \tag{5.12}$$

Stirling's approximation at $N \to \infty$ reads

$$\log(N!) = N\log(N) - N. \tag{5.13}$$

We use it to obtain the large N limit of the degeneracies

$$\begin{aligned} \log\left(D^N_{n,n_2}\right) = &N \log\left(1 + \frac{2n-n_2}{N}\right) + (2n-n_2)\log\left(1 + \frac{N}{2n-n_2}\right) \\ &+ N\log\left(1 + \frac{n_2}{N}\right) + n_2 \log\left(1 + \frac{N}{n_2}\right) + \log\left(\frac{2n-2n_2+1}{2n-n_2}\right) . \end{aligned} \tag{5.14}$$

To account for the possibility of $n = n_2$ we cannot neglect the $+1$ in the last term. In the analysis of S_{eff} it is important to know not just the scaling behavior of n_2 with N, but the behavior of the difference $\epsilon = n - n_2$. Written in terms of ϵ the effective action takes the form

$$\begin{aligned} S_{\text{eff}} = &-\frac{\beta j}{2}\frac{\epsilon^2}{n} + \frac{\beta j}{2}\frac{n-\epsilon}{n} - N\log\left(1 + \frac{n+\epsilon}{N}\right) - (n+\epsilon)\log\left(1 + \frac{N}{n+\epsilon}\right) \\ &- N\log\left(1 + \frac{n-\epsilon}{N}\right) - (n-\epsilon)\log\left(1 + \frac{N}{n-\epsilon}\right) - \log\left(\frac{2\epsilon+1}{n+\epsilon}\right) . \end{aligned} \tag{5.15}$$

5.1. Method of steepest descent

It is revealed, that S_{eff} is of order $\mathcal{O}(N)$, therefore the partition function is dominated by the minimum of S_{eff} with respect to ϵ. Despite being of subleading order, the last logarithm diverges as ϵ goes to 0 due to the behavior of the degeneracy of the irreducible representations. We thus need to be careful when neglecting this term. Let us apply the method of steepest descent to evaluate the partition function. We aim to find the minima of $\tilde{S}_{\text{eff}} := S_{\text{eff}}/N$. In the case of $\epsilon = \mathcal{O}(N)$ we examine the derivative of $\tilde{S}_{\text{eff}}$ by ϵ/n to find the saddle-point equation

$$\begin{aligned}\frac{d\tilde{S}_{\text{eff}}}{d(\epsilon/n)} &= -\beta j\frac{\epsilon}{N} - \frac{n}{N}\log\left(1+\frac{N}{n+\epsilon}\right) + \frac{n}{N}\log\left(1+\frac{N}{n-\epsilon}\right) - \frac{2n^2}{N(2\epsilon+1)(n+\epsilon)} \\ &\stackrel{!}{=} 0.\end{aligned} \tag{5.16}$$

This expression is indeed of order $\mathcal{O}(1)$, just as one would expect. At $\epsilon \neq 0$ the last term can be neglected, due to it being of subleading order $\mathcal{O}(1/N)$. However, in the limit $\epsilon \to 0$ this ordering gets skewed and this term cannot be neglected. We conclude, that $\epsilon = 0$ is not a valid solution of the saddle-point equation and henceforth we work with

$$\frac{d\tilde{S}_{\text{eff}}}{d(\epsilon/n)} = -\beta j\frac{\epsilon}{N} - \frac{n}{N}\log\left(1+\frac{N}{n+\epsilon}\right) + \frac{n}{N}\log\left(1+\frac{N}{n-\epsilon}\right) \stackrel{!}{=} 0. \tag{5.17}$$

The question arises, whether $\tilde{S}_{\text{eff}}$ has a minimum for some $\epsilon > 0$ with $\epsilon = \mathcal{O}(N)$. By definition the maximal value of ϵ is 1. We know, that the ground state which dominates at zero temperature is fully symmetric for the ferromagnet, meaning $\epsilon \to n$ as $\beta \to \infty$.

$$\underbrace{\square\square\ldots\square}_{2n}, \tag{5.18}$$

In (5.17) this means that the diverging logarithmic term must compensate for $-\beta j$. Consider the two extremes of ϵ at finite values of β. At $\epsilon/n = 0$ the first derivative of $\tilde{S}_{\text{eff}}$ (5.17) vanishes. At $\epsilon/n = 0$ it diverges to $+\infty$. Also consider the second derivative of $\tilde{S}_{\text{eff}}$ while still neglecting the subleading order terms,

$$\frac{d^2\tilde{S}_{\text{eff}}}{d(\epsilon/n)^2} = -\beta j\frac{n}{N} + \frac{1}{1+\epsilon/n}\frac{1}{N/n+1+\epsilon/n} + \frac{1}{1-\epsilon/n}\frac{1}{N/n+1-\epsilon/n}. \tag{5.19}$$

At $\epsilon = 0$ we find, that

$$\left.\frac{d^2\tilde{S}_{\text{eff}}}{d(\epsilon/n)^2}\right|_{\epsilon=0} = \frac{n}{N}(\beta_c j - \beta j), \qquad \beta_c := \frac{2N}{N+n} \tag{5.20}$$

is negative at low temperatures, where $\beta > \beta_c$. By numerical evaluation one can check easily, that the second derivative is monotonically increasing between $\epsilon/n = 0$ and 1. We now know enough about the topology of the first derivative of $\tilde{S}_{\text{eff}}$ as a function of ϵ/n to argue that there exists exactly one minimum of $\tilde{S}_{\text{eff}}$ in the range $0 < \epsilon/n < 1$ for $\beta > \beta_c$, or $T < T_c = 1/\beta_c$ respectively. This solution breaks down at the critical temperature T_c, which implies the existence of a phase transition. Whenever a phase transition is encountered the question arises as

to what order this transition is. This particular phase transition is unusual in that it originates from an infinite number of flavors instead of an infinite volume. This in not something unprecedented. In 1980 Gross and Witten published the discovery of a third order phase transition in the large N limit of the two-dimensional $U(N)$ Wilson lattice gauge theory [7]. In the Ehrenfest classification the order of a phase transition is given by the derivative of the free energy

$$F = -\frac{\log(Z)}{\beta} \tag{5.21}$$

that exhibits a discontinuity. To determine this order would require us to understand the behavior of ϵ at the critical temperature. Due to time constraints this is not pursued further here. In Chapter 4 we have not found anything that would point towards a phase transition. This suggests that we do not yet fully understand the cluster description of the SU(N) quantum ferromagnet.

5.2. Magnetic susceptibility

In the previous chapter we discussed the importance of a finite expectation value of the cluster length $\langle|\mathcal{C}|\rangle_{SC}$, which is best measured in units of β, to the feasibility of using clusters to describe the large N limit of the SU(N) quantum ferromagnet. Considering that all spins in a cluster must be of the same flavor reveals that the cluster length is very closely related to the magnetic susceptibility

$$\langle T^a T^a \rangle, \qquad a \in \{1, 2, \ldots, N^2 - 1\}, \tag{5.22}$$

where T^a is the total spin $T^a_x + T^a_y$. The two quantities only differ by a factor. The magnetic susceptibility can be expressed as the quadratic Casimir (5.7). Since the magnetic susceptibility is independent of the value of a we find

$$\langle \sum_{a=1}^{N^2-1} T^a T^a \rangle = \langle C_2(n_1, n_2) \rangle \quad \Rightarrow \quad \langle T^a T^a \rangle = \frac{\langle C_2(n_1, n_2) \rangle}{N^2 - 1}. \tag{5.23}$$

Expressed in terms of ϵ the quadratic Casimir reads

$$C_2(\epsilon) = \frac{N-2}{N} n^2 + \epsilon^2 + (N-2)n + \epsilon. \tag{5.24}$$

In the low temperature phase, where $\epsilon \sim N$, the leading order of the expression is given by

$$C_2(\epsilon) = n^2 + \epsilon^2 + Nn, \qquad \beta > \beta_c. \tag{5.25}$$

The resulting magnetic susceptibilitiy at $N \to \infty$ with fixed n/N is

$$\langle T^a T^a \rangle_{\beta > \beta_c} = \frac{n}{N}\left(1 + \frac{n}{N}\right) + \frac{\epsilon^2}{N^2}, \tag{5.26}$$

where ϵ is the solution of the saddle-point equation (5.17). The factor between the cluster length and the magnetic susceptibility in the two-site case

$$\langle T^a T^a \rangle = \frac{n}{N} \langle|\mathcal{C}|\rangle_{SC} \tag{5.27}$$

is found by comparison at the limit $\beta \to 0$, where there can be no interaction between lattice sites and the cluster formalism becomes much simpler. We understand from the cluster formalism, that the two-site model at $\beta \to 0$ is equivalent to two independent single-site models. The calculation of the magnetic susceptibility in the single-site case is trivial, since we know the form of the quadratic Casimir (3.13). One then simply has to take into consideration that the magnetic susceptibility is an extensive quantity, while the cluster size is intensive.

6. Conclusion

Several developments in the understanding of two-dimensional $\mathbb{C}\mathrm{P}(N-1)$ non-linear σ models in the large N limit have been achieved.
In this thesis it has been confirmed that $\mathbb{C}\mathrm{P}(N-1)$ models are indeed topologically non-trivial, and the self-duality equations that define the instanton states, which are a result of the topological non-triviality, have been derived. Additionally, the leading order of the mass-gap equation in a $1/N$ expansion has been determined. The D-theory approach to regularize $\mathbb{C}\mathrm{P}(N-1)$ models has been investigated, and we have confirmed that the dimensionally reduced low-energy degrees of freedom of a $(2+1)$-dimensional ferromagnetic $\mathrm{SU}(N)$ quantum Heisenberg model indeed yield the $(1+1)$-dimensional $\mathbb{C}\mathrm{P}(N-1)$ model at zero vacuum angle. We have formulated the necessary conditions for dimensional reduction to take place and for the lattice model to yield meaningful results. Further, we have established how the limit $N \to \infty$ is to be approached in a way that is well defined.
The cluster formalism has proven to be an effective tool to describe lattice configurations at large values of N. While the idea originated from the search for efficient algorithms to perform Monte Carlo calculations, we have used clusters in order to progress towards an analytical finite effort description of $\mathrm{SU}(N)$ quantum ferromagnets at $N \to \infty$. So far this has worked for a single-site system. For larger systems the problem becomes more complicated. Some attempts have been made by letting a cluster propagate through the lattice in a diffusion process. However, so far our attempts at describing the probability to close the cluster have proven inadequate and further investigation is required. At this point it is also unclear how this approach could recreate the phase transition at finite temperature in a two-site system.
Finally, we have solved the two-site case analytically via a group theoretical approach. We have managed to calculate the expectation value of the magnetic susceptibility, which is directly related to the expectation value of the size of a cluster in units of β. Interestingly, a phase transitions appears at finite temperature $T_c = 1/\beta_c = j(N+n)/(2N)$. This phase transition shows a similarity to the third order phase transition discovered by Gross and Witten [7] in that it arises in the large N limit. Further investigation is required in order to identify the order of this phase transition in this two-site system.

A. Additional calculations for the field theory formulation of $\mathbb{C}\mathrm{P}(N-1)$

A.1. Equivalence of topological charge terms in the z- and P-formalism

We explicitly show that the expressions (2.15) and (2.16) for the topological charge in the two representations of the classical $\mathbb{C}\mathrm{P}(N-1)$ model are equivalent. First, let us repeat the definition of the two formulations, z with the gauge field A_μ and P:

$$\begin{aligned} z^\mathsf{T} = (z_1, ..., z_N), \quad z_i \in \mathbb{C}, \quad |z|^2 = z^\dagger z = 1, \\ A_\mu = \frac{1}{2} i \left(\overline{z_i} \partial_\mu z_i - (\partial_\mu \overline{z_i}) z_i \right), \end{aligned} \tag{A.1}$$

$$P = |z\rangle \langle z| = z z^\dagger, \qquad P_{ij} = z_i \overline{z_j}, \tag{A.2}$$

$$P^2 = P, \qquad \mathrm{Tr}\, P = 1, \qquad P^\dagger = P. \tag{A.3}$$

The topological charge term in z-notation is given by

$$\begin{aligned} Q[z] &= \frac{1}{2\pi} \int d^2x \epsilon_{\mu\nu} \partial_\mu A_\nu \\ &= \frac{1}{2\pi} \int d^2x \epsilon_{\mu\nu} \frac{i}{2} ((\partial_\mu \overline{z_i}) \partial_\nu z_i + \overline{z_i} \partial_\mu \partial_\nu z_i - (\partial_\mu \partial_\nu \overline{z_i}) z_i - (\partial_\nu \overline{z_i}) \partial_\mu z_i) \\ &= \frac{1}{2\pi} \int d^2x \epsilon_{\mu\nu} \frac{i}{2} ((\partial_\mu \overline{z_i}) \partial_\nu z_i - (\partial_\nu \overline{z_i}) \partial_\mu z_i). \end{aligned} \tag{A.4}$$

In the last step the two terms that are symmetric in μ and ν are eliminated by $\epsilon_{\mu\nu}$. We finally show the equivalence of the two formulations of Q by using the definition (A.2) of P and the two identities

$$\begin{aligned} \epsilon_{\mu\nu} \partial_\mu ((\partial_\nu \overline{z_i}) z_i) &= \epsilon_{\mu\nu} \partial_\mu (\overline{z_j} z_j (\partial_\nu \overline{z_i}) z_i) \\ &= \epsilon_{\mu\nu} (\partial_\mu \overline{z_j}) z_j (\partial_\nu \overline{z_i}) z_i + \epsilon_{\mu\nu} \overline{z_j} (\partial_\mu z_j) (\partial_\nu \overline{z_i}) z_i + \epsilon_{\mu\nu} \partial_\mu ((\partial_\nu \overline{z_i}) z_i) \\ &= \epsilon_{\mu\nu} \overline{z_j} (\partial_\mu z_j) (\partial_\nu \overline{z_i}) z_i + \epsilon_{\mu\nu} \partial_\mu ((\partial_\nu \overline{z_i}) z_i), \end{aligned} \tag{A.5}$$

and

$$\epsilon_{\mu\nu} (\partial_\mu z^\dagger) \partial_\nu z = \frac{\epsilon_{\mu\nu}}{2} ((\partial_\mu z^\dagger) \partial_\nu z - (\partial_\nu z^\dagger) \partial_\mu z), \tag{A.6}$$

which are built on the anti-symmetry of $\epsilon_{\mu\nu}$ and z being a unit vector. It is found that $Q[P]$ indeed reduces to $Q[z]$.

$$\begin{aligned}
Q[P] &= \frac{i}{2\pi}\int d^2x\epsilon_{\mu\nu}\,\mathrm{Tr}(P\partial_\mu P\partial_\nu P)\\
&= \frac{i}{2\pi}\int d^2x\epsilon_{\mu\nu}\,\mathrm{Tr}(zz^\dagger((\partial_\mu z)z^\dagger + z\partial_\mu z^\dagger)((\partial_\nu z)z^\dagger + z\partial_\nu z^\dagger))\\
&= \frac{i}{2\pi}\int d^2x\epsilon_{\mu\nu}\,\mathrm{Tr}(z^\dagger(\partial_\mu z)z^\dagger(\partial_\nu z)z^\dagger z + z^\dagger(\partial_\mu z)z^\dagger z(\partial_\nu z^\dagger)z\\
&\qquad\qquad + z^\dagger z(\partial_\mu z^\dagger)(\partial_\nu z)z^\dagger z + z^\dagger z(\partial_\mu z^\dagger)z(\partial_\nu z^\dagger)z)\\
&= \frac{i}{2\pi}\int d^2x\epsilon_{\mu\nu}\,\mathrm{Tr}((\partial_\mu z^\dagger)\partial_\nu z + z^\dagger(\partial_\mu z)(\partial_\nu z^\dagger)z)\\
&= \frac{i}{4\pi}\int d^2x\epsilon_{\mu\nu}\,\mathrm{Tr}((\partial_\mu z^\dagger)\partial_\nu z - (\partial_\mu z)\partial_\nu z^\dagger)\\
&= Q[z]
\end{aligned} \tag{A.7}$$

A.2. $\mathbb{C}\mathrm{P}(N-1)$ Euler-Lagrange equation

In this part of the appendix the classical Euler-Lagrange equations for $\mathbb{C}\mathrm{P}(N-1)$ models in the z-formalism are derived. We remind ourselves of the definition of the field variable

$$z \in \mathbb{C}^N \qquad \text{with} \qquad |z| = 1. \tag{A.8}$$

The Lagrange multiplier field $\lambda(x)$ is introduced to realize the constraint on $z(x)$ in the partition function

$$\int \mathcal{D}\overline{z}\mathcal{D}z\; e^{-S[z]} = \int \mathcal{D}\overline{z}\mathcal{D}z\mathcal{D}\lambda_\mu\; e^{-S[z,\lambda]}, \tag{A.9}$$

where

$$S[z,\lambda] = \int d^2x \frac{2}{g^2}\left(\overline{D_\mu z}D_\mu z + \lambda(\overline{z}z-1)\right). \tag{A.10}$$

By integrating out λ_μ one finds the delta function $\delta(\overline{z}_i z_i)$. We find the Euler-Lagrange equations

$$0 = \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\overline{z}} - \frac{\delta\mathcal{L}}{\delta\overline{z}} = D_\mu D_\mu z - \lambda z, \tag{A.11}$$

$$0 = \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\lambda} - \frac{\delta\mathcal{L}}{\delta\lambda} = \overline{z}z - 1. \tag{A.12}$$

The goal now is to use (A.12) to express λ in (A.11) in terms of z and A_μ. The identities

$$\partial_\mu(\overline{z}z) = 0 \Rightarrow z\partial_\mu\overline{z} = -\overline{z}\partial_\mu z \tag{A.13}$$

which follow directly from (A.12), and

$$A_\mu = \frac{i}{2}(\overline{z}\partial_\mu z - z\partial_\mu\overline{z}) \tag{A.14}$$

are used to show the equality

$$\begin{aligned}\overline{D_\mu z} D_\mu z &= (\partial_\mu - iA_\mu)\overline{z}(\partial_\mu + iA_\mu)z \\ &= \partial_\mu \overline{z} D_\mu z - \overline{z}\partial_\mu D_\mu z - 2iA_\mu \overline{z} D_\mu z + \overline{z} D_\mu D_\mu z \\ &= -\partial_\mu \overline{z} D_\mu z - \overline{z}\partial_\mu D_\mu z + \overline{z} D_\mu D_\mu z \\ &= -\partial_\mu(\overline{z} D_\mu z) + \overline{z} D_\mu D_\mu z \\ &= \overline{z} D_\mu D_\mu z.\end{aligned} \tag{A.15}$$

Therefore the second-order field equations read

$$D_\mu D_\mu z + (\overline{D_\mu z} D_\mu z) z = 0. \tag{A.16}$$

A.3. Terms in the large N expansion of $\mathbb{C}\mathrm{P}(N-1)$ models

In this section we derive the leading order terms in the power series expansion of the effective action encountered in Section 2.3. Accordingly, this still follows the paper by D'Adda et al. [5]. First we remind ourselves of the full expression for the effective action

$$S_{\text{eff}} = N \operatorname{Tr}(\log \Delta) + \frac{i\sqrt{N}}{2f} \int d^2x \alpha(x), \tag{A.17}$$

where

$$\Delta = -\partial_\mu \partial_\mu + \frac{1}{N}\lambda_\mu \lambda_\mu + \frac{i}{\sqrt{N}} \partial_\mu \cdot \lambda_\mu + \frac{i}{\sqrt{N}} \lambda_\mu \partial_\mu + \frac{i\alpha}{\sqrt{N}} + m^2. \tag{A.18}$$

We expand $\log \Delta$ in powers of $1/\sqrt{N}$. Note that in the first term of order $1/\sqrt{N}$ in (A.18) the derivative acts on both λ_μ and a test function. We order $\log \Delta$ by powers of $1/\sqrt{N}$ to find

$$\log \Delta = \log(-\Box + m^2) + \frac{1}{\sqrt{N}} \frac{1}{-\Box + m^2} (i\partial_\mu \cdot \lambda_\mu + i\lambda_\mu \partial_\mu + i\alpha) + \mathcal{O}\left(\frac{1}{N}\right). \tag{A.19}$$

Plugging this back into (A.17) we identify the two leading order contributions to S_{eff}.

$$\begin{aligned} S_{\text{eff}}^{(0)} &= \operatorname{Tr}\left(\log(-\Box + m^2)\right), \\ S_{\text{eff}}^{(1)} &= \frac{i}{2f} \int d^2x\, \alpha(x) - \operatorname{Tr}\left(\frac{i\partial_\mu \cdot \lambda_\mu + i\lambda_\mu \partial_\mu + i\alpha}{-\Box + m^2}\right). \end{aligned} \tag{A.20}$$

The zeroth order term $S_{\text{eff}}^{(0)}$ only gives a constant contribution to the partition function since it does not contain any field variables. The next-to-leading order term $S_{\text{eff}}^{(1)}$ however, does contain the Lagrange multiplier field α and the auxiliary fields λ_μ. In the limit $N \to \infty$ the partition function is dominated by field configurations that minimize $S_{\text{eff}}^{(1)}$. We can give the second term in (A.20) a more explicit form by writing out the trace over the physical Hilbert space. Both

terms containing λ_μ vanish in the trace due to being antisymmetric in momentum space

$$\begin{aligned}
\operatorname{Tr}\frac{i\partial_\mu\cdot\lambda_\mu}{-\Box+m^2} &= \int d^2x\,\langle x|\,\frac{i\partial_\mu\cdot\lambda_\mu}{-\Box+m^2}\,|x\rangle \\
&= \int d^2x\int d^2q\,\langle x|q\rangle\,\langle q|\,\frac{i\partial_\mu\cdot\lambda_\mu}{-\Box+m^2}\,|x\rangle \\
&= \int d^2q\,\frac{iq_\mu}{-q^2+m^2}\int d^2x\;\lambda_\mu(x) \\
&= 0, \\
\operatorname{Tr}\frac{i\lambda_\mu\partial_\mu}{-\Box+m^2} &= 0.
\end{aligned} \tag{A.21}$$

The only term that survives is

$$\begin{aligned}
\operatorname{Tr}\frac{i\alpha}{-\Box+m^2} &= \int d^2x\,\langle x|\,\frac{i\alpha}{-\Box+m^2}\,|x\rangle \\
&= \int d^2x\int d^2q\,\langle x|\,\frac{i\alpha}{-\Box+m^2}\,|q\rangle\,\langle q|x\rangle \\
&= \int d^2x\int d^2q\;i\alpha(x)\,\langle x|q\rangle\,\frac{1}{-q^2+m^2}\,\langle q|x\rangle \\
&= \int\frac{d^2q}{(2\pi)^2}\,\frac{1}{-q^2+m^2}\underbrace{\int d^2x\;\alpha(x)}_{\tilde\alpha(0)},
\end{aligned} \tag{A.22}$$

where $\tilde\alpha(0)$ is the zero-momentum Fourier transform of $\alpha(x)$. Putting everything together we finally end up with the expression in (2.37):

$$S^{(1)}_{\text{eff}} = i\tilde\alpha(0)\left(\frac{1}{2f}-\int\frac{d^2q}{(2\pi)^2}\frac{1}{q^2+m^2}\right). \tag{A.23}$$

B. Additional calculations for the D-theory formulation of $\mathbb{C}\mathrm{P}(N-1)$ models

B.1. $\mathrm{SU}(N)$ symmetry of the Heisenberg model

In this part of the appendix we show explicitly the $\mathrm{SU}(N)$ invariance of the ferromagnetic quantum Heisenberg model claimed in Section 2.1. We remind ourselves of the Hamiltonian

$$H = -J \sum_{\langle xy \rangle} T_x^a T_y^a, \tag{B.1}$$

with the Hermitian and traceless $N \times N$ matrices T_x^a. They are defined by the commutation and anti-commutation relations

$$\left[T_x^a, T_y^b\right] = i f^{abc} T_x^c \delta_{xy}, \tag{B.2}$$

$$\{T_x^a, T_y^b\} = \frac{1}{N} \mathbb{1} \delta_{ab} \delta_{xy} + i d^{abc} T_x^c \delta_{xy}. \tag{B.3}$$

and normalized by

$$\mathrm{Tr}(T^a T^b) = \delta_{ab}/2. \tag{B.4}$$

In a preparatory step we calculate the commutation relation of the Hamiltonian and the total spin

$$T^a = \sum_x T_x^a. \tag{B.5}$$

We find

$$\begin{aligned}
[H, T^a] &= \left[-J \sum_{\langle xy \rangle} T_x^b T_y^b, \sum_z T_z^a \right] \\
&= -J \sum_{\langle xy \rangle} \sum_z \left[T_x^b T_y^b, T_z^a \right] \\
&= -J \sum_{\langle xy \rangle} \sum_z \left(\left[T_x^b, T_z^a \right] T_y^b + T_x^b \left[T_y^b, T_z^a \right] \right) \\
&= -iJ f^{bac} \sum_{\langle xy \rangle} \left(T_x^c T_y^b + T_x^b T_y^c \right) \\
&= 0,
\end{aligned} \tag{B.6}$$

where in the last step we used that f^{abc} is antisymmetric under the exchange of indices b and c while the expression in the brackets is symmetric. Under global $\mathrm{SU}(N)$ transformations the Hamiltonian transforms with the unitary matrix

$$U = \exp\left(i\omega_a T^a\right). \tag{B.7}$$

The transformation matrix is generated by the total spin. By writing the exponential as a series and repeatedly applying (B.6) we show explicitly the invariance of H under $\mathrm{SU}(N)$ transformations

$$\begin{aligned} H' &= UHU^\dagger = UH\exp\left(-i\omega_a T^a\right) = UH\left(1 - i\omega_a T^a + ...\right) \\ &= U\left(1 - i\omega_a T^a + ...\right)H = UU^\dagger H = H. \end{aligned} \tag{B.8}$$

B.2. Dispersion relation of magnons in the $\mathrm{SU}(N)$ ferromagnet

In Section 3.1.2 a part of the calculation of magnon energies was skipped. This appendix completes the calculations. The aim is to find the effect of the shift operator part of the Hamiltonian H_x^s acting on the state $|0'_x\rangle = T_x^{-,1}|0\rangle$. We start by splitting H_x^s into $2d$ parts, one for each direction of the neighboring site. The directions are denoted by $\mu \in \{1, \ldots, d\}$, such that $x \pm \hat{\mu}$ are the neighboring sites to x. We write

$$H_x^s = -J \sum_{y|\langle xy\rangle} \sum_{b=1}^{N(N-1)/2} (T_x^{+,b}T_y^{-,b} + T_x^{-,b}T_y^{+,b}) = \sum_{\hat{\mu}} H_{x,+\hat{\mu}}^s + H_{x,-\hat{\mu}}^s, \tag{B.9}$$

where

$$H_{x,\pm\hat{\mu}}^s = -J \sum_{b=1}^{N(N-1)/2} (T_x^{+,b}T_{x\pm\hat{\mu}}^{-,b} + T_x^{-,b}T_{x\pm\hat{\mu}}^{+,b}). \tag{B.10}$$

Since we have chosen $|0\rangle$ to be the ground state of maximum spin projection $|uu\ldots u\rangle$ only shift operators with $b = 1$ survive. The operator $H_{x,\pm\hat{\mu}}^s$ acting on $|0'_x\rangle$ shifts the excitation from x to the neighboring site $x \pm \hat{\mu}$,

$$\begin{aligned} H_{x,\pm\hat{\mu}}^s |0'_x\rangle &= -J(T_x^{+,1}T_{x\pm\hat{\mu}}^{-,1} + T_x^{-,1}T_{x\pm\hat{\mu}}^{+,1})T_x^{-,1}|0\rangle = -JT_x^{+,1}T_{x\pm\hat{\mu}}^{-,1}T_x^{-,1}|0\rangle \\ &= -JT_{x\pm\hat{\mu}}^{-,1}T_x^{+,1}T_x^{-,1}|0\rangle = -JT_{x\pm\hat{\mu}}^{-,1}\left(\tilde{T}_x^1 + T_x^{-,1}T_x^{+,1}\right)|0\rangle \\ &= -J\frac{n}{2}|0'_{x\pm\hat{\mu}}\rangle. \end{aligned} \tag{B.11}$$

In the first line of this calculation $T^{+,1}$ acting on the up-flavor spin at $x \pm \hat{\mu}$ vanishes. In the second to last line we used the commutator of T^+ and T^-,

$$\left[T^{+,1}, T^{-,1}\right] = \frac{1}{2}\left[T^1 + iT^2, T^1 - iT^2\right] = i\left[T^2, T^1\right] = T^3 = \tilde{T}^1. \tag{B.12}$$

This result is used to let $H^s_{x,\pm\hat{\mu}}$ act on the full magnon state $|p\rangle$,

$$\begin{aligned} H^s_{x,\pm\hat{\mu}}|p\rangle &= \sum_x \exp\left(ix_ip_i\right) H^s_{x,\pm\hat{\mu}}|0'_x\rangle = -\frac{nJ}{2}\sum_x \exp\left(ix_ip_i\right)|0'_{x\pm\hat{\mu}}\rangle \\ &= -\frac{nJ}{2}\sum_x \exp\left(i(x\mp\hat{\mu})_ip_i\right)|0'_x\rangle = -\frac{nJ}{2}\sum_x \exp\left(ix_ip_i\right)\exp\left(\mp ip_\mu\right)|0'_x\rangle \\ &= -\frac{nJ}{2}\exp\left(\mp ip_\mu\right)|p\rangle\,. \end{aligned} \tag{B.13}$$

Thus we find the eigenvalue of $H^s_{x,\pm\hat{\mu}}$ to be

$$H^s_x|p\rangle = -\frac{nJ}{2}\sum_{\hat{\mu}}\left(\exp\left(ip_\mu\right)+\exp\left(-ip_\mu\right)\right)|p\rangle = -nJ\sum_{\mu=1}^{d}\cos(p_\mu)|p\rangle \tag{B.14}$$

B.3. Matching parameters in the EFT

The parameters of an effective theory carry information about the underlying microscopic theory. We therefore need to express the spin stiffness ρ_s and the magnetization density m in terms of the microscopic coupling J and n that defines the representation of $\mathrm{SU}(N)$. This is done by matching observable quantities of the two systems. To find m we study the effect of an external magnetic field on the fully symmetric ground state consisting only of up-flavor spins $|0\rangle = |uu\ldots u\rangle$. The external field is introduced via a chemical potential term

$$H' = H - B^a\sum_x T^a_x, \tag{B.15}$$

where we chose B^a such that only the first diagonal generator remains $B^aT^a = B\tilde{T}^1$. This chemical potential breaks the ground state degeneracy by encouraging up- and discouraging down-spins. In a configuration with n up-spins on each site the change of the ground state energy is given by

$$E'_0 = \langle 0| - B\sum_x \hat{T}^1_x|0\rangle = E_0 - BV\frac{n}{2}. \tag{B.16}$$

Let us now consider the EFT analogue. This is treated in a paper by Bär, Imboden and Wiese [21]; however, their calculations differ by a factor i compared to this thesis. The treatment of the thermodynamic potential by Freedman and McLerran [4] seems to agree with our choice of factors. The non-abelian chemical potential couples to the generator $\int d^{d-1}xj_0$. Therefore it behaves in the same way as an analytic continuation of the zero-component of a gauge field. In a field theory this implies the introduction of a covariant derivative

$$\begin{aligned} D_tP &= \partial_tP + i[W_t, P], \\ W_t &= iB^aT^a. \end{aligned} \tag{B.17}$$

The external field B^a is chosen in the same way as above. We write down the resulting action

$$\begin{aligned} S_{\text{eff}}[P,B] &= \int d^2x \left[\int_{S^1} dt \rho_s \operatorname{Tr}(\partial_i P \partial_i P) - 2m \int_{H^2} dt d\tau \operatorname{Tr}(P D_t P \partial_\tau P - P \partial_\tau P D_t P) \right] \\ &= S_{\text{eff}}[P] - 2mi \int d^2x \int_{H^2} dt d\tau \operatorname{Tr}(P[W_t,P]\partial_\tau P - P\partial_\tau P[W_t,P]) \\ &= S_{\text{eff}}[P] - 2mB \int d^2x \int_{S^1} dt \operatorname{Tr}(P(x,t,1)\hat{T}^1 - P(x,t,0)\hat{T}^1) \end{aligned} \tag{B.18}$$

We now use our freedom to choose the interpolation of P in τ such that only the term with the physical $P(x,t,\tau=1)$ survives. For positive values of Bm the effective action is minimized by the ground state $P_0(x,t) = \operatorname{diag}(1,0,0,\dots)$. This spawns a change in the effective action of

$$\delta S_{\text{eff}} = -\beta V m B. \tag{B.19}$$

We now match the two energies

$$\delta S_{\text{eff}} = \beta \delta E_0 \;\Rightarrow\; -\beta V B m = -\beta \frac{V}{a^2} B \frac{n}{2}, \tag{B.20}$$

to find

$$m = \frac{n}{2a^2}. \tag{B.21}$$

Here, we reintroduced the lattice constant a that was previously set to zero. The spin stiffness ρ_s we derive by matching the magnon dispersion relation. In the $\mathrm{SU}(N)$ model

$$E_{\mathrm{SU}(N)} = \frac{Jn}{2} a^2 p^2 \tag{B.22}$$

was already derived above. In the EFT the dispersion relation is given by

$$E_{\text{EFT}} = \frac{\rho_s}{m} p^2 \tag{B.23}$$

according to [21]. By plugging in the result for m above we finally get

$$\rho_s = \frac{Jn^2}{4}. \tag{B.24}$$

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