Dimensional Regularization and Asymptotic Freedom of the δ-Function Potential in Relativistic Quantum Mechanics

Bachelor Thesis

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Written by Igor Jurosevic

and supervised by

Prof. Dr. Uwe-Jens Wiese

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Abstract

This bachelor thesis is divided into three chapters. Chapter 1 treats a non-relativistically moving particle in a δ-potential well. In a first attempt, the setting is investigated in position and momentum space. The second chapter treats the same potential, but this time the particle is allowed to move relativistically. The appearing divergences are treated by applying a method known as "dimensional regularization". In the treatment of the so-called scattering states, the running coupling constant $\lambda(E, E_b)$ is studied together with the corresponding β function and the asymptotic freedom of the $\lambda(E, E_b)$ coupling is shown. Since the potential can be regarded as caused by a super heavy mass sitting in the origin, the whole setting can be thought of as a relativistic scattering experiment with a contact interaction. Since such contact interactions are not excluded by Leutwyler's classical non-interaction theorem, the above described setting could be a hint to a quantum loop hole in the theorem. Chapter 3 gives a small recapitulation of the conclusions and remarks made over the whole thesis.

Contents

Preface

In order to create an easier access for physics students to topics such as "dimensional regularization" and "renormalization" and to cross-check some results obtained through the theory of self-adjoint extensions of the pseudo-differential operator $H = \sqrt{p^2 + m^2}$, M. H. Al-Hashimi, A. M. Shalaby, and U. - J. Wiese wrote a paper entitled "Asymptotic Freedom, Dimensional Transmutation, and an Infra-red Conformal Fixed Point for the δ-Function Potential in 1-dimensional Relativistic Quantum Mechanics". Some problems investigated in the paper are also treated in this thesis. Retrospectively we can say, that the paper constituted a kind of "road map" for this thesis. We wish to emphasize the word "retrospectively" for the following reason: In order to give us an idea of the scientific work of a theoretical physicist and to keep us surrounded by the "fog" of exploration and mystery, our supervisor only gave us hints on the tasks we should perform and kept the above mentioned paper in secret. Therefore, we only gained knowledge of the existence of the above mentioned paper after we had performed our own computations and thoughts about these topics.

Chapter 1

Non-relativistic particle in the δ-potential well

1.1 Exploration in position space using the particle physicist's approach

1.1.1 The attractive case

1.1.1.1 The bound state

In this chapter we will begin with the study of the Dirac δ [5] potential well. To understand a particle's behaviour in this well, we have to solve the one-dimensional Schrödinger equation. In order to do so, we assume that the particle is moving in a non-relativistic way (i.e. the particle's kinetic energy is small compared to its rest mass energy). The potential has the following form, where λ is in $\mathbb{R}$

$$V(x) = \lambda\delta(x). \tag{1.1}$$

There are two possible types of such wells. The attractive ones, where $\lambda < 0$ and the repulsive ones, where $\lambda > 0$. The case $\lambda = 0$ demands a special treatment. We will consider it in more detail in a few pages.
As we can see, the potential doesn't depend on time and therefore we are allowed to use the separation of variables approach for solving the Schrödinger equation. In this case the time-dependence just affects a phase factor to the stationary part. Therefore, we shall focus on the explicit solution of the stationary equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + \lambda\delta(x)\psi(x) = E\psi(x). \tag{1.2}$$

For now we will consider an attractive potential, avoid the point $x = 0$ and focus on the sections where $x \neq 0$. Further we shall define $\lambda \in \mathbb{R}_+$ and attach the sign which corresponds to one of the two possible potential types (i.e. a negative sign for the attractive case and a positive one for the repulsive case). We look for solutions, whose energy binds the particle to the well (E <0). The term "bind" may sound strange, but if one thinks of the δ-potential as a sequence of ever deepening and narrowing finite box potentials, where a particle may indeed be inside the box, the term becomes self-explanatory.

For $x < 0, E < 0$

$$\frac{d^2}{dx^2}\psi(x) = -\frac{2mE}{\hbar^2}\psi(x). \tag{1.3}$$

This differential equation can be solved by using an exponential ansatz

$$\psi_-(x) = Ae^{kx} + Be^{-kx}, k = \sqrt{\frac{-2mE}{\hbar^2}}. \tag{1.4}$$

Since the unitarity principle demands normalizable solutions (i.e. the probability to find the particle somewhere has to be 1), we have to abandon the second term in the sum. Otherwise, the function will blow up as x goes

to $-\infty$ and we will fail at normalizing it.

For $x > 0, E < 0$ the situation is analogous

$$\psi_+(x) = Ce^{kx} + De^{-kx}, k = \sqrt{\frac{-2mE}{\hbar^2}}. \tag{1.5}$$

Again the unitarity principle forces us to drop a term. This time it has to be the first term, or the function will blow up, as x goes to ∞ and this would, again, screw up the normalization of $\psi(x)$. The solution is now of the form

$$\psi(x) = \begin{cases} Ae^{kx} & , x < 0 \\ De^{-kx} & , x > 0 \end{cases}. \tag{1.6}$$

So far the solution is a function with a discontinuity at $x = 0$. In order to patch it up at $x = 0$, we need conditions how to do this. Since we solved a second order differential equation we have also two degrees of freedom which we can fix with appropriate boundary conditions. The "standard" set of boundary conditions for problems like ours is

1.) $\psi(x)$ continuous.

2.) $\frac{d\psi(x)}{dx}$ continuous almost everywhere.

The continuity of $\psi(x)$ can be achieved easily

$$\lim_{x_- \to 0} \psi(x) = \lim_{x_- \to 0} \psi_-(x) = \lim_{x_- \to 0} Ae^{kx} = A, \tag{1.7}$$

$$\lim_{x_+ \to 0} \psi(x) = \lim_{x_+ \to 0} \psi_+(x) = \lim_{x_+ \to 0} De^{-kx} = D. \tag{1.8}$$

So if we want $\psi(x)$ to be continuous at $x = 0$, we have to impose the condition $A = D$.

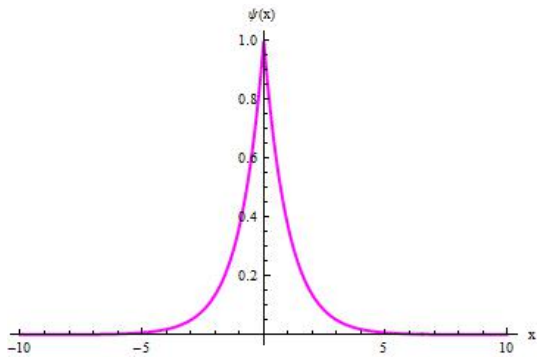


Figure 1.1: The form of $\psi(x)$. The spike at $x = 0$ indicates that the derivative of $\psi(x)$ at $x = 0$ can't be continuous.

When one looks at the plot in Figure 1.1, one can see that the continuity of $\frac{d\psi(x)}{dx}$ will, at least for $x = 0$, be violated. Not being continuous, doesn't mean, that there is no way to characterize the wave function's behaviour, as we now shall see:

Consider the Hamiltonian with $x \in]-\epsilon, \epsilon[$:

$$H\psi(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) - \lambda\delta(x)\psi(x) = E\psi(x). \tag{1.9}$$

Unleashing an integral on this relation, implies

$$\int_{-\epsilon}^{\epsilon} H\psi(x)dx = \int_{-\epsilon}^{\epsilon} -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x)dx + \int_{-\epsilon}^{\epsilon} -\lambda\delta(x)\psi(x)dx = E\int_{-\epsilon}^{\epsilon}\psi(x)dx. \tag{1.10}$$

With the integral on the Hamiltonian we obtain two simplifications. First we get an expression with the derivative of $\psi(x)$ in it. Second, the part of the sum containing the δ function (strictly speaking, it is a distribution, but physicist's language conventions label it as a "function") becomes treatable in the sense, that it is no longer infinite!

$$\int_{-\epsilon}^{\epsilon} H\psi(x)dx = -\frac{\hbar^2}{2m}\frac{d}{dx}\psi(x)\Big|_{-\epsilon}^{\epsilon} - \lambda\psi(0) = E\int_{-\epsilon}^{\epsilon}\psi(x)dx. \tag{1.11}$$

The equation yields a few more things for harvesting. To bring in this harvest, we shall take a closer look at the first term on the left side and the term on the right side of the equation simultaneously

$$-\frac{\hbar^2}{2m}\frac{d}{dx}\psi(x)\Big|_{-\epsilon}^{\epsilon} = \frac{d}{dx}\psi(\epsilon) - \frac{d}{dx}\psi(-\epsilon), \tag{1.12}$$

$$E\int_{-\epsilon}^{-\epsilon}\psi(x)dx = E\left(\int_{-\epsilon}^{0}\psi(x)dx + \int_{0}^{\epsilon}\psi(x)dx\right). \tag{1.13}$$

Considering the limit $\epsilon \to 0$ yields

$$-\frac{\hbar^2}{2m}\lim_{\epsilon\to 0}\psi(x) = -\frac{\hbar^2}{2m}\lim_{\epsilon\to 0}\left(\frac{d}{dx}\psi(-\epsilon) - \frac{d}{dx}\psi(\epsilon)\right) = -\frac{\hbar^2}{2m}\lim_{\epsilon\to 0}\left(-Ake^{k\epsilon} - Ake^{-k\epsilon}\right) = \frac{\hbar^2 Ak}{m}, \tag{1.14}$$

$$\lim_{\epsilon\to 0} E\left(\int_{-\epsilon}^{0} Ae^{kx}dx + \int_{0}^{\epsilon} Ae^{-kx}dx\right) = E\lim_{\epsilon\to 0}\left(Ake^{kx}\Big|_{-\epsilon}^{0} - Ake^{-kx}\Big|_{0}^{\epsilon}\right) = 0. \tag{1.15}$$

By inserting equations (1.14) and (1.15) into equation (1.11), we get

$$\frac{\hbar^2 Ak}{m} - \lambda\psi(0) = 0 \Leftrightarrow \frac{\hbar^2 Ak}{m} = \lambda\psi(0). \tag{1.16}$$

With the computation above, we determined the behaviour of the derivative of ψ. Although it is clear that integrating over a function with a finite number of discontinuities will deliver a "smoother" function (integrating has smoothing properties), it is still very fascinating, that the singularity at $x = 0$ disturbs the continuity of the derivative only in such a way, that the corresponding function receives just a "kink" at this point.

If we recall that $\psi(0)$ is the value of the wave function at the contact point $x = 0$, which we determined to be A, we can simplify further

$$k = \frac{m\lambda}{\hbar^2}. \tag{1.17}$$

Again we have to remember something. While using the exponential ansatz for solving the Schrödinger equation we defined k, or the wave number, in dependence of E. Therefore, we are now able to determine its form, which is

$$k^2 = \frac{-2mE}{\hbar^2} = \frac{m^2\lambda^2}{\hbar^4} \Leftrightarrow E = \frac{-m\lambda^2}{2\hbar^2}. \tag{1.18}$$

So the (not yet normalized) solution to the stationary Schrödinger equation looks the following way

$$\psi(x) = \begin{cases} Ae^{\frac{m\lambda}{\hbar}x} & , x < 0 \\ A & , x = 0 \\ Ae^{-\frac{m\lambda}{\hbar}x} & , x > 0 \end{cases}. \tag{1.19}$$

It is possible to write it more compactly. Then, it reads

$$\psi(x) = Ae^{-\frac{m\lambda}{\hbar}|x|}. \tag{1.20}$$

Since we would like to have a physically meaningful description, we have to normalize the solution. A normalized solution means, that the probability, which is the integral of the modulus of ψ squared, for finding the particle somewhere on the x-axis is one. Formalized, the normalization looks the following way, where $\psi(x)^*$ is the complex conjugate of $\psi(x)$.

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_{-\infty}^{\infty} \psi(x)^* \psi(x) dx = \int_{-\infty}^{0} \psi(x)^* \psi(x) dx + \int_{0}^{\infty} \psi(x)^* \psi(x) dx = 1. \tag{1.21}$$

Once more we will consider both terms simultaneously

$$\int_{-\infty}^{0} |A|^2 e^{\frac{2m\lambda}{\hbar^2}x} dx = \frac{\hbar^2}{2m\lambda} |A|^2 e^{\frac{2m\lambda}{\hbar^2}x}\Big|_{-\infty}^{0} = |A|^2 \frac{\hbar^2}{2m\lambda}, \tag{1.22}$$

$$\int_{0}^{\infty} |A|^2 e^{-\frac{2m\lambda}{\hbar^2}x} dx = -\frac{\hbar^2}{2m\lambda} |A|^2 e^{-\frac{2m\lambda}{\hbar^2}x}\Big|_{0}^{\infty} = |A|^2 \frac{\hbar^2}{2m\lambda}. \tag{1.23}$$

Hence we obtain

$$|A|^2 \left(\frac{\hbar^2}{2m\lambda} + \frac{\hbar^2}{2m\lambda} \right) = |A|^2 \frac{\hbar^2}{m\lambda} = 1 \Leftrightarrow |A|^2 = \frac{m\lambda}{\hbar^2}. \tag{1.24}$$

Therefore we arrive at

$$A = |A| = \pm \frac{\sqrt{m\lambda}}{\hbar}. \tag{1.25}$$

By convention we take the positive value as A. So the solution of the Schrödinger equation, or of the eigenvalue problem, takes the following form

$$\psi(x) = \frac{\sqrt{m\lambda}}{\hbar} e^{-\frac{m\lambda}{\hbar^2}|x|}, \; E = -\frac{m\lambda^2}{2\hbar}. \tag{1.26}$$

Summarizing the thoughts of the last few pages we can say that we looked at a very special potential known as the Dirac δ-potential. We investigated its properties in the sense that we considered it in the stationary Schrödinger equation, and there we asked whether it provides bound states. Indeed, we found one bound state, which we then normalized. Figure 1.2 provides examples with concrete values of m and λ.

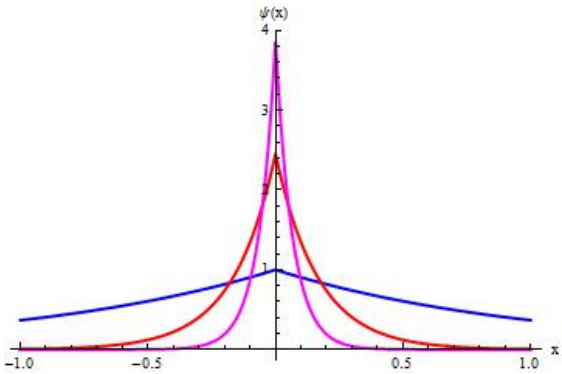


Figure 1.2: Examples of properly normalized bound state wave functions.

1.1.1.2 The scattering states

Now we shall set our attention on an attractive potential, where the energy of the particle is greater than zero (i.e. $E > 0$, $\lambda < 0$). The procedure is analogous to the solution of the bound case. Again we avoid the point $x = 0$ and look at the two sectors, where $x < 0$ respectively $x > 0$.

For $x < 0$, $E > 0$ the Schrödinger equation reads

$$H\psi(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x) \Leftrightarrow \frac{d^2}{dx^2}\psi(x) = -\frac{2mE}{\hbar^2}\psi(x). \tag{1.27}$$

This time there is a small complication. The factor in front of $\psi(x)$ is negative. Such differential equations can be solved by using a complex exponential ansatz

$$\psi_-(x) = Ae^{ikx} + Be^{-ikx},\ k = \sqrt{\frac{2mE}{\hbar^2}}. \tag{1.28}$$

There is one big problem with this procedure: Since the solution represents cosine and sine which are oscillating functions, without any damping, normalization, in the usual sense, can not be obtained. Nevertheless, it can still be of great use, if one is just interested in the behaviour of the system. The solution is describing a free particle and therefore there is a physical motivation for not discarding it. Furthermore, one can construct normalizable wave packages out of a bunch of given k's or wave numbers (for the procedure take for instance [6]), if desired. So if we constrain ourselves to the study of the behaviour of the system, we can continue with our work.

For $x > 0$, $E > 0$ we obtain

$$\psi(x)_+ = Ce^{ikx} + De^{-ikx},\ k = \sqrt{\frac{2mE}{\hbar^2}}. \tag{1.29}$$

This time our basic solution reads

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & ,x < 0 \\ Ce^{ikx} + De^{-ikx} & ,x > 0 \end{cases}. \tag{1.30}$$

Again ψ has a discontinuity at $x = 0$. We shall use our standard set of boundary conditions to patch up the function at $x = 0$. First, we investigate the continuity of $\psi(x)$

$$\lim_{x_-\to 0}\psi(x) = \lim_{x_-\to 0}\psi_-(x) = \lim_{x_-\to 0}(Ae^{ikx} + Be^{-ikx}) = A + B, \tag{1.31}$$

$$\lim_{x_+\to 0}\psi(x) = \lim_{x_+\to 0}\psi_+(x) = \lim_{x_+\to 0}(Ce^{ikx+}De^{-ikx}) = C + D. \tag{1.32}$$

If we demand continuity at $x = 0$, we have to impose $A + B = C + D$. Now we shall focus on the derivative of ψ. The same procedure as in the bound state case yields the following expression

$$-\frac{\hbar^2}{2m}ik(C - D - A + B) = \lambda(A + B) \Leftrightarrow (C - D - A + B) = -\frac{2m}{ik\hbar^2}(A + B) = \frac{2im\lambda}{k\hbar^2}(A + B). \tag{1.33}$$

With further simplifications, we can get an expression in which we can determine how C and D depend on A and B

$$C - D = \frac{2im\lambda}{k\hbar^2}A + A + \frac{2im\lambda}{k\hbar^2}B - B = A\left(1 + \frac{2im\lambda}{k\hbar^2}\right) - B\left(1 - \frac{2im\lambda}{k\hbar^2}\right). \tag{1.34}$$

To maintain some degree of simplicity in our notation, we define $\beta = \frac{m\lambda}{k\hbar^2}$. This alters equation (1.34) a bit

$$C - D = A(1 + i2\beta) - B(1 - i2\beta). \tag{1.35}$$

Up to this point we perhaps used algebra and standard procedures for solving differential equations. The reader has already noticed that we have five parameters (A, B, C, D and β) and not enough information to fix them all. In order to circumvent this problem we will now take a more physical approach to the problem. Let's imagine a

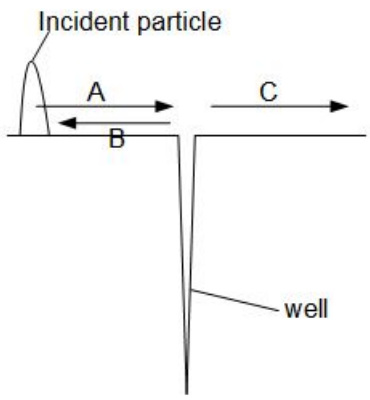


Figure 1.3: In order to reduce the number of free parameters, we imagine a scattering experiment, where the particle is fired from the left on to the well. A, B, C represent the corresponding amplitudes of our ansatz.

scattering experiment, where we fire a particle from the left on to the potential. As we already mentioned, our ansatz represents unbound particles. The terms with a positive sign in their exponent are moving to the right, and the terms with the negative sign in the exponent are moving to the left.
Since we fire from the left as shown in Figure 1.3, there will be no incoming particles from the right side. Therefore, the amplitude D can be set to zero. What remains is the incoming particle which can be reflected at the "edges" of the δ-peak. Further, the particle could also just pass through the well. Beware of thinking of the last statement as the tunnel-effect! The particle has a bigger energy than foreseen by the potential! Tunnelling describes the fact, that a particle with a smaller energy than the maximal well-energy still may be found outside the well. In contrast to this, our particle has enough energy to pass the potential without "bending" to it. Formalizing the statements above delivers

$$C - D = A(1 + i2\beta) - B(1 - i2\beta) \overset{D=0}{\Rightarrow} C = A(1 + i2\beta) - B(1 - i2\beta). \tag{1.36}$$

If we recall the continuity condition for $\psi(x)$, we can express equation (1.36) still a bit differently,

$$C = A(1 + i2\beta) - B(1 - i2\beta) \Leftrightarrow A + B = A(1 + i2\beta) - B(1 - i2\beta). \tag{1.37}$$

Now, we are able to determine B (or A) in dependence of A (respectively B). Of course we are completely free in our choice which parameter shall be expressed in dependence of which. In our scattering set up we chose to fire the particle from the left side on to the well and therefore it appears somehow "natural" to set A as the free parameter. The dependency is obtained as follows

$$A+B = A(1+i2\beta)-B(1-i2\beta) \Leftrightarrow B+B(1-i2\beta) = A(1+2i\beta)-A \Leftrightarrow B = A\frac{2i\beta}{2-2i\beta} \Leftrightarrow B = A\frac{i\beta}{1-i\beta}. \tag{1.38}$$

Remembering the fact that $C = A + B$ delivers an expression for C

$$C = A + B = A + A\frac{i\beta}{1-i\beta} \Leftrightarrow \frac{A - Ai\beta + Ai\beta}{1-i\beta} = A\frac{1}{1-i\beta}. \tag{1.39}$$

We managed to reduce the number of free parameters from 5 to 2, which is the best we can do. We could, however, create the already mentioned wave package and so fix A as a function of k, the energy of the particle. In our way of treatment we have to chose the incident amplitude A and the energy of the particle. With the computations above, we are even able to calculate the transmission and reflection coefficients for the scattering process, which depend on only one parameter, as we shall see now:

The transmission coefficient is defined as the modulus of the quotient of the passed amplitude and the incident amplitude squared. Formally this is

$$|T|^2 = \left|\frac{C}{A}\right|^2 = \frac{|C|^2}{|A|^2} = \left|\frac{1}{1-i\beta}\right|^2 = \frac{1}{1+\beta^2}. \tag{1.40}$$

The definition of the reflection coefficient is similarly to the one for the transmission

$$|R|^2 = \left|\frac{B}{A}\right|^2 = \frac{|B|^2}{|A|^2} = \left|\frac{i\beta}{1-i\beta}\right|^2 = \frac{\beta^2}{1+\beta^2}. \tag{1.41}$$

If we remember that $\beta = \frac{m\lambda}{k\hbar^2}$, we can see better how energy, mass, and the "depth" λ of the well influence the reflection and transmission behaviour of the well.

Before we proceed further, we shall, as promised on page 3, investigate the case $\lambda = 0$. To have a formally clear and firm statement, we will anticipate a procedure, which will be used and introduced properly in the next section (the reader may please forgive the disruption of the "reading flow"). If we use a Fourier transformation on the Schrödinger equation with our potential plugged in, we get the expression given below

$$\frac{\hbar^2}{2m}k^2\tilde{\psi}(k) + \int_{-\infty}^{\infty} -\lambda\delta(x)\psi(x)e^{-ikx}dx = E\tilde{\psi}(k). \tag{1.42}$$

Evaluating the integral in (1.42) generates

$$\frac{\hbar^2}{2m}k^2\tilde{\psi}(k) - \lambda\psi(0) = E\tilde{\psi}(k). \tag{1.43}$$

Since $\lambda = 0$, the second term vanishes and we obtain the description of an object which corresponds to a planar wave or an unbound particle (as will be shown in the next sub-chapter). Therefore, the case $\lambda = 0$ is the first unbound or scattering state.

1.1.2 The repulsive case

As already mentioned, a repulsive potential is defined with the positive sign of λ. We assume that our particle moves, again, in a non relativistic manner and we avoid the point $x = 0$ and look at the regions $x < 0$ and $x > 0$. Now there is a big difference between the attractive and the repulsive potential. Since the repulsive potential is greater than zero, there can not exist any bound states. In order to see why this fact holds, we recall the generic stationary Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x). \tag{1.44}$$

Rewriting it a bit, leads us to

$$\frac{d^2}{dx^2}\psi(x) = \frac{2m}{\hbar^2}(V(x) - E)\psi(x). \tag{1.45}$$

$E < V_{min}$ implies, that the sign of the second derivative of ψ can never change. Moreover, the sign is also always the same as that of ψ itself. If we recall what the second derivative of a function means (i.e. the way a function is being bent), we can see, that we will have trouble to normalize ψ. It is, in any case, bent away from the x-axis, which is equivalent to a function without damping. Therefore, the minimal energy has to be bigger than V_{min}, if we want normalizable (physical) solutions.

Now we shall take a look at the Schrödinger equation for $x < 0, E > 0$. It reads

$$\frac{d^2}{dx^2}\psi(x) = -\frac{2mE}{\hbar^2}\psi(x), \tag{1.46}$$

which is the same expression as for the scattering states in the attractive case. Since further processing would just produce a copy of the scattering solutions, we will not think the repulsive potential through. One final (amazing) remark has to be made, though. Since both cases (scattering states in the attractive case and all solutions to the repulsive potential) lead to plane waves, we can construct for both scenarios transmission and reflection coefficients. Since the procedure for those coefficients does not change and they depend only on λ^2, there would be no difference in the results! Therefore we can say that, as far as the transmission and reflection probabilities are concerned, we can not distinguish a well from a barrier!

1.1.3 Considering the scattering states by using the spatial symmetry of the Dirac δ-potential

On the last few pages we solved the problem by eliminating parameters through introducing a physically meaningful experimental setting. As we now shall see, there is a way to solve the problem without taking recourse to an experimental setting. Before we start, lets again take a look at the potential. As it can be seen in Figure 1.4, the potential is spatially symmetric around the peak. This allows us to consider even and odd functions as solutions [6]. Those solutions we shall figure out on the next few pages. Since we have the above mentioned parity symmetry, it is sufficient to consider one side of the well. The opposite side is then fixed automatically by mirroring the obtained solution properly.

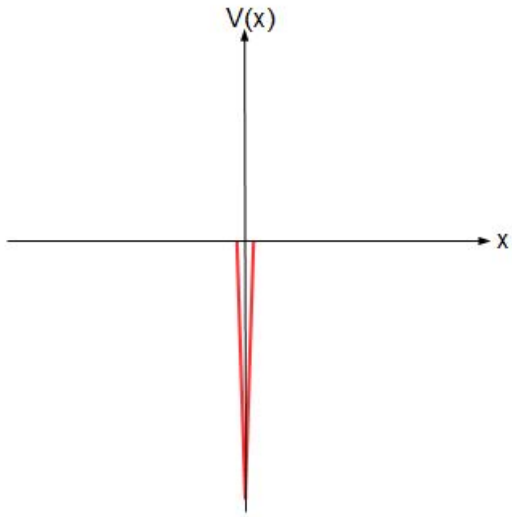


Figure 1.4: If one sets up the reference frame in such a way that the δ peak is at the origin, one gets a spatially symmetric setting around 0. This allows the construction of solutions which respect the parity symmetry.

We will now take a look on the right hand side of the well and consider the Schrödinger equation there. With the same assumptions as in the previous sub-sections and for $x > 0$ it reads

$$\frac{d^2}{dx^2}\psi(x) = -\frac{2Em}{\hbar^2}\psi(x). \tag{1.47}$$

This equation can be solved, if we try a $\sin(x)$ ansatz.

$$\psi(x) = A\sin\left(\frac{\sqrt{2Em}}{\hbar}\right). \tag{1.48}$$

Of course, the reader may take the ansatz and check by differentiating two times that it solves the Schrödinger equation. To patch together the solution for both regions we will reflect the solution. To reflect it properly we have to recall that $\sin(x)$ is an odd function. One now may ask what "odd" or "even" may mean for functions. Formalized, behind "odd" and "even" are the following definitions:

$$\text{f is odd} :\Leftrightarrow f(-x) = -f(x). \tag{1.49}$$

$$\text{f is even} :\Leftrightarrow f(-x) = f(x). \tag{1.50}$$

With the previous definition we can now reflect the solution. Since $x < 0$ provides already a sign, we have to attach another to it (so the odd property is compensated). After this, we can attach the regular mirroring sign to the solution. The process combined with the usual boundary conditions (i.e. continuity of ψ and continuity almost everywhere of ψ') yields

$$\psi(x) = \begin{cases} -A\sin(-kx) & , x < 0, \\ 0 & , x = 0, \\ A\sin(kx) & , x > 0. \end{cases} \tag{1.51}$$

Now we will focus on the derivative of ψ. To determine its behaviour, we will use the same trick as in the previous sub-sections.

$$\int_{-\epsilon}^{\epsilon} -\frac{\hbar^2}{2m}\psi^{''}(x)dx + \int_{-\epsilon}^{\epsilon} \lambda\delta(x)\psi(x)dx = E\int_{-\epsilon}^{\epsilon} \psi(x)dx. \tag{1.52}$$

The right-hand side and the second term of the equation are zero. The first term demands a closer look

$$-\frac{\hbar^2}{2m}\psi^{'}(x)|_{-\epsilon}^{\epsilon} = \frac{\hbar^2}{2m}(Ak\cos(-k\epsilon) - Ak\cos(k\epsilon)) = 0. \tag{1.53}$$

In the line above we exploited the fact, that $\cos(x)$ is an even function and so the sign of the argument does not matter at all. Therefore also the first term becomes zero and hence the derivative of ψ at $x = 0$ is zero. With the knowledge we gathered above, we can say that odd functions do not sense the δ-peak.
Let us recall the Schrödinger equation for $x > 0$ again

$$\frac{d^2}{dx^2}\psi(x) = -\frac{E2m}{\hbar^2}\psi(x). \tag{1.54}$$

On the previous pages we used a $\sin(x)$, which fulfills the equation. However, as the reader can convince himself, also a $\cos(x)$ solves the equation. We will now try the following ansatz

$$\psi(x) = B\cos(kx + \phi), x > 0. \tag{1.55}$$

The constant phase ϕ can be justified the following heuristic way. The cosine has a maximum at $x = 0$. If we put there an "obstacle" (e.g. a repulsive δ peak), it would be wrong if the modulus of $\psi(x = 0)$ squared, would be one still. After all, there is a force at $x = 0$ that pushes the particle aside. Reflecting our ansatz properly delivers

$$\psi(x) = \begin{cases} B\cos(-(kx - \phi)) & , x < 0, \\ B\cos(\phi) & , x = 0, \\ B\cos(kx + \phi) & , x > 0. \end{cases} \tag{1.56}$$

Once more, we use our trick to look at the derivative of ψ

$$\int_{-\epsilon}^{\epsilon} -\frac{\hbar^2}{2m}\psi^{''}(x)dx + \int_{-\epsilon}^{\epsilon} \lambda\delta(x)\psi(x)dx = E\int_{-\epsilon}^{\epsilon} \psi(x)dx. \tag{1.57}$$

Computation of the three terms yields

$$-\frac{\hbar^2}{2m}(-k\sin(k\epsilon + \phi) - k\sin(k\epsilon + \phi)) + \lambda\cos(\phi) = 0. \tag{1.58}$$

The first term can be simplified further

$$-\frac{\hbar^2}{2m}(-2k\sin(k\epsilon + \phi)) = \frac{\hbar^2 k}{m}\sin(k\epsilon + \phi). \tag{1.59}$$

$\epsilon \to 0$ generates

$$\lim_{\epsilon\to 0}\frac{\hbar^2 k}{m}\sin(k\epsilon + \phi) = \frac{\hbar^2 k}{m}\sin(\phi). \tag{1.60}$$

Inserting the result (1.60) into equation (1.58) we get

$$\frac{\hbar^2 k}{m}\sin(\phi) = -\lambda\cos(\phi). \tag{1.61}$$

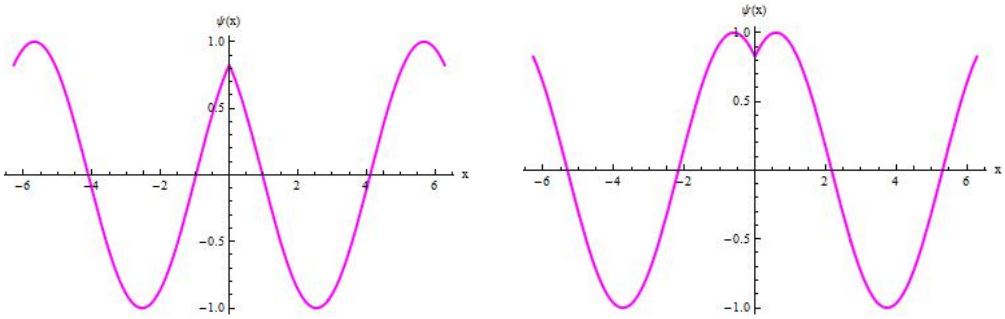
If we push further towards an expression for ϕ we can take the ϕ-containing factors on one side and the other things on the other. In addition, if we recall that $\frac{\sin(x)}{\cos(x)} = \tan(x)$ [1, p. 52] we obtain a nice and simple expression

$$\tan(\phi) = -\frac{\lambda m}{\hbar^2 k}. \tag{1.62}$$

Unleashing the arctan on equation (1.62), we obtain an expression in which ϕ is explicit

$$\phi = \arctan\left(-\frac{\lambda m}{\hbar^2 k}\right) = -\arctan\left(\frac{\lambda m}{\hbar^2 k}\right) \tag{1.63}$$

Until now, we spent not a single word on the properties of the potential. Since the property of repulsiveness (attractiveness) is hidden in the sign of λ, we have now to consider it. According to equation (1.63), a repulsive potential produces a negative ϕ, while an attractive one generates a positive ϕ.



(a) Attractive case (b) Repulsive case

Figure 1.5: The influence of the sign of λ on the phase ϕ and the behaviour of the wavefunction around $x = 0$.

If we take a look at Figure 1.5, we can see that even in the attractive case the probability to find the particle at $x = 0$ is only a local maximum (despite the fact, that there sits an attracting force). The probability to find it at some points away from the well is higher. The right picture also shows very impressively how the repulsive potential "pushes" the wave function away from $x = 0$. One may ask now, what the difference between the approach of the previous sub-sections and the actual procedure is. In the previous sub-sections we had to use an experimental setting in order to fix or eliminate free parameters. In the actual approach we just generated two fundamental solutions without any recourse to an experimental setting, which is not only useful for a mathematical purist. If we recall the Euler relation ($e^{ix} = \cos(x) + i\sin(x)$)[1, p. 31], we are able to construct the experimental approach from the present one.

1.2 Exploration of the attractive δ-potential in momentum space

The last section was dedicated to the exploration of our problem in position space. In order to prepare ourselves for the consideration of the more sophisticated relativistic problem, we shall now explore the same problem as in the previous section. However, this time we will explore it in momentum space. Since momentum and position space are linked by a Fourier transformation [5], we will first check how the Schrödinger equation behaves under such a transformation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x), \tag{1.64}$$

where $V(x)$ is a generic potential. Applying a Fourier transformation to the expression above, yields the equation

$$\int_{-\infty}^{\infty}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x)\right)e^{-ikx}dx = \int_{-\infty}^{\infty} E\psi(x)e^{-ikx}dx. \tag{1.65}$$

Since the Fourier transformation is a linear map, we can simplify equation (1.65) to

$$-\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\frac{d^2}{dx^2}\psi(x)e^{-ikx}dx + \int_{-\infty}^{\infty} V(x)\psi(x)e^{-ikx}dx = \int_{-\infty}^{\infty} E\psi(x)e^{-ikx}dx. \tag{1.66}$$

Exploiting the property of the Fourier transformation, that derivatives of the variable, which is being transformed, turn into factors of the new variable, equation (1.66) becomes

$$\frac{\hbar^2}{2m}k^2\tilde{\psi}(k) + \int_{-\infty}^{\infty} V(x)\psi(x)e^{-ikx}dx = E\tilde{\psi}(k). \tag{1.67}$$

Further simplifications are not possible, until one knows the properties of the potential $V(x)$. With the insights we gained above, we will now focus on the same problem as in the previous section. All assumptions and conventions we made there shall now be used once more.

1.2.1 The bound state

Since the Dirac δ-potential is a concrete one, we can plug it into equation (1.67) and see what we get

$$\frac{\hbar^2}{2m}k^2\tilde{\psi}(k) - \int_{-\infty}^{\infty} \lambda\delta(x)\psi(x)e^{-ikx}dx = E\tilde{\psi}(k). \tag{1.68}$$

The second term contains a δ-function and therefore we can compute it by using the "computation rules" [5] for the δ-function.

$$-\int_{-\infty}^{\infty} \lambda\delta(x)\psi(x)e^{-ikx}dx = -\lambda\psi(0) = c_b. \tag{1.69}$$

With the considerations above (c_b is the bound case constant), equation (1.68) can be written as

$$\frac{\hbar^2}{2m}k^2\tilde{\psi}(k) + c_b = E\tilde{\psi}(k). \tag{1.70}$$

The last result we gained is the Schrödinger equation we have to solve in momentum space, if we are interested in the wave function. Executing some algebraic manipulations in (1.70) delivers an expression for the wave function in momentum space

$$\frac{c_b}{E - \frac{\hbar^2k^2}{2m}} = \tilde{\psi}(k). \tag{1.71}$$

This may somehow be a bit surprising. After all, in position space we had to solve a second order differential equation. However, we must not forget that (thanks to the properties of the Fourier transformation) we transformed the derivatives into factors! Let us now figure out the energy eigenvalue for the bound state. For this task we shall again use a well known property of the Fourier transformation

$$\psi(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\psi}(k)e^{ik0}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\psi}(k)dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{c_b}{E - \frac{\hbar^2k^2}{2m}}dk. \tag{1.72}$$

Recalling the expression for c_b, we get

$$\psi(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-\lambda\psi(0)}{E - \frac{\hbar^2k^2}{2m}}dk. \tag{1.73}$$

Further processing delivers a useful relation:

$$1 = \frac{\lambda}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\frac{\hbar^2k^2}{2m} + (-E)}dk. \tag{1.74}$$

This relation is known as the "gap equation", which we shall now investigate further. Since we are considering the bound state case, the energy E is smaller than zero and therefore we write the additional sign in front of E explicitly in order to prevent confusion about the square root of negative values

$$\frac{\lambda}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\frac{\hbar^2k^2}{2m} + (-E)}dk = \frac{\lambda}{2\pi(-E)}\int_{-\infty}^{\infty} \frac{1}{\frac{\hbar^2k^2}{2(-E)m} + 1}dk. \tag{1.75}$$

Making the substitution $u = \frac{\hbar k}{\sqrt{2(-E)m}}$, $\frac{du}{dk} = \frac{\hbar}{\sqrt{2(-E)m}}$ brings us to a well known integral, which we can process further

$$\frac{\lambda\sqrt{2(-E)m}}{2\pi E\hbar}\int_{-\infty}^{\infty}\frac{1}{u^2+1}du = \frac{\lambda\sqrt{2(-E)m}}{2\pi E\hbar}\arctan(u)|_{-\infty}^{\infty} = \frac{\lambda\sqrt{2(-E)m}}{2\pi E\hbar}\pi. \tag{1.76}$$

Recalling the relation of equation (1.74), we get

$$1 = \frac{\lambda\sqrt{2(-E)m}}{2\pi E\hbar}\pi. \tag{1.77}$$

From here on we obtain an expression for E, which states

$$E = -\frac{\lambda^2 m}{2\hbar^2} =: E_b. \tag{1.78}$$

This is the same eigenvalue that we obtained in the previous section.
To summarize, we have a wave function and an eigenvalue, which take the form

$$\tilde{\psi}(k) = \frac{c_b}{E - \frac{\hbar^2 k^2}{2m}}, E_b = -\frac{m\lambda^2}{2\hbar^2}. \tag{1.79}$$

1.2.2 The scattering states

As in the exploration in position space, we shall now look at states with positive energy (or scattering states). Since the parity approach was an elementary one, we would like to use it also in momentum space. If we want to keep an analogous procedure to that one in position space, we will hit on a problem: In order to keep up compatibility between the solutions in position and momentum space, we have to use the position space functions equivalents in momentum space. More concretely: We require Fourier transforms of functions, which are not exactly Fourier transformable (sine and cosine are not L_2 [4] functions). However, there is a way to handle this issue. Before we do it, we shall conduct a small exercise, which will (hopefully) clarify the procedure on the next few pages (and minimize confusion).

Although the sine function is not a member of the L_2 space, we can still apply the Fourier transformation to it and look at what we get

$$\int_{-\infty}^{\infty}\sin(k_0x)e^{-ikx}dx. \tag{1.80}$$

This expression is not very encouraging, but if we remember once more the Euler relation, we can rewrite the sine and insert its new form into equation (1.80):

$$\frac{1}{2i}\int_{-\infty}^{\infty}(e^{ik_0x} - e^{-ik_0x})e^{-ikx}dx = \frac{1}{2i}\left(\int_{-\infty}^{\infty}e^{i(k_0-k)x}dx - \int_{-\infty}^{\infty}e^{-i(k_0+k)}dx\right). \tag{1.81}$$

The right-hand side of the equation above corresponds to the Fourier transformation of the constant function $f(x) = 1$, which is by "definition" the Dirac δ [5].

$$\frac{1}{2i}\left(\int_{-\infty}^{\infty}e^{-i(k-k_0)x}dx - \int_{-\infty}^{\infty}e^{-i(k_0+k)}dx\right) = \frac{1}{2i}(\delta(k-k_0) - \delta(k+k_0)). \tag{1.82}$$

Executing the inverse Fourier transformation yields

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2i}(\delta(k-k_0) - \delta(k+k_0))e^{ikx}dk = \frac{1}{4\pi i}\left(\int_{-\infty}^{\infty}\delta(k-k_0)e^{ikx}dk - \int_{-\infty}^{\infty}\delta(k+k_0)e^{ikx}dk\right), \tag{1.83}$$

which after evaluating the integrals becomes

$$\frac{1}{4\pi i}(e^{ik_0x} - e^{-ik_0x}) = \frac{1}{4\pi i}(2i\sin(k_0x)) = \frac{1}{2\pi}\sin(k_0x). \tag{1.84}$$

Hence, if we want a proper Fourier transformation of a sine (the cosine is examined analogously), we have to attach some factors. With the exercise in mind, we can now write down the Fourier transforms of sine and cosine

$$\mathcal{F}(\sin(k_0 x)) = \frac{\pi}{i}\left(\delta(k-k_0) - \delta(k+k_0)\right), \tag{1.85}$$

$$\mathcal{F}(\cos(k_0 x)) = \pi(\delta(k-k_0) + \delta(k+k_0)), \tag{1.86}$$

where the $\mathcal{F}$ stands for the Fourier transformation.

With the above insights, we now write down an ansatz for the odd solution of the transformed Schrödinger equation

$$\tilde{\psi}_E^{odd}(p) = \frac{\pi}{i}(\delta(p-k) - \delta(p+k)). \tag{1.87}$$

Inserting this ansatz in the transformed Schrödinger equation yields

$$\frac{\hbar^2}{2m}p^2\frac{\pi}{i}(\delta(p-k) - \delta(p+k)) + c_s = E\frac{\pi}{i}(\delta(p-k) - \delta(p+k)). \tag{1.88}$$

Recalling that c_s (where the s stands for "scattering") is $-\lambda\psi(0)$ (or remembering that odd solutions do not "sense" the δ-peak), we can further simplify the expression above

$$\frac{\hbar^2}{2m}p^2\frac{\pi}{i}(\delta(p-k) - \delta(p+k)) = E\frac{\pi}{i}(\delta(p-k) - \delta(p+k)). \tag{1.89}$$

If we wish our ansatz to solve the transformed equation, we have to identify E with $\frac{\hbar^2 k^2}{2m}$. If we do so, we have the odd solution to the transformed Schrödinger equation. Now we make an ansatz for the even solution

$$\tilde{\psi}_E^{even}(p) = \pi(\delta(p-k) + \delta(p+k)) + \tilde{\phi}(p). \tag{1.90}$$

Inserting it in the transformed Schrödinger equation we get an expression which is a bit more complicated than in the odd case

$$\frac{\hbar^2}{2m}p^2\left(\pi(\delta(p-k) + \delta(p+k)) + \tilde{\phi}(p)\right) + c_s = E(\pi(\delta(p-k) + \delta(p+k)) + \tilde{\phi}(p)). \tag{1.91}$$

This time, there is no reason to set c_s to zero, since the corresponding ansatz in position space is not zero (in general). While the sum of the δ-functions would provide the cosine in position space, the term which contains $\tilde{\phi}(p)$ is a puzzle, to which we will now give further thoughts. If we recall a not correct but "pragmatic" definition of the Dirac δ (i.e. everywhere zero, except at the origin) we can exploit the fact that it is zero for values of p which are not equal $\pm k$. If we look at those values, expression (1.91) becomes much simpler to work with. The δ terms vanish and what survives, after further elementary algebraic operations, looks the following way

$$\frac{\hbar^2 p^2}{m}\tilde{\phi}(p) + 2c_s = 2E\tilde{\phi}(p). \tag{1.92}$$

From this equation, we can derive an expression for $\tilde{\phi}(p)$, which states that

$$\tilde{\phi}(p) = \frac{c_s}{E - \frac{\hbar^2 p^2}{2m}}. \tag{1.93}$$

Summarizing the last few lines, we now have the solution to the Fourier transformed Schrödinger equation. This solution takes the form

$$\tilde{\psi}_E^{even}(p) = \pi(\delta(p-k) + \delta(p+k)) + \frac{c_s}{E - \frac{\hbar^2 p^2}{2m}}. \tag{1.94}$$

If we want to check the compatibility of our ansatz, we need the inverse Fourier transform of $\tilde{\phi}(p)$. Formalized, this means

$$\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{c_s}{E - \frac{\hbar^2 p^2}{2m}} e^{ipx} dp = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{c_s}{\frac{\hbar^2 k^2}{2m} - \frac{\hbar^2 p^2}{2m}} e^{ipx} dp. \tag{1.95}$$

Recalling how it is possible to express $\phi(0)$ in terms of its Fourier transform $\tilde{\phi}(p)$, we can gain an expression for c_s which depends just on ϕ

$$c_s = -\lambda\psi(0) = \frac{-\lambda}{2\pi}\int_{-\infty}^{\infty} \tilde{\psi}(p) e^{ip0} dp = \frac{-\lambda}{2\pi}\int_{-\infty}^{\infty} (\pi(\delta(p-k) + \delta(p+k)) + \tilde{\phi}(p)) dp. \tag{1.96}$$

Picking the right-hand side of (1.96) and further processing yields

$$c_s = \frac{-\lambda}{2\pi}\left(2\pi + \int_{-\infty}^{\infty} \tilde{\phi}(p) dp\right) = -(\lambda + \lambda\phi(0)). \tag{1.97}$$

The reader may now ask himself, why we derived such an expression. After all, we already have one which works well. The expression above is completely independent of any pre-knowledge of things in position space and therefore we are able to develop a solution in momentum space and compare it with the one we constructed in position space. So this path presents a very good cross check opportunity. We will keep the alternative description of c_s in mind but now we shall return to the form of $\phi(x)$.

As seen in expression (1.95), we have to solve the following integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{c_s}{\frac{\hbar^2 k^2}{2m} - \frac{\hbar^2 p^2}{2m}} e^{ipx} dp = \frac{2mc_s}{2\pi\hbar^2}\int_{-\infty}^{\infty} \frac{1}{k^2 - p^2} e^{ipx} dp = \frac{mc_s}{\pi\hbar^2}\int_{-\infty}^{\infty} \frac{1}{k^2 - p^2} e^{ipx} dp. \tag{1.98}$$

Perhaps the reader has already noticed, that despite its simple appearance, the integrand has very sharp teeth. It contains two singular points at $p = \pm k$ over which we must integrate. If we can circumvent them, we may even be able to fix them in a further step. Indeed, there is a way to avoid these points. Since the integrand is a quotient of two polynomials, it has an analytic continuation to $\mathbb{C}$, which looks the same as the original function. Therefore, we shall consider the integrand as a map from $\mathbb{C}$ to $\mathbb{C}$ and try to integrate it there. Since we now have one more dimension of spatial freedom, we can move around the critical points. As for every integration in $\mathbb{C}$, we have to patch together an integration path. Figure 1.6 provides an illustration of the path we will take.

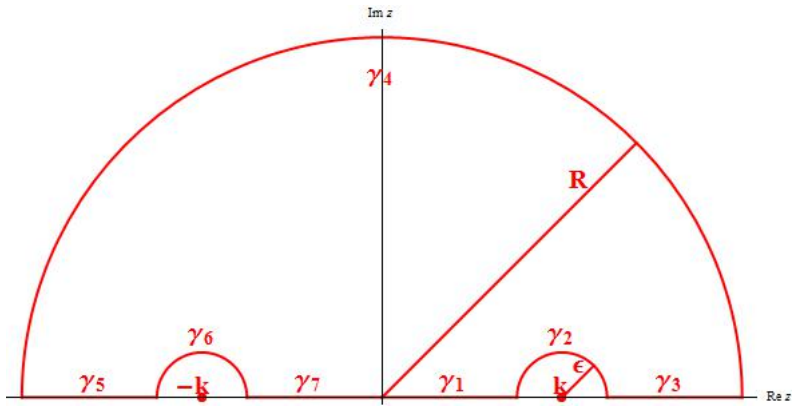


Figure 1.6: The integration path we shall take (orientation: counter-clockwise) in order to evade the singular points and to determine $\phi(x)$.

A way to parametrize the path above is given on the next page.

$$\gamma_1 : [0,k] \to \mathbb{C}, \qquad \gamma_1(t) = t - \epsilon, \qquad \gamma_1' = 1. \tag{1.99}$$
$$\gamma_2 : [\pi,0] \to \mathbb{C}, \qquad \gamma_2(t) = k + \epsilon e^{it}, \qquad \gamma_2' = i\epsilon e^{it}. \tag{1.100}$$
$$\gamma_3 : [k,R] \to \mathbb{C}, \qquad \gamma_3(t) = t + \epsilon, \qquad \gamma_3' = 1. \tag{1.101}$$
$$\gamma_4 : [0,\pi] \to \mathbb{C}, \qquad \gamma_4(t) = Re^{it}, \qquad \gamma_4' = iRe^{it}. \tag{1.102}$$
$$\gamma_5 : [-R,-k] \to \mathbb{C}, \qquad \gamma_5(t) = t - \epsilon, \qquad \gamma_5' = 1. \tag{1.103}$$
$$\gamma_6 : [\pi,0] \to \mathbb{C}, \qquad \gamma_6(t) = -k + \epsilon e^{it}, \qquad \gamma_6' = i\epsilon e^{it}. \tag{1.104}$$
$$\gamma_7 : [-k,0] \to \mathbb{C}, \qquad \gamma_7(t) = t + \epsilon, \qquad \gamma_7' = 1. \tag{1.105}$$

Therefore, the integral takes the form

$$\int_\gamma \frac{e^{itz}}{k^2 - z^2} dz = \int_0^k \frac{e^{i(t-\epsilon)x}}{k^2 - (t-\epsilon)^2} dt + \int_\pi^0 \frac{e^{i(k+\epsilon e^{it})x}}{k^2 - (k+\epsilon e^{it})^2} i\epsilon e^{it} dt + \int_k^R \frac{e^{i(t+\epsilon)x}}{k^2 - (t+\epsilon)^2} dt +$$
$$+ \int_0^\pi \frac{e^{iRe^{it}x}}{k^2 - (Re^{it})^2} iRe^{it} dt + \int_{-R}^{-k} \frac{e^{i(t-\epsilon)x}}{k^2 - (t-\epsilon)^2} dt + \int_\pi^0 \frac{e^{i(-k+\epsilon e^{it})x}}{k^2 - (-k+\epsilon e^{it})^2} i\epsilon e^{it} dt + \int_{-k}^0 \frac{e^{i(t+\epsilon)x}}{k^2 - (t+\epsilon)^2} dt. \tag{1.106}$$

Since the function is holomrophic on the domain we defined with the closed integration path, writing the terms in a different order, considering the limit $\epsilon \to 0$ and using the residue theorem [3] provides

$$\int_{-R}^R \frac{e^{itx}}{k^2 - t^2} dt + \lim_{\epsilon \to 0} \left(\int_\pi^0 \frac{e^{i(k+\epsilon e^{it})x}}{k^2 - (k+\epsilon e^{it})^2} i\epsilon e^{it} dt + \int_\pi^0 \frac{e^{i(-k+\epsilon e^{it})x}}{k^2 - (-k+\epsilon e^{it})^2} i\epsilon e^{it} dt \right) + \int_0^\pi \frac{e^{iRe^{it}x}}{k^2 - (Re^{it})^2} iRe^{it} dt = 0. \tag{1.107}$$

The first term is the integral we wish to determine, while the fourth term will vanish, when we send R to infinity (the reader may convince himself quite easily that the integrand is bounded and will vanish). Terms two and three arise from the paths we used to circumvent the critical points. In the limit $\epsilon \to 0$ we have to take a closer look at them. Let us investigate term number two first

$$\lim_{\epsilon \to 0} \left(\int_\pi^0 \frac{e^{i(k+\epsilon e^{it})x}}{k^2 - (k+\epsilon e^{it})^2} i\epsilon e^{it} dt \right) = \lim_{\epsilon \to 0} \left(\int_\pi^0 \frac{e^{i(k+\epsilon e^{it})x}}{k^2 - (k^2 + 2k\epsilon e^{it} + (\epsilon e^{it})^2)} i\epsilon e^{it} dt \right). \tag{1.108}$$

Further simplifying yields

$$\lim_{\epsilon \to 0} \left(\int_\pi^0 \frac{e^{i(k+\epsilon e^{it})x}}{-2k - \epsilon e^{it})} i dt \right) = -\int_\pi^0 \frac{ie^{ikx}}{2k} dt = \frac{i\pi}{2k} e^{ikx}. \tag{1.109}$$

An analogous procedure simplifies the third term to

$$\lim_{\epsilon \to 0} \left(\int_\pi^0 \frac{e^{i(-k+\epsilon e^{it})x}}{k^2 - (-k+\epsilon e^{it})^2} i\epsilon e^{it} dt \right) = -\frac{i\pi}{2k} e^{-ikx}. \tag{1.110}$$

The second term is valid for $x > 0$ and the third holds for $x < 0$. Since the third term contains an additional sign in the exponent, we have to take the modulus of x in order to patch together the correct behaviour

$$\frac{i\pi}{2k} e^{ikx} - \frac{i\pi}{2k} e^{-ikx} = -\frac{i\pi}{2k} i2\sin(k|x|) = -\frac{\pi}{k}\sin(k|x|). \tag{1.111}$$

Hence, $\phi(x)$ has the following form

$$\phi(x) = \frac{mc_s}{\hbar^2 k} \sin(k|x|). \tag{1.112}$$

With the insights we gathered above, we are now able to write down the inverse Fourier transform of $\tilde{\psi}_E^{even}$.

$$\psi_E^{even}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\pi(\delta(k-p)+\delta(k-p)) + \frac{c_s}{E-\frac{\hbar^2 p^2}{2m}}\right)e^{ipx}dp = \cos(kx) + \frac{mc_s}{\hbar^2 k}\sin(k|x|). \tag{1.113}$$

Remembering expression (1.97) we can further simplify the above equation, which yields

$$\cos(kx) + \frac{mc_s}{\hbar^2 k}\sin(k|x|) = \cos(kx) + \frac{-m\lambda}{\hbar^2 k}\sin(k|x|). \tag{1.114}$$

Remarkably the factor in front of the sine is nothing else than $-tan(\phi)$ (see equation (1.62) and do not forget the sign convention for the attractive potential)! Therefore, we can write (1.114) still a bit differently

$$\cos(kx) + \frac{m\lambda}{\hbar^2 k}\sin(k|x|) = \cos(kx) - \frac{\sin(\phi)}{\cos(\phi)}\sin(k|x|). \tag{1.115}$$

Multiplying the last equation with $\cos(\phi)$ yields

$$\cos(\phi)(\cos(kx) - \frac{\sin(\phi)}{\cos(\phi)}\sin(k|x|)) = \cos(kx)\cos(\phi) - \sin(\phi)\sin(k|x|). \tag{1.116}$$

Invoking the relation $\cos(\alpha\pm\beta) = \cos(\alpha)\cos(\beta)\mp\sin(\alpha)\sin(\beta)(see[1, \text{p. } 54])$ would allow further simplifications. In order to prevent a mess with signs (after all, there is a modulus to be considered), we shall look at $x > 0$ first. So equation (116) becomes

$$\cos(kx)\cos(\phi) - \sin(\phi)\sin(kx) = \cos(kx+\phi). \tag{1.117}$$

The same relation applied for $x < 0$ yields

$$\begin{aligned}&\cos(kx)\cos(\phi) - \sin(\phi)\sin(-kx) = \cos(kx)\cos(\phi) + \sin(\phi)\sin(kx) = \cos(kx-\phi) = \\ &= \cos(-(-kx+\phi)) = \cos(-kx+\phi).\end{aligned} \tag{1.118}$$

Equations (1.117) and (1.118) can be written compactly as

$$\cos(kx)\cos(\phi) - \sin(\phi)\sin(k|x|) = \cos(k|x|+\phi), \tag{1.119}$$

which is nothing else than the ansatz we used to solve the Schrödinger equation in position space! Therefore, if we add the "miraculous" factor $\cos(\phi)$ to the ansatz in momentum space, we achieve full compatibility of the approaches in momentum and position space.

1.3 Orthogonality in momentum space

Before we progress any further, we have to check if the acquired solutions in momentum space fulfill orthogonality (after all we consider them to be eigenstates of a physical and therefore self-adjoint system). Therefore, we shall dedicate this section to the checking of orthogonality of the solutions we gathered in the previous section. First, we will check if the wave function of the bound state is orthogonal to the unbound states. For the odd wave function this looks like

$$\left\langle \psi_E^{odd} \middle| \psi_{E_b} \right\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\frac{\pi}{i}(\delta(p-k)-\delta(p+k)))^*\left(\frac{c_b}{E_b - \frac{\hbar^2 p^2}{2m}}\right)dp. \tag{1.120}$$

Further simplifications lead us to the following expression

$$\left\langle \psi_E^{odd} \middle| \psi_{E_b} \right\rangle = -\frac{c_b}{2i}\left(\frac{1}{E_b - \frac{\hbar^2 k^2}{2m}} - \frac{1}{E_b - \frac{\hbar^2(-k)^2}{2m}}\right) = 0. \tag{1.121}$$

Therefore, the odd scattering solutions are orthogonal to the bound state. For the even solutions the same procedure as above, yields

$$\left\langle \psi_E^{even} \middle| \psi_{E_b} \right\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\pi(\delta(p-k) + \delta(p+k)) + \frac{c_s}{E - \frac{\hbar^2 p^2}{2m}} \right)^* \left(\frac{c_b}{E_b - \frac{\hbar^2 p^2}{2m}} \right) dp. \tag{1.122}$$

This time, further (elementary) processing generates an expression, which looks a bit more complicated than in the odd case

$$\left\langle \psi_E^{even} \middle| \psi_{E_b} \right\rangle = \frac{c_b}{2} \left(\frac{2}{E_b - \frac{\hbar^2 k^2}{2m}} \right) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{c_s}{E - \frac{\hbar^2 p^2}{2m}} \frac{c_b}{E_b - \frac{\hbar^2 p^2}{2m}} dp. \tag{1.123}$$

The second term on the right-hand side demands further treatment. Since we have a product of two rational polynomials we can conduct a partial fraction expansion. The corresponding computation is not complicated but tiresome and bulky (we encourage the reader though, still to control the result for himself). Therefore, here we provide just the result.

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{c_s}{E - \frac{\hbar^2 p^2}{2m}} \frac{c_b}{E_b - \frac{\hbar^2 p^2}{2m}} dp = \frac{c_s c_b}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(\frac{2mE}{\hbar^2} - p^2)(\frac{2mE_b}{\hbar^2} - p^2)\frac{\hbar^4}{4m^2}} dp. \tag{1.124}$$

After the partial fraction expansion we have

$$\frac{4m^2 c_s c_{c_b}}{\hbar^4 2\pi} \int_{-\infty}^{\infty} \frac{1}{(\alpha - p^2)(\beta - p^2)} = \frac{4m^2 c_s c_b}{\hbar^4 2\pi} \frac{1}{\beta - \alpha} \left(\int_{-\infty}^{\infty} \frac{1}{\alpha - p^2} dp - \int_{-\infty}^{\infty} \frac{1}{\beta - p^2} dp \right). \tag{1.125}$$

If we recall the integral of equation (1.98) and its result (1.112) we can use the fact that the first term is nothing else than $\phi(0)$, which is 0. Hence, equation (1.125) gets reduced to

$$\frac{4m^2 c_s c_b}{\hbar^4 2\pi} \frac{1}{\beta - \alpha} (-1) \int_{-\infty}^{\infty} \frac{1}{\beta - p^2} dp = \frac{4m^2 c_s c_b}{\hbar^4 2\pi} \frac{1}{\beta - \alpha} \int_{-\infty}^{\infty} \frac{1}{p^2 + \tilde{\beta}} dp, \tag{1.126}$$

where we absorbed the sign into the negative β which is now the positive $\tilde{\beta}$. Evaluating the integral analogously to (1.75), the expression above becomes

$$\frac{4m^2 c_s c_b}{\hbar^4 2\pi} \frac{1}{\beta - \alpha} \int_{-\infty}^{\infty} \frac{1}{p^2 + \tilde{\beta}} dp = \frac{4m^2 c_s c_b}{\hbar^4 2\pi} \frac{1}{\beta - \alpha} \frac{\pi}{\sqrt{\tilde{\beta}}}. \tag{1.127}$$

Simplifying and reconstituting $\alpha,\ \beta,\ \tilde{\beta}$ yields

$$\frac{4m^2 c_s c_b}{\hbar^4 2\pi} \frac{1}{\beta - \alpha} \frac{\pi}{\sqrt{\tilde{\beta}}} = \frac{4m^2 c_s c_b}{\hbar^4 2} \frac{1}{\frac{2mE_b}{\hbar^2} - \frac{2mE}{\hbar^2}} \frac{1}{\sqrt{\frac{-2mE_b}{\hbar^2}}}. \tag{1.128}$$

Further processing delivers

$$\frac{4m^2 c_s c_b}{\hbar^4 2} \frac{1}{\frac{2mE_b}{\hbar^2} - \frac{2mE}{\hbar^2}} \frac{1}{\sqrt{\frac{-2mE_b}{\hbar^2}}} = \frac{m c_s c_b}{\hbar} \frac{1}{E_b - E} \frac{1}{\sqrt{-2mE_b}}. \tag{1.129}$$

Now our knowledge of c_s c_b comes into play. Remembering their form and recalling (1.26) in order to express $\sqrt{-2mE_b}$, allows further refining of expression (1.129).

$$\frac{m c_s c_b}{\hbar} \frac{1}{E_b - E} \frac{1}{\sqrt{-2mE_b}} = \frac{m(-\lambda)(-\lambda\sqrt{\lambda m})}{\hbar^2} \frac{1}{E_b - E} \frac{\hbar}{m\lambda} = \frac{\lambda\sqrt{m\lambda}}{\hbar} \frac{1}{E_b - E}. \tag{1.130}$$

The expression on the right of (1.130) is nothing else than

$$-c_b \frac{1}{E_b - E}. \tag{1.131}$$

Plugging this into equation (1.123) results in

$$\frac{c_b}{E_b - E} + \frac{-c_b}{E_b - E} = 0. \tag{1.132}$$

At this point, we exploited the fact that $E = \frac{\hbar^2 k^2}{2m}$. Therefore, also the even solution is orthogonal to the bound state. As next we shall check the orthogonality between two different odd, respectively even, wave functions. Taking the scalar product between two different (i.e. solutions have not the same energy or wave number) odd wave functions looks the following way.

$$\left\langle \psi_{E_1}^{odd} \middle| \psi_{E_2}^{odd} \right\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\frac{\pi}{i}(\delta(p-k_1) - \delta(p+k_1)))^* (\frac{\pi}{i}(\delta(p-k_2) - \delta(p+k_2)))dp. \tag{1.133}$$

Expanding and simplifying the product leads to

$$\left\langle \psi_{E_1}^{odd} \middle| \psi_{E_2}^{odd} \right\rangle = \frac{\pi}{2} \int_{-\infty}^{\infty} (\delta(p-k_1)\delta(p-k_2) - \delta(p-k_1)\delta(p+k_2) - \delta(p+k_1)\delta(p-k_2) + \delta(p+k_1)\delta(p+k_2))dp. \tag{1.134}$$

Recalling how a δ-function acts on another function allows us to rewrite (1.134) as

$$\left\langle \psi_{E_1}^{odd} \middle| \psi_{E_2}^{odd} \right\rangle = \frac{\pi}{2}(\delta(k_2 - k_1) - \delta(k_1 + k_2) - \delta(k_1 + k_2) + \delta(k_1 - k_2)), \tag{1.135}$$

which is nothing else, than

$$\left\langle \psi_{E_1}^{odd} \middle| \psi_{E_2}^{odd} \right\rangle = \pi(\delta(k_1 - k_2) - \delta(k_1 + k_2)). \tag{1.136}$$

The expression above is zero as long as $k_1 \neq k_2$. Therefore, odd wave functions are orthogonal to each other. Now we take a look at even functions

$$\left\langle \psi_{E_1}^{even} \middle| \psi_{E_2}^{even} \right\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\pi(\delta(p-k_1) + \delta(p+k_1)) + \tilde{\phi}_1)^* (\pi(\delta(p-k_2) + \delta(p+k_2)) + \tilde{\phi}_2)dp. \tag{1.137}$$

Again, further processing delivers a not as simple expression as in the odd case

$$\left\langle \psi_{E_1}^{even} \middle| \psi_{E_2}^{even} \right\rangle = \frac{1}{2\pi}(2\delta(k_1 - k_2) + 2\delta(k_1 + k_2) + \tilde{\phi}_1(k_2) + \tilde{\phi}_2(k_1) + \int_{-\infty}^{\infty} \tilde{\phi}_1(p)\tilde{\phi}_2(p)dp). \tag{1.138}$$

Inserting the explicit form for $\tilde{\phi}_i(p)$ (see equation (1.93)), eliminating the first two terms analogous to (1.136), the above expression gets morphed in

$$\left\langle \psi_{E_1}^{even} \middle| \psi_{E_2}^{even} \right\rangle = \frac{1}{2\pi} \left(0 + \frac{c_{E_1}}{E_1 - E_2} + \frac{c_{E_2}}{E_2 - E_1} + \int_{-\infty}^{\infty} \frac{c_{E_1}}{E_1 - \frac{\hbar^2 p^2}{2m}} \frac{c_{E_2}}{E_2 - \frac{\hbar^2 p^2}{2m}} dp \right). \tag{1.139}$$

Since the form of c_{s_i} only depends on λ, the terms two and three can be considered to be antisymmetric and therefore cancel each other out. What has survived to this point is the integral, which demands additional thoughts. Once more, the path over a partial fraction expansion provides the key to the computation of it. The terms generated by the procedure can be further treated if one recalls how $\phi_i(0)$ can be computed. Since this special value of ϕ_i is zero (see equation (1.112)), all terms vanish and the orthogonality of two even wave functions has been shown. What remains to be examined is the orthogonality of an even to an odd wave function. Fortunately in this case we can argue the following way: The product of an even and an odd function remains odd. An odd function integrated over a symmetric interval (in our case the interval is $]\infty, \infty[$) is zero and therefore orthogonality holds. As we already mentioned, position space and momentum space are connected through a Fourier transformation. Since the Fourier transformation belongs to the family of unitary maps, it preserves the previously computed scalar product between the eigenstates. Therefore we can further say that our system remains orthogonal also in position space.

Chapter 2

Relativistic particle in the δ-potential well

The last chapter was dedicated to the Dirac δ potential for a particle, which moved in a non-relativistic manner. This chapter is devoted to the same potential, but the particle now has the freedom to move relativistically. While the procedure in chapter one was quite straightforward, the small change has, as we shall see, grave impacts on the whole scenario. Where in the previous chapter our greatest challenge was the treatment of two poles, here we well hit on divergences, whose treatment will demand the usage of a most sophisticated method known as dimensional regularisation. We will also revert to computer programs in order to solve some, otherwise difficult to evaluate integrals! Fortunately our playing ground (the last chapter) provides a solid foundation on which we can build chapter two.

2.1 The bound state

2.1.1 Dimensional regularisation and renormalization of the δ-potential and construction of the wave function

As first action, we shall formulate the Hamiltonian for the new problem. Since the particle now moves in a relativistic way and the potential remains an attractive Dirac δ well, we can add it to the Hamiltonian of a free and relativistically moving particle. All things patched together lead us to the exprssion

$$\sqrt{m^2c^4 + p^2c^2}\tilde{\psi}(p) - \int_{-\infty}^{\infty} \lambda\delta(x)\psi(x)e^{ipx}dp = E\tilde{\psi}(p). \tag{2.1}$$

In order to keep a simple notation we shall set $\hbar = c = 1$. For the same reason, we just keep in mind that we consider the attractive case and drop therefore also the sign in front of λ. With these conventions, which will be applied until the end of this thesis, equation (2.1) takes the form

$$\sqrt{m^2 + p^2}\tilde{\psi}(p) + \lambda\psi(0) = E_b\tilde{\psi}(p). \tag{2.2}$$

The observant reader may now ask herself, why we formulated the problem directly in momentum space. Besides the fact that the relation between relativistic momentum and energy uses explicitly the momentum quantity, there is a subtle mathematical reason which makes quite an impact. If we want to express momentum in position space, we have to use the corresponding position space operator. It reads $-i\frac{\partial}{\partial x}$. Inserting it into equation (2.2) yields

$$\sqrt{m^2 - \frac{\partial^2}{\partial x^2}}\tilde{\psi}(p) + \lambda\psi(0) = E_b\tilde{\psi}(p). \tag{2.3}$$

Now, if we want to proceed in position space, we must first understand what the square root of a partial derivative means and how it can be handled. Although the task is not completely hopeless it relies on mathematics we, unfortunately, do not have at our disposal. Thus, we circumvent the issue by formulating the problem directly in momentum space.

In analogy to the bound non-relativistic case, we will formulate again a gap equation. For this task, we take equation (2.2) and after some algebraic manipulations we obtain

$$\tilde{\psi}_{E_b}(p) = \frac{\lambda\psi(0)}{E_b - \sqrt{m^2+p^2}}. \tag{2.4}$$

A similar procedure to the non-relativistic case generates the following relation, which is the gap equation we sought after

$$\frac{1}{\lambda} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{E_b - \sqrt{m^2+p^2}} dp. \tag{2.5}$$

As we can see, the integral contains now a square root. Since the integral is logarithmically divergent (once more, we encourage the reader to convince himself of the latter statement), the modification through the square root destroys any hope to solve it. As it looks now, we hit a dead end. Fortunately, there are still ways to handle the problem. In order to blow in a breach in the obstacle above, we shall add zero to equation (2.5).

$$\frac{1}{\lambda} + 0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{E_b - \sqrt{m^2+p^2}} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\sqrt{p^2+m^2}} dp - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\sqrt{p^2+m^2}} dp. \tag{2.6}$$

The manner in which we added zero to the above equation may look a bit weird. The full reach of the action will become clear, when we process (2.6) further. For now let us just say, that we are trying to separate the problematic part from the rest of the integral and that the way above provides a successful procedure for this. Since the left side is quite trivial, we shall focus on the processing of the right side. Taking the first and second term together generates

$$\frac{1}{2\pi}\left(\int_{-\infty}^{\infty} \frac{E_b}{E_b\sqrt{p^2+m^2} - (p^2+m^2)} dp - \int_{-\infty}^{\infty} \frac{1}{\sqrt{p^2+m^2}} dp\right). \tag{2.7}$$

While the first term of the expression above "smells" somehow like an $\arctan(x)$, the second term now contains the bare divergence. Although we can "smell" the $\arctan(x)$ in the first term its solution would consume much time. The reader may do the computation by hand, we however, fed it to a computation program (we used "Mathematica"). The program spit out the following result

$$\frac{1}{\lambda} = \frac{-E_b}{2\pi}\left(\frac{\pi}{\sqrt{m^2-E_b^2}} + \frac{2\arctan\left(\frac{E_b}{\sqrt{m^2-E_b^2}}\right)}{\sqrt{m^2-E_b^2}}\right) - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\sqrt{p^2+m^2}} dp. \tag{2.8}$$

Since the bound state energy has to be smaller than the rest mass of the particle (otherwise the particle would behave like an unbound one), we can define a new parameter $x = \frac{E_b}{m}$, which fulfills the condition $x^2 < 1$. While the latter constraint would allow the consideration of negative x (which implies also states that we call "strong bound" states with $E_b < 0$) the focus of this thesis is only on the bound states with $0 < x < 1$. The above condition allows the use of the relation [2, p. 86]

$$\arctan\left(\frac{x}{\sqrt{1-x^2}}\right) = \arcsin(x). \tag{2.9}$$

Therefore, (2.8) can be rewritten in a more compact manner as

$$\frac{1}{\lambda} = \frac{-E_b}{2\pi\sqrt{m^2-E_b^2}}\left(\pi + 2\arcsin\left(\frac{E_b}{m}\right)\right) - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\sqrt{p^2+m^2}} dp. \tag{2.10}$$

So far we have characterized the first term of equation (2.7). As already mentioned, the second term contains the divergence, which we shall now investigate further. In order to do so, we first take the integral to $n = D$ dimensions (i.e. looking at the integral in $\mathbb{R}^{n=D}$). There, it reads

$$\left(\frac{1}{2\pi}\right)^D \int_{\mathbb{R}^D} \frac{1}{\sqrt{\vec{p}^{\,2}+m^2}} d^D p. \tag{2.11}$$

Since the integrand depends only on the euclidean norm of $\vec{p}$ squared, we can simplify it further by switching to spherical coordinates.

$$\left(\frac{1}{2\pi}\right)^D \int_{\infty}^{\infty} \frac{1}{\sqrt{\vec{p}^2+m^2}} d^D p = \left(\frac{1}{2\pi}\right)^D \int_0^{\infty} \frac{S_{D-1} p^{D-1}}{\sqrt{p^2+m^2}} dp = \frac{S_{D-1}}{(2\pi)^D} \int_0^{\infty} \frac{p^{D-1}}{\sqrt{p^2+m^2}} dp. \tag{2.12}$$

S_{D-1} is nothing else than the volume of the $D-1$-dimensional sphere. So far, the integral looks not very helpful, since for any dimension $D \geq 1$ it remains divergent. Just ignoring the divergence is not an option (after all it is a part of the total energy of the particle). A rude but viable way would be the introduction of an upper bound for the momentum, the particle is allowed to take. This method is often referred to as "momentum cut-off". It is very simple, but in general its application does not preserve all informations about the system [7]. At this point, we introduce the method known as "dimensional regularisation" [7]. Instead of continuing the integral just to additional real dimensions, it allows the dimension, by introduction of a parameter, to become any (even complex) number. Included in this are also dimensions between zero and one. In this range the procedure exploits one peculiarity of some divergent integrals. Those integrals can, however, be parametrized in such a way that their integration yields a result which contains the Γ function and the parameter as its argument. Figure 2.1 shows the behaviour of the Γ function in a certain range.

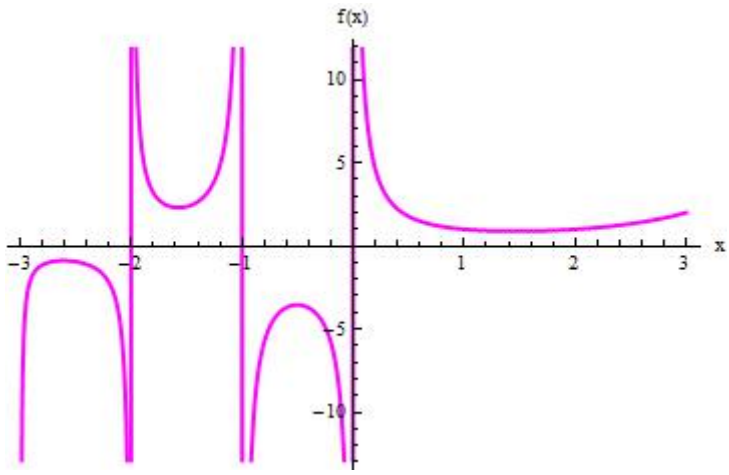


Figure 2.1: The behaviour of the Γ function for $x \in [-3, 3]$.

As long as the argument stays between minus one and zero, the Γ function and therefore also the integral which it represents, stays finite. It blows up to infinity when the introduced parameter hits the value, which corresponds to the original dimension of the integral. In a next step (often through some power expansion in terms of the parameter) it is possible to formulate an expression which allows direct access to the parameter. Summarized, the procedure gives a way to represent the divergence with a parameter, the so called "regulator" and therefore control the divergence's behaviour directly. Moreover, as long as the divergence is switched off, it is still possible to examine the physics, where otherwise we could only see ∞!

So much for the idea. Now we shall apply the ideas described above to our situation. We will not try to parametrize (2.12) manually. We fed the integral to our program, which gave the response below

$$\frac{S_{D-1}}{(2\pi)^D} \int_0^{\infty} \frac{p^{D-1}}{\sqrt{p^2+m^2}} dp = \frac{2\pi^{\frac{D}{2}}}{2(2\pi)^D \sqrt{\pi}} m^{D-1} \Gamma\left(\frac{1-D}{2}\right), \ 0 < Re(D) < 1, \tag{2.13}$$

In the above expression a mass term (m^{D-1}) appeared. We shall give a heuristic explanation for this appearance. If we perform the substitution $y^2 = \frac{p^2}{m^2}$, we must also adjust the term p^{D-1} in the integral above. This yields the term m^{D-1}. Therefore we can say that the mass term has to appear for dimensional compatibility reasons. Interestingly the coupling λ is a dimensionless number only in one spatial dimension. If we want to keep it that way, we have to attach a term m^{1-D} to the coupling. All further computations were carried out with the dimensionless coupling. Now let us focus again on the processing of (2.13). As long as $D < 1$, the Γ function remains finite and everything is finite. For $D = 1$ it blows up to infinity, exactly as our original integral. At this point we shall introduce the regulator, $\epsilon := D - 1$ which will do the trick, since it is zero when $D = 1$ and therefore it generates the full scale divergence, according to the behaviour of (2.12). Now we have to find a

way to access the regulator directly. Since we are interested in the behaviour of the Γ function around 0, we will conduct a Laurent expansion. The result, which once more was generated by the computation program, is given below.

$$\frac{2\pi^{\frac{D}{2}}}{2(2\pi)^D\sqrt{\pi}}\Gamma\left(\frac{1-D}{2}\right) \overset{Laurentexp.}{=} \frac{-1}{\pi\epsilon} + \frac{-\gamma + 2\ln(2) + \ln(\pi)}{2\pi} + \mathcal{O}(\epsilon). \tag{2.14}$$

The right-hand side of the equation above states that the integral diverges with one over ϵ as ϵ goes to zero (respectively D goes to one). Plugging it into (2.10) yields

$$\frac{1}{\lambda(\epsilon, E_b)} = \frac{-E_b}{2\pi\sqrt{m^2 - E_b^2}}\left(\pi + 2\arcsin\left(\frac{E_b}{m}\right)\right) + \frac{1}{\pi\epsilon} + \frac{\gamma - 2\ln(2) - \ln(\pi)}{2\pi} - \mathcal{O}(\epsilon). \tag{2.15}$$

In analogy to the non-relativistic setting, (2.15) can be considered as the solution to the gap equation. But there are big differences! While in chapter one λ parametrized the bound state energy, here we have no chance to do it this way, since λ would just blow up. We choose a bound energy E_b and fix the regulator's behaviour in this way. This procedure is known as renormalization. We could also say that the regulator's behaviour is set in relation to E_b and therefore renormalized to it. We characterize the "depth" or coupling strength with ϵ and E_b, which has now become a parameter (the opposite way of the non-relativistic setting)! Instead of an eigenvalue we have gained a parametrized coupling strength $\lambda(E_b, \epsilon)$.

As far as the wave function is concerned, we can write down a (not yet normalized) wave function.

$$\tilde{\psi}_{E_b}(p) = \frac{A}{E_b - \sqrt{p^2 + m^2}}, \tag{2.16}$$

where A is the normalization constant. Normalizing the wave function leads to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{A}{E_b - \sqrt{p^2+m^2}}\right)^*\left(\frac{A}{E_b - \sqrt{p^2+m^2}}\right)dp = \frac{|A|^2}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(E_b - \sqrt{p^2+m^2})^2}dp = 1. \tag{2.17}$$

Once more Mathematica helped us solving the above expression.

$$\frac{A^2}{2\pi}\left(\frac{2E_b}{m^2 - E_b^2} + \frac{m^2}{(m^2 - E_b^2)^{\frac{3}{2}}}\left(\pi + 2\arcsin\left(\frac{E_b}{m}\right)\right)\right) = 1. \tag{2.18}$$

Further algebraic processing yields

$$|A| = \sqrt{\frac{2\pi(m^2 - E_b^2)^{\frac{3}{2}}}{2E_b\sqrt{m^2 - E_b^2} + m^2\left(\pi + 2\arcsin\left(\frac{E_b}{m}\right)\right)}}. \tag{2.19}$$

Hence, the normalized momentum space wave function has the form

$$\tilde{\psi}_{E_b}(p) = \sqrt{\frac{2\pi(m^2 - E_b^2)^{\frac{3}{2}}}{2E_b\sqrt{m^2 - E_b^2} + m^2\left(\pi + 2\arcsin\left(\frac{E_b}{m}\right)\right)}}\frac{1}{E_b - \sqrt{p^2+m^2}}. \tag{2.20}$$

As mentioned above, (2.20) is the representation in momentum space. Of course, it would be very interesting to see its form in position space. Therefore, we will now apply the inverse Fourier transform on expression (2.20)

$$\psi_{E_b}(x) = \frac{1}{2\pi}\sqrt{\frac{2\pi(m^2 - E_b^2)^{\frac{3}{2}}}{2E_b\sqrt{m^2 - E_b^2} + m^2\left(\pi + 2\arcsin\left(\frac{E_b}{m}\right)\right)}}\int_{-\infty}^{\infty}\frac{e^{ipx}}{E_b - \sqrt{p^2+m^2}}dp. \tag{2.21}$$

In order to solve the integral we perform an analytic continuation to $\mathbb{C}$ (the whole procedure is similar to the one in chapter 1). Fortunately, also square-root functions have a continuation which looks the same way as the function in $\mathbb{R}$. There is one problem with square roots in the complex plain. In order to keep them bijective,

we have to do a cut in the complex plane [3]. In our case the cut begins at im and goes to infinity. Further dissecting of the denominator reveals also two pols at

$$p_1 = i\sqrt{m^2 - E_b^2},\ p_2 = -i\sqrt{m^2 - E_b^2}. \tag{2.22}$$

Before we can execute finally the integral, we have to patch together an integration path. Figure 2.2 provides us with visual support for generating the wished contour.

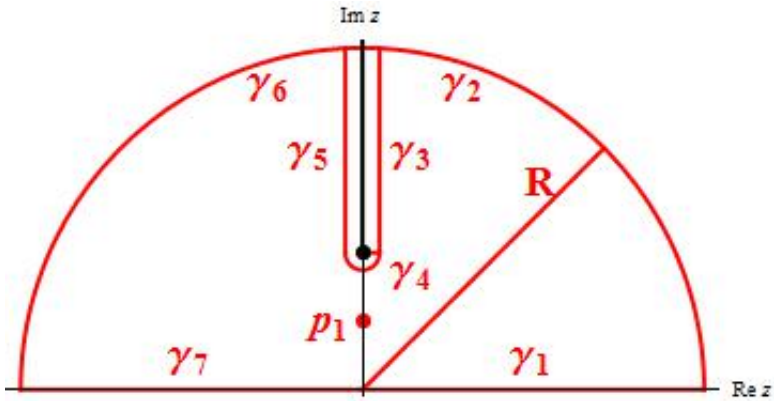


Figure 2.2: The integration path (orientation: counter-clockwise) we shall take. The black point and the corresponding black line represent the cut, which has to be considered, while p_1 is the pole enclosed by the entire path γ. γ_3 and γ_5 have to be considered being right on the cut, but are, for visualisation′s sake, depicted parallel to it.

$$\gamma_1 : [0, R] \to \mathbb{C}, \qquad \gamma_1(t) = t, \qquad \gamma_1'(t) = 1. \tag{2.23}$$

$$\gamma_2 : [0, \frac{\pi}{2}] \to \mathbb{C}, \qquad \gamma_2(t) = Re^{it}, \qquad \gamma_2'(t) = iRe^{it}. \tag{2.24}$$

$$\gamma_3 : [R, m] \to \mathbb{C}, \qquad \gamma_3(t) = i(t+\epsilon), \qquad \gamma_3'(t) = i. \tag{2.25}$$

$$\gamma_4 : [\frac{\pi}{2}, \frac{-3\pi}{2}] \to \mathbb{C}, \qquad \gamma_4(t) = im + \epsilon e^{it}, \qquad \gamma_4'(t) = i\epsilon e^{it}. \tag{2.26}$$

$$\gamma_5 : [m, R] \to \mathbb{C}, \qquad \gamma_5(t) = i(t+\epsilon), \qquad \gamma_5'(t) = i. \tag{2.27}$$

$$\gamma_6 : [\frac{\pi}{2}, \pi] \to \mathbb{C}, \qquad \gamma_6(t) = Re^{it}, \qquad \gamma_6'(t) = iR\epsilon e^{it}. \tag{2.28}$$

$$\gamma_7 : [-R, 0] \to \mathbb{C}, \qquad \gamma_7(t) = t, \qquad \gamma_7'(t) = 1. \tag{2.29}$$

With the parametrization above and the fact that the integrand is meromorphic on the domain defined through γ we can finally assault the integral by using the Residue theorem [3].

$$\begin{aligned}
&\frac{A}{2\pi}\int_\gamma \frac{e^{izx}}{E_b - \sqrt{p^2+m^2}}dz = \frac{A}{2\pi}\Bigg(\int_0^R \frac{e^{itx}}{E_b - \sqrt{t^2+m^2}}dt + \int_0^{\frac{\pi}{2}} \frac{e^{iRe^{it}x} iRe^{it}}{E_b - \sqrt{(Re^{it})^2+m^2}}dt + \\
&+ \int_R^m \frac{e^{i(i(t+\epsilon))x} i}{E_b - \sqrt{(i(t+\epsilon))^2+m^2}}dt + \int_{\frac{\pi}{2}}^{-\frac{3\pi}{2}} \frac{e^{i(im+\epsilon e^{it})x} i\epsilon e^{it}}{E_b - \sqrt{(im+\epsilon e^{it})^2+m^2}}dt + \int_m^R \frac{e^{i(i(t+\epsilon))x} i}{E_b - \sqrt{(i(t+\epsilon))^2+m^2}}dt \quad + \\
&+ \int_{\frac{\pi}{2}}^{\pi} \frac{e^{iRe^{it}x} iRe^{it}}{E_b - \sqrt{(Re^{it})^2+m^2}}dt + \int_{-R}^0 \frac{e^{itx}}{E_b - \sqrt{t^2+m^2}}dt\Bigg) = iARes(f(z), p_1).
\end{aligned} \tag{2.30}$$

A is the normalization constant (2.19). In the above expression the symbol for the square root has to be considered as purely symbolic! This is because we have terms which live on both sides of the branch cut. This issue we shall solve by adding another sign to the square roots whose integration path lies on the left side of the cut. Reassembling, reorganizing certain terms and considering the limit $\epsilon \to 0$ yields

$$\frac{A}{2\pi}\left(\int_{-R}^{R}\frac{e^{itx}}{E_b-\sqrt{t^2+m^2}}dt+\int_{R}^{m}\frac{e^{-tx}i}{\sqrt{m^2-t^2}-E_b}dt+\int_{m}^{R}\frac{e^{-tx}i}{E_b+\sqrt{m^2-t^2}}dt+0+\right.$$
$$\left.+\int_{0}^{\frac{\pi}{2}}\frac{e^{iRe^{it}x}iRe^{it}}{E_b-\sqrt{(Re^{it})^2+m^2}}dt+\int_{\frac{\pi}{2}}^{\pi}\frac{e^{iRe^{it}x}iRe^{it}}{E_b+\sqrt{(Re^{it})^2+m^2}}dt\right)=iARes(f(z),p_1), \tag{2.31}$$

where 0 stands for γ_4 (geometrically: Circle with radius ϵ around im). Analogous to the computation of (1.95) the terms four and five vanish in the limit $R\to\infty$. Hence, the expression above gets reduced to

$$\frac{A}{2\pi}\left(\int_{-\infty}^{\infty}\frac{e^{itx}}{E_b-\sqrt{t^2+m^2}}dt+\int_{m}^{\infty}\frac{e^{-tx}i}{\sqrt{m^2-t^2}-E_b}dt+\int_{m}^{\infty}\frac{e^{-tx}i}{E_b+\sqrt{m^2-t^2}}dt\right)=iARes(f(z),p_1). \tag{2.32}$$

Since the terms two and three of (2.32) are just valid for $x>0$, both can be summed to

$$\int_{m}^{\infty}\frac{e^{-tx}i}{\sqrt{m^2-t^2}-E_b}dt+\int_{m}^{\infty}\frac{e^{-tx}i}{E_b+\sqrt{m^2-t^2}}dt=i\int_{m}^{\infty}\frac{2e^{-|x|t}\sqrt{m^2-t^2}}{m^2-t^2-E_b^2}dt. \tag{2.33}$$

Since $t\geq m$, the square root on the right-hand side of (2.33) can be made real at the cost of another imaginary unit. With this simplification we can write (2.33) as

$$-\int_{m}^{\infty}\frac{2e^{-|x|t}\sqrt{t^2-m^2}}{m^2-t^2-E_b^2}dt=:\chi(x). \tag{2.34}$$

What remains to be done is the determination of the residue in p_1. Since the function is a fraction, we can compute residue in p_1 the following way [3].

$$Res(f(z),p_1)=Res\left(\frac{g}{h}(z),p_1\right)=\frac{g(p_1)}{h'(p_1)}, \tag{2.35}$$

where h' is the complex derivative of the denominator, g stands for the numerator, and p_1 is the pole we encountered. In our case $g(p_1)$ reads

$$e^{ii\sqrt{m^2-E_b^2}x}=e^{-x\sqrt{m^2-E_b^2}}. \tag{2.36}$$

For $h'(p_1)$ we obtain

$$h'(p_1)=\frac{d}{dz}\left(E_b-\sqrt{z^2+m^2}\right)\Big|_{z=p_1}=\frac{-i\sqrt{m^2-E_b^2}}{E_b}. \tag{2.37}$$

Plugging the two results into (2.35), we get

$$iARes(f(z),p_1)=\frac{-iAE_be^{-|x|\sqrt{m^2-E_b^2}}}{i\sqrt{m^2-E_b^2}}. \tag{2.38}$$

Before we reveal the representation of the wave function in position space, we summarize the previous results as

$$\frac{A}{2\pi}\int_{\gamma}\frac{e^{izx}}{E_b-\sqrt{z^2+m^2}}dz=\frac{A}{2\pi}\left(\int_{-\infty}^{\infty}\frac{e^{itx}}{E_b-\sqrt{t^2+m^2}}dt-\int_{m}^{\infty}\frac{2e^{-|x|t}\sqrt{t^2-m^2}}{m^2-t^2-E_b^2}dt\right)=\frac{-iAE_be^{-|x|\sqrt{m^2-E_b^2}}}{i\sqrt{m^2-E_b^2}}. \tag{2.39}$$

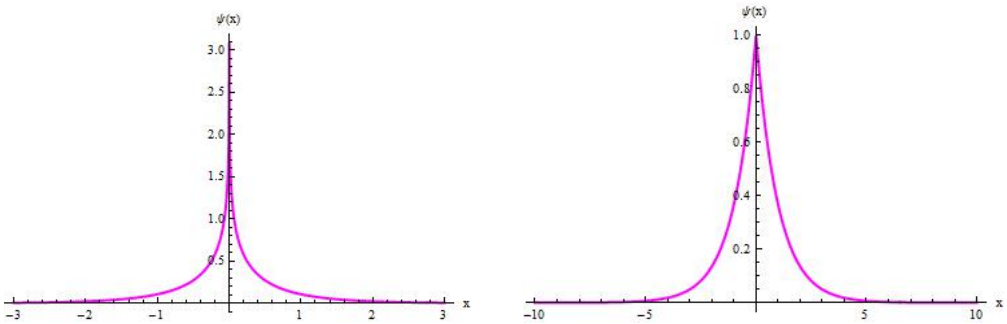
Expression (2.39) is the result for the integration path γ, obtained with the residue theorem. A few simple algebraic manipulations lead us to

$$\frac{A}{2\pi}\int_{-\infty}^{\infty}\frac{e^{itx}}{E_b-\sqrt{t^2+m^2}}dt=\frac{-AE_be^{-|x|\sqrt{m^2-E_b^2}}}{\sqrt{m^2-E_b^2}}-\frac{A}{\pi}\int_{m}^{\infty}\frac{e^{-t|x|}\sqrt{t^2-m^2}}{E_b^2+t^2-m^2}dt. \tag{2.40}$$

With the knowledge from the previous page, we finally write down the representation of the wave function in position space

$$\psi_{E_b}(x) = \frac{A}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ipx}}{E_b - \sqrt{p^2+m^2}} \overset{(2.40)}{=} A\left(\frac{-E_b e^{-|x|\sqrt{m^2-E_b^2}}}{\sqrt{m^2-E_b^2}} - \frac{1}{\pi}\int_m^{\infty}\frac{e^{-t|x|}\sqrt{t^2-m^2}}{E_b^2+t^2-m^2}dt\right). \tag{2.41}$$

The modulus in the first term arises because of the different way of closing the contour if one takes the "direction" to $-\infty$. Finally, let us take a closer look at the above expression and compare it with its non-relativistic pendant. The first thing we can see is the additional sign in both terms. Since it occurs in both terms, we can consider it as an overall phase factor and therefore we do not have to care about it further. More interesting and amazing is the second term. It is the contribution from the branch cut. Besides the fact that we are only able to evaluate it numerically, it produces infinity at $x = 0$. The special thing about $x = 0$ is that it supports the Dirac δ. Strange enough, despite the singular point, the wave function remains normalizable. Further, the term depends on the modulus of x. If we send x to $\pm\infty$ the term becomes zero! Additionally it also contains E_b. Therefore we can say that term number two is a non-locality effect of the potential! Even if the particle is away from the δ-peak it "feels" its effect. This effect is strongest, when the particle is right at the δ peak! If we know that the relativistic Hamiltonian, as we formulated it, is a non-local entity, then the above non-locality effect is not surprising. As the square root in the Hamiltonian is the reason for the branch cut, it is also responsible for the non-locality! One possible reason for the latter statement is the following. If we would expand the Hamiltonian in a Taylor series, we would gain a mess of polynomials and derivatives of square roots. Every order contains also a polynomial. Therefore we can say that one has to know a lot about the surrounding area of the square root if one wants to describe it. Hence the reason for the non-locality effect, a feature the non-relativistic scenario does not know in any way. Figure 2.3 gives an example for $\psi_{E_b}(x)$.



(a) Relativistic case (b) Non-relativistic case

Figure 2.3: Shapes of the relativistic bound state wave function and its non relativistic pendant. While the non-relativistic wave function has just a finite "kink" at $x = 0$, the relativistic wave function blows up to infinity at the same point.

2.1.2 The non-relativistic limit, $\lambda(E_b, \epsilon)$, and the wave function

Since the last sub-section consisted of mathematically demanding concepts and calculations, we shall now do relaxing things. Conveniently they will also serve as a cross-check opportunity for the most important results we obtained until now. Since chapter two is all about the relativistic behaviour of a particle, it would be very interesting to investigate the effects of non-relativistic speeds on the results we obtained in the previous sub-section. If our considerations were right, then the non-relativistic scenario should be somehow "packed" in the relativistic one.

As a first step, we shall investigate relation (2.15).

$$\frac{1}{\lambda} = \frac{-E_b}{2\pi\sqrt{m^2-E_b^2}}\left(\pi + 2\arcsin\left(\frac{E_b}{m}\right)\right) + \frac{1}{\pi\epsilon} + \frac{\gamma - 2\ln(2) - \ln(\pi)}{2\pi} - \mathcal{O}(\epsilon). \tag{2.42}$$

As we already mentioned, it describes the coupling strength of the potential in dependence of the bound state energy. Before we can make any attempts to unpack the non-relativistic case, we have to find a way to formalize

what "non-relativistic" could possibly mean. In order to do this we will clarify what E_b represents. The reader may now say "E_b is the energy of the bound state!" While the statement is certainly right, we might still give it another thought. E_b describes the energy, a particle has, if it is in the bound state (analogous to the energy E_n an electron needs, if it wants to be on the n'th level in the hydrogen atom).

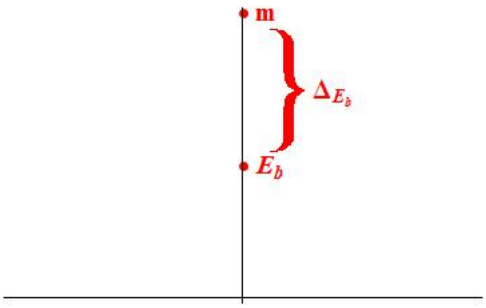


Figure 2.4: Illustration of the relative "positions" of rest mass energy m and the bound state energy E_b. The gap between the two energies is the binding energy ΔE_b.

As shown in Figure 2.4, the bound state E_b is smaller than m. Analogous to the ionization energy of the hydrogen atom, there is also the energy the particle has to "invest", if it wants to be considered as a free particle. This energy we call binding energy ΔE_b. If the particle moves non-relativistically, then its total energy should be very close to its rest mass. Therefore, a bound particle's binding energy ΔE_b should also be very small (i.e. so small that E_b is almost m), with respect to m. Let us now apply this statement to (2.15). If we express the bound state energy in terms of rest mass and binding energy, expression (2.15) becomes

$$\frac{1}{\lambda(\epsilon)} = \frac{-(\Delta E_b + m)}{2\pi\sqrt{m^2 - (\Delta E_b + m)^2}}\left(\pi + 2\arcsin\left(\frac{\Delta E_b + m}{m}\right)\right) + \frac{1}{\pi\epsilon} + \frac{\gamma - 2\ln(2) - \ln(\pi)}{2\pi} - \mathcal{O}(\epsilon). \tag{2.43}$$

Further algebraic processing of the expression above generates

$$\frac{1}{\lambda(\epsilon)} = \frac{-(\Delta E_b + m)}{2\pi m\sqrt{1 - \frac{(\Delta E_b + m)^2}{m^2}}}\left(\pi + 2\arcsin\left(\frac{\Delta E_b}{m} + 1\right)\right) + \frac{1}{\pi\epsilon} + \frac{\gamma - 2\ln(2) - \ln(\pi)}{2\pi} - \mathcal{O}(\epsilon). \tag{2.44}$$

Before we execute the non-relativistic limit, we shall expand the binomial in the denominator of the factor in the above expression

$$\frac{1}{\lambda(\epsilon)} = \frac{-(\Delta E_b + m)}{2\pi m\sqrt{1 - \frac{(\Delta E_b)^2 + 2m\Delta E_b + m^2}{m^2}}}\left(\pi + 2\arcsin\left(\frac{\Delta E_b}{m} + 1\right)\right) + \frac{1}{\pi\epsilon} + \frac{\gamma - 2\ln(2) - \ln(\pi)}{2\pi} - \mathcal{O}(\epsilon). \tag{2.45}$$

Now we shall consider ΔE_b very small. Since any powers of ΔE_b are even smaller, we shall neglect all higher powers of it (the same argument is applicable to ratios of ΔE_b divided by m). Under those circumstances the latter expression becomes

$$\frac{1}{\lambda(\epsilon)} = -\sqrt{\frac{m}{-2\Delta E_b}} + \frac{1}{\pi\epsilon} + \frac{\gamma - 2\ln(2) - \ln(\pi)}{2\pi} - \mathcal{O}(\epsilon). \tag{2.46}$$

If we recall that the non-relativistic coupling defined the bound state energy completely without need of being regularized and renormalized in any way, we shall now cut off all things which appeared through the relativistic setting. This means that we now have a simple relation between λ and ΔE_b (analogous to the non-relativistic setting). With this small redefinition, (2.46) becomes

$$\frac{1}{\lambda} = -\sqrt{\frac{m}{-2\Delta E_b}}. \tag{2.47}$$

Performing further algebraic processing reveals

$$\Delta E_b = \frac{-m\lambda^2}{2}, \tag{2.48}$$

which is nothing else than the energy eigenvalue of the bound state in the non-relativistic scenario! With the knowledge we gathered above we can draw one amazing conclusion. While in the non-relativistic case bound energy and binding energy have the same unique value, a relativistically moving particle makes a difference between the bound state and binding energy. Another fascinating thing is the fact, that the binding energy is "drained" particle rest mass energy.

Now we shall focus on the wave function

$$\psi_{E_b}(x) = A\left(\frac{-E_b e^{-|x|\sqrt{m^2-E_b^2}}}{\sqrt{m^2-E_b^2}} - \frac{1}{\pi}\int_m^\infty \frac{e^{-t|x|}\sqrt{t^2-m^2}}{E_b^2+t^2-m^2}dt\right), \tag{2.49}$$

where A is still the normalization constant. The constant is a mess of symbols an we will therefore consider it separately. With the same idea as for the examination of (2.15) we rewrite A in terms of ΔE_b

$$\begin{aligned} A &= \sqrt{\frac{2\pi(m^2-E_b^2)^{\frac{3}{2}}}{2E_b\sqrt{m^2-E_b^2}+m^2\left(\pi+2\arcsin\left(\frac{E_b}{m}\right)\right)}} = \\ &= \sqrt{\frac{2\pi(m^2-(\Delta E_b+m)^2)^{\frac{3}{2}}}{2(\Delta E_b+m)\sqrt{m^2-(\Delta E_b+m)^2}+m^2\left(\pi+2\arcsin\left(\frac{(\Delta E_b+m)}{m}\right)\right)}}. \end{aligned} \tag{2.50}$$

Considering ΔE_b small and comparing the first term of the denominator to the second one, allows us to drop the first term in the expression above. Expanding all remaining ΔE_b containing binomials and neglecting powers of ΔE_b higher than one lead us to

$$A = \sqrt{\frac{2\pi m^3\left(\frac{-2\Delta E_b}{m}\right)^{\frac{3}{2}}}{m^2\left(\pi+2\arcsin\left(\frac{(\Delta E_b+m)}{m}\right)\right)}}. \tag{2.51}$$

If we recall (2.47), we can rewrite (2.51) as

$$A_{non\ rel} = \sqrt{\frac{2\pi m^3\lambda^3}{2\pi m^2}} = \sqrt{m\lambda^3}. \tag{2.52}$$

The last result is on its own not of great use. Therefore, we will now take a closer look at the first term of the wave function. Rewriting the factor in front of the exponential function in terms of the binding energy leads to the following expression.

$$\frac{-E_b e^{-|x|\sqrt{m^2-E_b^2}}}{\sqrt{m^2-E_b^2}} = \frac{-(\Delta E_b+m)}{\sqrt{m^2-(\Delta E_b^2+m^2)^2}}e^{-|x|\sqrt{m^2-(\Delta E_b+m)^2}}. \tag{2.53}$$

If we use again the same procedures and simplifications as for A, the above expression reduces to

$$\frac{-(\Delta E_b+m)}{\sqrt{m^2-(\Delta E_b^2+m^2)^2}}e^{-|x|\sqrt{m^2-(\Delta E_b+m)^2}} = \sqrt{\frac{m}{-2\Delta E_b}}e^{-|x|\sqrt{-2\Delta E_b}}. \tag{2.54}$$

Rewriting the expression above with help of (2.47) we obtain

$$\sqrt{\frac{m}{-2\Delta E_b}}e^{-|x|\sqrt{-2\Delta E_b}} = \frac{1}{\lambda}e^{-|x|m\lambda}. \tag{2.55}$$

Multiplying this entity with the expression we obtained for the non-relativistic amplitude in (2.52) yields

$$\frac{\sqrt{m\lambda^3}}{\lambda}e^{-m\lambda|x|} = \sqrt{m\lambda}e^{-m\lambda|x|}. \tag{2.56}$$

This is nothing else than the non-relativistic wave function we obtained in chapter one! Of course now the question arises what the second term stands for? As already stated, in its function for the relativistic behaviour it is a manifestation of the interaction between potential and particle at non-zero distances. Relying on our studies in chapter one, we can say that such a non-local interaction is not existent in the non-relativistic setting. Therefore we dare to say further that the term will vanish in the non-relativistic limit. Once more, rewriting the bound state energy in terms of binding energy and rest mass yields

$$-\frac{1}{\pi}\int_m^\infty \frac{e^{-t|x|}\sqrt{t^2-m^2}}{(\Delta E_b+m)^2+t^2-m^2}dt = -\frac{1}{\pi}\int_m^\infty \frac{e^{-t|x|}\sqrt{t^2-m^2}}{\Delta E_b^2+2\Delta E_b m+m^2+t^2-m^2}dt \tag{2.57}$$

Again neglecting higher powers of ΔE_b and attaching $A_{non\ rel}$, lead to the expression below

$$-\frac{\sqrt{m\lambda^3}}{\pi}\int_m^\infty \frac{e^{-t|x|}\sqrt{t^2-m^2}}{2\Delta E_b m+t^2}dt = \frac{\sqrt{m(\frac{-2\Delta E_b}{m})^{\frac{3}{2}}}}{\pi}\int_m^\infty \frac{e^{-t|x|}\sqrt{t^2-m^2}}{2\Delta E_b m+t^2}dt \tag{2.58}$$

If we now send the ratio between rest mass and binding energy to zero (i.e. taking the non-relativistic limit), we suppress $\chi(x)$ entirely. With the considerations above, we can say that the bound state non-relativistic wave function and its eigenvalue do indeed emerge from the relativistic bound state wave function and its, although regularized energy. Given the strange features of the relativistic setting (non-local interaction, infinity at the point $x=0$ to name some of them) such cogging between the two settings is amazing!

2.2 The scattering states

In the last section we examined the relativistic bound state with a bound state energy between zero and the rest mass of the involved particle. As already mentioned, this thesis will not treat strong or ultra-strong bound states. However, there is one energy regime we shall take a closer look at. We consider now the region, where the particle's total energy is bigger than the rest mass of the particle. Those states are (analogous to the non-relativistic setting) nothing else than scattering states. They shall be investigated on the next few pages.

2.2.1 Regularisation, renormalization, and construction of the wave function

Since we are in momentum space and the parity symmetry of the potential still holds, we shall use the same approach as in chapter 1. After expressing the momentum in terms of total energy and rest mass, we can formulate the ansatz below.

$$\tilde{\psi}_E^{odd}(p) = \frac{\pi}{i}\left(\delta(p-\sqrt{E^2-m^2})-\delta(p+\sqrt{E^2-m^2})\right). \tag{2.59}$$

Inserting it in the relativistic Schrödinger equation (equation (2.2)) we already used for the investigation of the bound state yields

$$\sqrt{p^2+m^2}\frac{\pi}{i}\left(\delta(p-\sqrt{E^2-m^2})-\delta(p+\sqrt{E^2-m^2})\right)+c_s = E\frac{\pi}{i}\left(\delta(p-\sqrt{E^2-m^2})-\delta(p+\sqrt{E^2-m^2})\right), \tag{2.60}$$

where c_s stands for $\lambda\psi(0)$. Recalling that odd states do not feel the potential, we can set c_s to zero. Identifying E with $\sqrt{k^2+m^2}$ confirms the ansatz as a solution of (2.2). The ansatz for the even solution takes the form

$$\tilde{\psi}_E^{even}(p) = \pi(\delta(p-\sqrt{E^2-m^2})+\delta(p+\sqrt{E^2-m^2}))+\tilde{\phi}_E(p). \tag{2.61}$$

Plugging the above ansatz into the relativistic Schrödinger equation and using the same manipulations as in the non-relativistic case yields an expression for $\tilde{\phi}_E(p)$.

$$\tilde{\phi}_E(p) = \frac{c_s}{E-\sqrt{p^2+m^2}} \tag{2.62}$$

As in the non-relativistic case, the above thoughts are valid for the even solution and therefore we have no reason to set c_s to zero. Recalling how we can express $\psi_E^{even}(0)$ with the help of its Fourier transform delivers an expression for c_s.

$$\lambda\psi_E^{even}(0) = c_s = \frac{\lambda}{2\pi}\int_{-\infty}^{\infty}\tilde{\psi}_E^{even}(p)e^{ip0} = \frac{\lambda}{2\pi}\left(\int_{-\infty}^{\infty}\pi(\delta(p-\sqrt{E^2-m^2})+\delta(p+\sqrt{E^2-m^2}))dp+\right.$$
$$\left.+\int_{-\infty}^{\infty}\tilde{\phi}_E(p)e^{ip0}dp\right) = \lambda + \lambda\phi_E(0). \tag{2.63}$$

As we know, $x = 0$ is the contact point, which we characterized in the bound state case. Therefore we shall direct our attention to the properties of $\phi_E(0)$. Using the same idea as for the determination of c_s, we write down the relation

$$\phi_E(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{c_s}{E-\sqrt{p^2+m^2}}e^{ip0}dp = (\lambda+\lambda\phi_E(0))\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{E-\sqrt{p^2+m^2}}dp. \tag{2.64}$$

For notation's sake we define the place holder $I(p)$ for the integral (including the factor $\frac{1}{2\pi}$). With it and some algebraic manipulations we obtain

$$(1-\lambda I(p))\phi_E(0) = \lambda I(p). \tag{2.65}$$

Hence, $\phi_E(0)$ takes the form

$$\phi_E(0) = \lambda\frac{I(p)}{1-\lambda I(p)} = \frac{I(p)}{\frac{1}{\lambda}-I(p)}. \tag{2.66}$$

If we want to evaluate $\phi_E(0)$, we have to know the properties of $I(p)$. Therefore, we shall further investigate the integral

$$I(p) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{E-\sqrt{p^2+m^2}}dp. \tag{2.67}$$

If this integral looks somehow familiar to the reader, then she has the previous section quite well in mind. It is the same divergent expression that we encountered in the investigation of the bound state. In order to handle it, we shall deploy the same dimensional regularisation machinery as in the bound state case. Since we conducted the whole procedure step by step in the bound state case, we shall here just give the crucial results. After dividing the integral into a finite part and the bare divergence, we write down the following result for the finite term (it was obtained with the help of Mathematica).

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{E}{E\sqrt{p^2+m^2}-(p^2+m^2)}dp = \frac{-E}{\sqrt{m^2-E^2}}\frac{1}{2\pi}\left(\pi+2\arctan\left(\frac{E}{\sqrt{m^2-E^2}}\right)\right). \tag{2.68}$$

This time the energy E is bigger than m. Therefore we can define the new parameter $x = \frac{E}{m}$ and use with it the following relation.

$$\arctan\left(\frac{x}{\sqrt{1-x^2}}\right) = \frac{-\pi}{2} + \operatorname{arctanh}\left(\frac{i\sqrt{1-x^2}}{x}\right). \tag{2.69}$$

With the relation above, we morph the result of the finite part the following way

$$\frac{-E}{\sqrt{m^2-E^2}}\frac{1}{2\pi}\left(\pi+2\arctan\left(\frac{E}{\sqrt{m^2-E^2}}\right)\right) = \frac{-x}{\sqrt{x^2-1}}\left(2\operatorname{arctanh}\left(\frac{-\sqrt{x^2-1}}{x}\right)\right). \tag{2.70}$$

Using the fact that arctanh is an odd function and replacing x morphs the result into

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{E}{E\sqrt{p^2+m^2}-(p^2+m^2)}dp = \frac{E}{\pi\sqrt{E^2-m^2}}\operatorname{arctanh}\left(\frac{\sqrt{E^2-m^2}}{E}\right). \tag{2.71}$$

This time we fix E as the renormalization scale. As far as the finite term is concerned, the above expression is the final result. Now we shall focus on the divergent term. Applying the same dimensional regularisation procedure as in the bound state case generates

$$I(p) = \frac{-1}{\pi\epsilon} + \frac{2\ln(2) + \ln(\pi) - \gamma}{2\pi} + \mathcal{O}(\epsilon). \tag{2.72}$$

With both terms put together, the integral reads

$$\frac{E}{\pi\sqrt{E^2 - m^2}} \operatorname{arctanh}\left(\frac{\sqrt{E^2 - m^2}}{E}\right) + \frac{1}{\pi\epsilon} + \frac{\gamma - 2\ln(2) - \ln(\pi)}{2\pi} - \mathcal{O}(\epsilon). \tag{2.73}$$

At this point we shall recall relation (2.66)

$$\phi_E(0) = \lambda \frac{I(p)}{1 - \lambda I(p)} = \frac{I(p)}{\frac{1}{\lambda} - I(p)}. \tag{2.74}$$

With the regularisation above we just regulated the divergent integral (not once more λ!). Inserting the corresponding expressions (equation (2.15) for $\frac{1}{\lambda}$ and expression (2.73) for $I(p)$), $\phi_E(0)$ reads

$$\phi_E(0) = \frac{\frac{E}{\pi\sqrt{E^2-m^2}} \operatorname{arctanh}\left(\frac{\sqrt{E^2-m^2}}{E}\right) + \frac{1}{\pi\epsilon} + \frac{\gamma - 2\ln(2) - \ln(\pi)}{2\pi} - \mathcal{O}(\epsilon)}{\frac{-E_b}{2\pi\sqrt{m^2-E_b^2}}\left(\pi + 2\arcsin\left(\frac{E_b}{m}\right)\right) - \frac{E}{\pi\sqrt{E^2-m^2}} \operatorname{arctanh}\left(\frac{\sqrt{E^2-m^2}}{E}\right)}. \tag{2.75}$$

If we take a closer look at the denominator, we can see that the regulators cancelled each other out! However, with the above computations we determined c_s and therefore also the even scattering state wave function in momentum space is completely fixed. Now we shall focus once more on the search for the position space representation of the momentum space scattering state wave functions.

The odd case is quite simple to transform and shall therefore not be written down explicitly. Analogous to the non-relativistic scenario, the even case will demand the most thinking. In order to understand why this is so, we shall recall (2.61). Its full form reads

$$\tilde{\psi}_E^{even}(p) = \pi(\delta(p - \sqrt{E^2 - m^2}) + \delta(p + \sqrt{E^2 - m^2})) + \frac{c_s}{E - \sqrt{p^2 + m^2}}. \tag{2.76}$$

Applying an inverse Fourier transformation generates

$$\psi_E^{even}(x) = \cos(\sqrt{E^2 - m^2}x) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{c_s}{E - \sqrt{p^2 + m^2}} dp. \tag{2.77}$$

The integral is exactly the same that we faced it in the relativistic bound state case. It has some new features, though. Since we are considering scattering states, the pole between im and 0 is no more present. Instead, the integral now contains two poles on the real axis. Those can be found at

$$p_1 = \sqrt{E^2 - m^2},\ p_2 = -\sqrt{E^2 - m^2}. \tag{2.78}$$

We will assault the integral the same way as we did by the inverse Fourier transformation of the bound state. Therefore, the analytic continuation to $\mathbb{C}$ brings in the same branch cut, we already encountered. Of course we will need again an integration path and the parametrization of it. On the next page Figure 2.5 shall support the visualisation of the integration path we will take and therefore also parametrize.

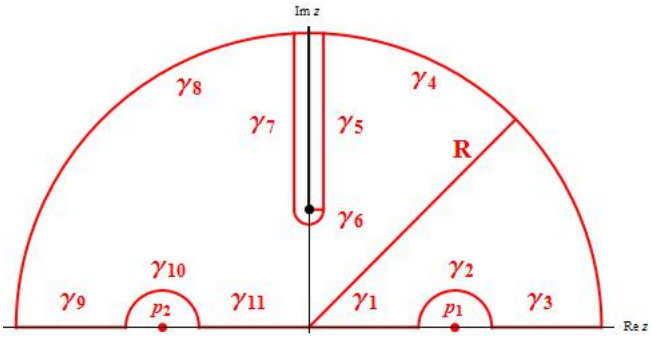


Figure 2.5: The integration path γ (orientation: counter-clockwise) which we shall use in order to compute the inverse Fourier transform of $\tilde{\phi}_E(p)$. The black line and the corresponding black point represent the branch cut. Analogous to the non-relativistic case the poles shall be evaded with the help of γ_2 and γ_{10}.

$$\gamma_1 : [0, \sqrt{E^2 - m^2}] \to \mathbb{C}, \qquad \gamma_1(t) = t - \epsilon, \qquad \gamma_1' = 1. \tag{2.79}$$
$$\gamma_2 : [\pi, 0] \to \mathbb{C}, \qquad \gamma_2(t) = \sqrt{E^2 - m^2} + \epsilon e^{it}, \qquad \gamma_2' = i\epsilon e^{it}. \tag{2.80}$$
$$\gamma_3 : [\sqrt{E^2 - m^2}, R] \to \mathbb{C}, \qquad \gamma_3(t) = t + \epsilon, \qquad \gamma_3' = 1. \tag{2.81}$$
$$\gamma_4 : [0, \frac{\pi}{2}] \to \mathbb{C}, \qquad \gamma_4(t) = Re^{it}, \qquad \gamma_4' = iRe^{it}. \tag{2.82}$$
$$\gamma_5 : [R, m] \to \mathbb{C}, \qquad \gamma_5(t) = i(t + \epsilon), \qquad \gamma_5' = i. \tag{2.83}$$
$$\gamma_6 : [\frac{\pi}{2}, -\frac{3\pi}{2}] \to \mathbb{C}, \qquad \gamma_6(t) = im + \epsilon e^{it}, \qquad \gamma_6' = i\epsilon e^{it}. \tag{2.84}$$
$$\gamma_7 : [m, R] \to \mathbb{C}, \qquad \gamma_7(t) = i(t + \epsilon), \qquad \gamma_7' = i. \tag{2.85}$$
$$\gamma_8 : [\frac{\pi}{2}, \pi] \to \mathbb{C}, \qquad \gamma_8(t) = Re^{it}, \qquad \gamma_7' = iRe^{it}. \tag{2.86}$$
$$\gamma_9 : [-R, -\sqrt{E^2 - m^2}] \to \mathbb{C}, \qquad \gamma_9(t) = t - \epsilon, \qquad \gamma_9' = 1. \tag{2.87}$$
$$\gamma_{10} : [\pi, 0] \to \mathbb{C}, \qquad \gamma_{10}(t) = -\sqrt{E^2 - m^2} + \epsilon e^{it}, \qquad \gamma_{10}' = i\epsilon e^{it}. \tag{2.88}$$
$$\gamma_{11} : [m, R] \to \mathbb{C}, \qquad \gamma_{11}(t) = t + \epsilon, \qquad \gamma_{11}' = 1. \tag{2.89}$$

Using the fact, that the integrand is holomorphic on the domain enclosed by γ, we can use the residue theorem in order to solve the integral. The integral then takes the form

$$\begin{aligned}
\frac{c_s}{2\pi}\int_\gamma \frac{1}{E - \sqrt{z^2 - m^2}} e^{izx} dz = \frac{c_s}{2\pi}\Bigg(&\int_{\gamma_1} \frac{1}{E - \sqrt{z^2 - m^2}} e^{izx} dz + \int_{\gamma_2} \frac{1}{E - \sqrt{z^2 - m^2}} e^{izx} dz + \\
+ &\int_{\gamma_3} \frac{1}{E - \sqrt{z^2 - m^2}} e^{izx} dz + \int_{\gamma_4} \frac{1}{E - \sqrt{z^2 - m^2}} e^{izx} dz + \int_{\gamma_5} \frac{1}{E - \sqrt{z^2 - m^2}} e^{izx} dz + \\
+ &\int_{\gamma_6} \frac{1}{E - \sqrt{z^2 - m^2}} e^{izx} dz + \int_{\gamma_7} \frac{1}{E - \sqrt{z^2 - m^2}} e^{izx} dz + \int_{\gamma_8} \frac{1}{E - \sqrt{z^2 - m^2}} e^{izx} dz + \\
+ &\int_{\gamma_9} \frac{1}{E - \sqrt{z^2 - m^2}} e^{izx} dz + \int_{\gamma_{10}} \frac{1}{E - \sqrt{z^2 - m^2}} e^{izx} dz + \int_{\gamma_{11}} \frac{1}{E - \sqrt{z^2 - m^2}} e^{izx} dz \Bigg) = ic_s Res = 0.
\end{aligned} \tag{2.90}$$

We already performed most of the integrals while computing the inverse Fourier transform of the bound state wave function. Because of this, we will not consider them in detail one more time. Instead, we shall just use their results. The semicircle which consists of γ_4 and γ_5 will disappear in the limit $R \to \infty$. After rearranging them, the terms $\gamma_1, \gamma_3, \gamma_9$ and γ_{11} build up the integral we desire. The circle described by γ_6 vanishes when we send ϵ to zero. What further remains to be written down, is the integral of the branch cut and the paths γ_2 and γ_{10}, which need deeper investigations. Let us firstly take a look at γ_2

$$\lim_{\epsilon\to 0}\int_{\gamma_2}\frac{1}{E-\sqrt{z^2-m^2}}e^{izx}dz=\lim_{\epsilon\to 0}\int_{\pi}^{0}\frac{e^{i(\sqrt{E^2-m^2}+\epsilon e^{it})x}}{E-\sqrt{(\sqrt{E^2-m^2}+\epsilon e^{it})^2-m^2}}i\epsilon e^{it}dt. \tag{2.91}$$

As we can see, sending ϵ towards zero becomes problematic, since we produce zero in the denominator. If we look at the closest surrounding area of $\epsilon=0$ we can approximate the denominator with a Taylor series

$$\lim_{\epsilon\to 0}\int_{\pi}^{0}\frac{e^{i(\sqrt{E^2-m^2}+\epsilon e^{it})x}}{E-\sqrt{(\sqrt{E^2-m^2}+\epsilon e^{it})^2-m^2}}i\epsilon e^{it}dt=\lim_{\epsilon\to 0}\int_{\pi}^{0}\frac{e^{i(\sqrt{E^2-m^2}+\epsilon e^{it})x}}{\frac{-\sqrt{E^2-m^2}}{E}\epsilon e^{it}+\mathcal{O}(\epsilon^2)}i\epsilon e^{it}dt. \tag{2.92}$$

Further algebraic processing yields

$$\lim_{\epsilon\to 0}\int_{\pi}^{0}\frac{e^{i(\sqrt{E^2-m^2}+\epsilon e^{it})x}}{\frac{-\sqrt{E^2-m^2}}{E}+\frac{\mathcal{O}(\epsilon^2)}{\epsilon e^{it}}}idt=\lim_{\epsilon\to 0}\int_{\pi}^{0}\frac{e^{i(\sqrt{E^2-m^2}+\epsilon e^{it})x}}{\frac{-\sqrt{E^2-m^2}}{E}+\mathcal{O}(\epsilon)}idt. \tag{2.93}$$

Since the remainder term of order ϵ^2 becomes order ϵ through the division with ϵ, we can now send ϵ to zero without any peril. What remains is

$$\int_{\pi}^{0}e^{i(\sqrt{E^2-m^2})x}\frac{E}{\sqrt{E^2-m^2}}idt=\frac{i\pi E}{\sqrt{E^2-m^2}}e^{i\sqrt{E^2-m^2}x}. \tag{2.94}$$

The same procedure as above applied to γ_{10} yields

$$\int_{\pi}^{0}e^{-i(\sqrt{E^2-m^2})x}\frac{E}{\sqrt{E^2-m^2}}idt=\frac{-i\pi E}{\sqrt{E^2-m^2}}e^{-i\sqrt{E^2-m^2}x}. \tag{2.95}$$

With the same argument as in (1.111) γ_2 and γ_{10} put together take the form

$$\frac{i\pi E}{\sqrt{E^2-m^2}}e^{i\sqrt{E^2-m^2}x}-\frac{i\pi E}{\sqrt{E^2-m^2}}e^{-i\sqrt{E^2-m^2}x}=\frac{-2\pi E}{\sqrt{E^2-m^2}}\sin(\sqrt{E^2-m^2}|x|). \tag{2.96}$$

The branch cut integral is

$$\chi(t)=-\int_{m}^{\infty}\frac{2e^{-|x|t}\sqrt{t^2-m^2}}{m^2-t^2-E^2}dt. \tag{2.97}$$

All considerations we did on the last page put into (2.90) and processed further with elementary algebraic operations yield the form of $\phi_E(x)$

$$\begin{aligned}\frac{c_s}{2\pi}\int_{-\infty}^{\infty}\frac{1}{E-\sqrt{p^2+m^2}}dp&=\frac{c_s}{2\pi}\left(\frac{2\pi E}{\sqrt{E^2-m^2}}\sin(\sqrt{E^2-m^2}|x|)+\int_{m}^{\infty}\frac{2e^{-|x|t}\sqrt{t^2-m^2}}{m^2-t^2-E^2}dt\right)=\\&=\frac{c_sE}{\sqrt{E^2-m^2}}\sin(\sqrt{E^2-m^2}|x|)+\frac{c_s}{\pi}\int_{m}^{\infty}\frac{e^{-|x|t}\sqrt{t^2-m^2}}{m^2-t^2-E^2}dt=\phi_E(x).\end{aligned} \tag{2.98}$$

Hence, the inverse Fourier transform of $\tilde{\psi}_E^{even}(p)$ is

$$\psi_E^{even}(x)=\cos(\sqrt{E^2-m^2}x)+\frac{c_sE}{\sqrt{E^2-m^2}}\sin(\sqrt{E^2-m^2}|x|)+\frac{c_s}{\pi}\int_{m}^{\infty}\frac{e^{-|x|t}\sqrt{t^2-m^2}}{m^2-t^2-E^2}dt, \tag{2.99}$$

where the full form of c_s is

$$c_s=\lambda(1+\phi_E(0))=\left(\frac{-E_b}{2\pi\sqrt{m^2-E_b^2}}\left(\pi+2\arcsin\left(\frac{E_b}{m}\right)\right)-\frac{E}{\pi\sqrt{E^2-m^2}}\operatorname{arctanh}\left(\frac{\sqrt{E^2-m^2}}{E}\right)\right)^{-1}. \tag{2.100}$$

This expression has quite interesting properties. Those properties are that interesting, that the study of the expression has its own sub-section! However, in order to prevent a loss of overview of the considerations we made in this sub-section, we shall briefly summarize what we did in it. In a first step, we focussed on the study of states, where the particle has a total energy bigger than its rest mass. Analogous to the non-relativistic setting, those states are the scattering states in the relativistic scenario. After constructing them in momentum space we used the inverse Fourier transformation to obtain representations in position space.

2.2.2 The running coupling constant $\lambda(E, E_b)$, asymptotic freedom, and the β-function

As we already stated in the previous section, c_s has very interesting properties. To give us some visual support, we shall recall (2.100).

$$c_s = \lambda(1+\phi_E(0)) = \left(\frac{-E_b}{2\pi\sqrt{m^2-E_b^2}}\left(\pi + 2\arcsin\left(\frac{E_b}{m}\right)\right) - \frac{E}{\pi\sqrt{E^2-m^2}}\text{arctanh}\left(\frac{\sqrt{E^2-m^2}}{E}\right)\right)^{-1}. \quad (2.101)$$

The first thing we can see, is the fact that c_s, although it contained the regulated divergent integrals for $\frac{1}{\lambda}$ and $I(p)$, does not depend on them completely. There is merely a dependence on the finite parts of them. Therefore, if we send the regulator to zero, c_s remains finite. If we take another look at c_s we can see that it contains the bound state energy and the energy of the scattering state. Therefore, if we would fix (for instance) E_b and vary E, c_s would vary. Given the above properties we can look at c_s as a description of the coupling of a scattering state with energy E to the bound state with energy E_b. This so called running coupling implies (in contrast to the non-relativistic case) that a scattering state with any finite energy E can never be considered as uninfluenced by the bound state! Although the coupling to the bound state may be weak it is still there! Let us now consider the following scenario. We fix the energy E_b and consider what happens when we send E to infinity. Since we are speaking of the running coupling, we shall rename c_s into $\lambda(E_b, E)$ and formalize the latter statement

$$\lim_{E\to\infty}\lambda(E,E_b) = \lim_{E\to\infty}\left(\frac{-E_b}{2\pi\sqrt{m^2-E_b^2}}\left(\pi + 2\arcsin\left(\frac{E_b}{m}\right)\right) - \frac{E}{\pi\sqrt{E^2-m^2}}\text{arctanh}\left(\frac{\sqrt{E^2-m^2}}{E}\right)\right)^{-1}. \quad (2.102)$$

Since we are considering the limit $E \to \infty$, for the rest mass m holds the fact $m \ll E$. Therefore, we are allowed to neglect m, which leads to

$$\lim_{E\to\infty}\lambda(E,E_b) = \left(\frac{-E_b}{2\pi\sqrt{m^2-E_b^2}}\left(\pi + 2\arcsin\left(\frac{E_b}{m}\right)\right) - \frac{1}{\pi}\text{arctanh}(1)\right)^{-1}. \quad (2.103)$$

Since the $\text{arctanh}(x)$ diverges to infinity as x approaches one, the sum inside the inverting brackets blows up to minus infinity. Considering the inverting brackets, the above expression becomes zero as E goes to infinity. Hence

$$\lim_{E\to\infty}\lambda(E,E_b) = 0. \quad (2.104)$$

This means nothing else than that the running coupling vanishes, if the scattering state has infinite much energy. Alternatively, we could also say that a particle does not feel the coupling any more, if the particle's energy is infinitely high. This behaviour is well known under the name "asymptotic freedom". One famous example which shows this behaviour, is the strong force which describes the interaction between quarks and gluons. Figure 2.6 (see next page) illustrates the behaviour of $\lambda(E, E_b)$ for different values of E_b.

In order to further characterize the running coupling constant, we shall construct its β-function. This function is defined as follows

$$\beta(\lambda(E,E_b)) = E\frac{\partial|\lambda(E,E_b)|}{\partial E}. \quad (2.105)$$

Plugging (2.100) into the the above definition yields

$$E\frac{\partial|\lambda(E,E_b)|}{\partial E} = E\frac{\partial\left|\left(\frac{-E_b}{2\pi\sqrt{m^2-E_b^2}}\left(\pi + 2\arcsin\left(\frac{E_b}{m}\right)\right) - \frac{E}{\pi\sqrt{E^2-m^2}}\text{arctanh}\left(\frac{\sqrt{E^2-m^2}}{E}\right)\right)^{-1}\right|}{\partial E}. \quad (2.106)$$

Before we let the partial derivative act on $\lambda(E, E_b)$, we have to investigate the role of the modulus. E_b and E are "by default" positive values, therefore the sign in front of them renders them negative. Since arcsin

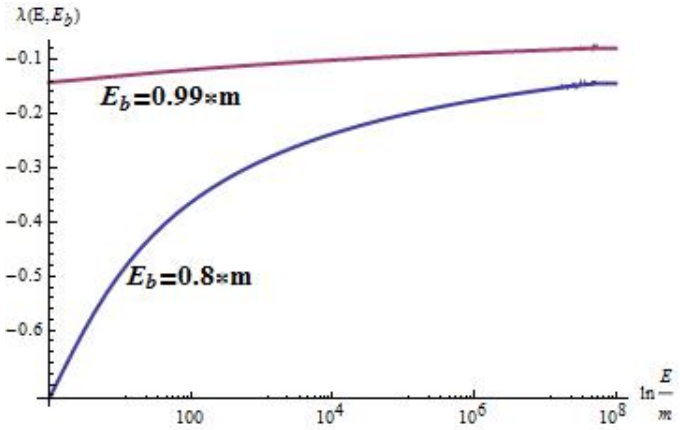


Figure 2.6: The behaviour of the running coupling constant in terms of the rest mass m of the particle. The asymptotic approximation towards the x-axis for large values of $\frac{E}{m}$ has physical relevance and is known as "asymptotic freedom".

and arctanh are positive in the domains we are operating, the sum remains negative and the modulus attaches therefore an additional sign. With these considerations the latter expression becomes

$$E\frac{\partial|\lambda(E,E_b)|}{\partial E} = E\frac{\partial\left(\frac{E_b}{2\pi\sqrt{m^2-E_b^2}}\left(\pi+2\arcsin\left(\frac{E_b}{m}\right)\right)+\frac{E}{\pi\sqrt{E^2-m^2}}\text{arctanh}\left(\frac{\sqrt{E^2-m^2}}{E}\right)\right)^{-1}}{\partial E}. \tag{2.107}$$

$\lambda(E,E_b)$ contains two trigonometric inverse functions, chained with square roots and fractions. Therefore we shall only give the result of the computations

$$\frac{E\partial|\lambda(E,E_b)|}{\partial E} = E(\lambda(E,E_b))^2\left(\frac{m^2}{(E^2-m^2)^{\frac{3}{2}}\pi}\text{arctanh}\left(\frac{\sqrt{E^2-m^2}}{E}\right)-\frac{E}{(E^2-m^2)\pi}\right). \tag{2.108}$$

Since the definition (2.105) uses only $\lambda(E,E_b)$ as its argument, we shall rewrite the above expression

$$\beta(\lambda(E,E_b),E) = (\lambda(E,E_b))^2\left(\frac{Em^2\text{arctanh}\left(\frac{\sqrt{E^2-m^2}}{E}\right)-E^2\sqrt{E^2-m^2}}{(E^2-m^2)^{\frac{3}{2}}\pi}\right). \tag{2.109}$$

Factorizing E out of the terms inside (2.109) yields

$$\beta(\lambda(E,E_b),E) = (\lambda(E,E_b))^2\left(\frac{Em^2\text{arctanh}(\sqrt{1-\tau^2})-E^3\sqrt{1-\tau^2}}{E^3(1-\tau^2)^{\frac{3}{2}}\pi}\right),\ \tau=\frac{m}{E}. \tag{2.110}$$

With the defintion of τ we can further simplifiy the latter expression

$$\beta(\lambda(E,E_b),\tau) = (\lambda(E,E_b))^2\left(\frac{\tau^2\text{arctanh}(\sqrt{1-\tau^2})-\sqrt{1-\tau^2}}{(1-\tau^2)^{\frac{3}{2}}\pi}\right). \tag{2.111}$$

Some more algebraic operations lead to

$$\beta(\lambda(E,E_b),\tau) = (\lambda(E,E_b))^2\left(\frac{\tau^2}{(1-\tau^2)}\frac{\text{arctanh}(\sqrt{1-\tau^2})}{\sqrt{1-\tau^2}\pi}-\frac{1}{(1-\tau^2)\pi}\right). \tag{2.112}$$

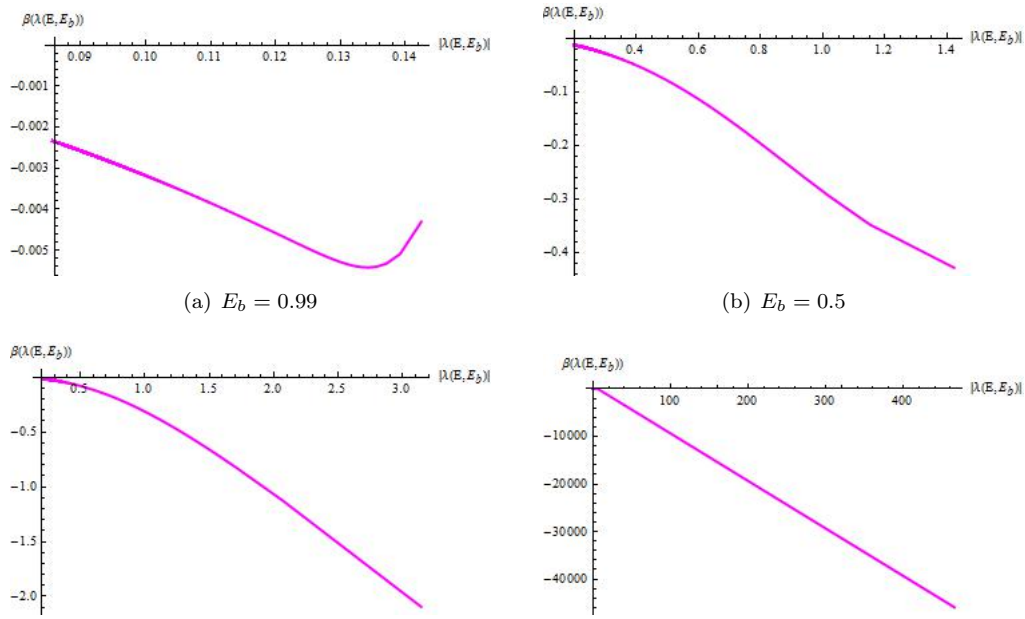
Recalling the form (2.15), cutting off the ϵ dependence in it and using the inverse of (2.101), leads to

$$\beta(\lambda(E,E_b),\tau) = (\lambda(E,E_b))^2\left(\frac{\tau^2}{(1-\tau^2)}\left(\frac{1}{\lambda(E_b)}-\frac{1}{\lambda(E,E_b)}\right)-\frac{1}{(1-\tau^2)\pi}\right), \tag{2.113}$$

which is equivalent to

$$\beta(\lambda(E,E_b),\tau) = \frac{-(\lambda(E,E_b))^2}{\pi} + \frac{(\lambda(E,E_b))^2\tau^2}{1-\tau^2}\left(\frac{1}{\lambda(E_b)} - \frac{1}{\lambda(E,E_b)} - \frac{1}{\pi}\right). \tag{2.114}$$

Unfortunately it is not possible to conduct further manipulations since (2.108) is an implicit expression. However, since the β-function describes the behaviour of the running coupling in dependence of energy change of the scattering state (recall (2.105)) it would be very interesting to plot it and take a look at the curve.



(a) $E_b = 0.99$ (b) $E_b = 0.5$

(c) $E_b = 0$ (d) $E_b = -0.99$

Figure 2.7: The behaviour of the β-function for several values of E_b. The end points of the curves correspond to the maximal value of $|\lambda(E,E_b)|$.

As already shown, $\lambda(E,E_b)$ becomes zero in the limit $E \to \infty$. Additionally $\beta(E,E_b)$ is continuous on $]m,\infty[$. Therefore all curves should begin at the origin. The curve for $E_b = 0.99$ is there no exception (it's generating was due to the slow convergence speed problematic and therefore cut off). It's CPU-time devouring computation put aside, the case $E_b = 0.99$ has another interesting property. If we take a closer look at the corresponding sub-figure, we can see that this curve is the only one with a minimum. The reasons for this feature demand further investigations, which unfortunately, will not be a part of this thesis.

2.2.3 Transmission and reflection coefficients

In chapter one we mentioned that it is possible to construct the plane wave ansatz by using linear combinations of the odd and even scattering states. It would be very interesting to see how the scattering behaviour of the relativistic setting looks like. Therefore, we shall now conduct the same scattering experiment as in chapter 1. This also means that we will construct wave functions which come close to the plane waves of chapter 1. Since the even relativistic scattering state is affected by the branch cut, we will not be able to fix the point $x = 0$ using continuity conditions. But if we go away from the critical point, the branch cut contribution is exponentially suppressed and we can consider the particle being (almost) a plane wave. Therefore, for the region where $x > 0$ we shall construct an outgoing wave the following way

$$T(k)e^{ikx} + C(k)\lambda(E,E_b)\chi(x) = A(k)\psi_E^{even}(x) + B(k)\psi_E^{odd}(x),\ k = \sqrt{E^2 - m^2}. \tag{2.115}$$

Using the Euler relation [1, p. 31] and recalling the work of the last few sections, we can rewrite the above expression into

$$T(k)(\cos(kx) + i\sin(kx)) + C(k)\lambda(E, E_b)\chi(x) = A(k)\Bigg(\cos(kx) + \frac{\lambda(E, E_b)\sqrt{k^2+m^2}}{k}\sin(kx) +$$
$$- \frac{\lambda(E, E_b)}{\pi}\int_m^\infty \frac{e^{-|x|t}\sqrt{t^2-m^2}}{E^2-m^2+t^2}dt\Bigg) + B(k)\sin(kx). \tag{2.116}$$

Through comparison of the terms, we can see that $C(k) = -A(k)$. Further comparing and simplifications lead to

$$T(k)(\cos(kx) + i\sin(kx)) = A(k)\cos(kx) + \left(\frac{A\lambda(E, E_b)\sqrt{k^2+m^2}}{k} + B\right)\sin(kx). \tag{2.117}$$

One thing we can pull out of the latter expression is $A(k) = T(k)$. Another relation looks a bit more complicated

$$B = \left(\frac{ik - \lambda(E, E_b)\sqrt{k^2+m^2}}{k}\right)A \tag{2.118}$$

This is the most we can take of the first side. Now we focus on the incident side (i.e. $x < 0$) of the well. We make the ansatz

$$e^{ikx} + R(k)e^{-ikx} + C(k)\lambda(E, E_b)\chi(x) = A(k)\cos(kx) - \frac{A\lambda(E, E_b)\sqrt{k^2+m^2}}{k}\sin(kx)$$
$$+ C(k)\lambda\chi(x) + B\sin(kx). \tag{2.119}$$

Rewriting the left-hand side a bit generates

$$(1 + R(k))\cos(kx) + i(1 - R)\sin(kx) = A(k)\cos(kx) + \left(B(k) - \frac{A(k)\lambda(E, E_b)\sqrt{k^2+m^2}}{k}\right). \tag{2.120}$$

Through comparing the terms on the left-hand and right-hand side of the last equation and inserting what we already know about $B(k)$, we can extract the following relations

$$A(k) = 1 + R(k),\ (1 - R(k)) = \frac{kA(k) + i2A(k)\lambda(E, E_b)\sqrt{k^2+m^2}}{k}. \tag{2.121}$$

Therefore we can determine $A(k)$ in dependence of k

$$A(k) = \frac{k}{k + i\lambda(E, E_b)\sqrt{k^2+m^2}}. \tag{2.122}$$

In the considerations above, we expressed all other coefficients in terms of A. Now we give their concrete forms

$$A(k) = \frac{k}{k + i\lambda(E, E_b)\sqrt{k^2+m^2}}, \tag{2.123}$$
$$B(k) = i, \tag{2.124}$$
$$C(k) = \frac{-k}{k + i\lambda(E, E_b)\sqrt{k^2+m^2}}, \tag{2.125}$$
$$R(k) = \frac{-i\lambda(E, E_b)\sqrt{k^2+m^2}}{k + i\lambda(E, E_b)\sqrt{k^2+m^2}}, \tag{2.126}$$
$$T(k) = \frac{k}{k + i\lambda(E, E_b)\sqrt{k^2+m^2}}. \tag{2.127}$$

2.3 Orthogonality in momentum space

Analogous to the non-relativistic case we have to ensure that the wave functions we constructed behave like a self-adjoint system (i.e. orthogonality between the different eigenstates has to hold). First, let us check the orthogonality of the bound state the odd scattering state.

$$\left\langle \psi_E^{odd} \middle| \psi_{E_b} \right\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\pi}{i} (\delta(p - \sqrt{E^2 - m^2}) - \delta(p + \sqrt{E^2 - m^2})) \right)^* \left(\frac{A}{E_b - \sqrt{p^2 + m^2}} \right) dp. \tag{2.128}$$

Algebraic processing and evaluating the resulting δ function containing integrals yields

$$\left\langle \psi_E^{odd} \middle| \psi_{E_b} \right\rangle = \frac{-1}{2i} \left(\frac{A}{E_b - E} - \frac{A}{E_b - E} \right) = 0, \tag{2.129}$$

which shows, that orthogonality for the bound state and the odd scattering state holds. Now let us take a look at the situation for even scattering states and the bound state

$$\left\langle \psi_E^{even} \middle| \psi_{E_b} \right\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\pi(\delta(p - \sqrt{E^2 - m^2}) + \delta(p + \sqrt{E^2 - m^2})) + \tilde{\phi}_E(p) \right)^* \left(\frac{A}{E_b - \sqrt{p^2 + m^2}} \right) dp. \tag{2.130}$$

Evaluating the δ terms results in the expression below.

$$\left\langle \psi_E^{even} \middle| \psi_{E_b} \right\rangle = \frac{A}{E_b - E} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{c_s}{E - \sqrt{p^2 + m^2}} \frac{A}{E_b - \sqrt{p^2 + m^2}} dp. \tag{2.131}$$

Once more $\tilde{\phi}_E(p)$ seems to cause trouble. Conducting a partial fraction expansion solves the problem and leads to

$$\left\langle \psi_E^{even} \middle| \psi_{E_b} \right\rangle = \frac{A}{E_b - E} + \frac{A}{E_b - E} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{c_s}{E - \sqrt{p^2 - m^2}} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{c_s}{E_b - \sqrt{p^2 + m^2}} dp \right). \tag{2.132}$$

Recalling expressions (2.64) and (2.5), we are able to write the following expression

$$\left\langle \psi_E^{even} \middle| \psi_{E_b} \right\rangle = \frac{A}{E_b - E} + \frac{A}{E_b - E} \left(\phi_E(0) - \frac{\lambda(1 + \phi_E(0))}{\lambda} \right) = 0. \tag{2.133}$$

Therefore, orthogonality holds also for even scattering states and the bound state. Now we shall investigate the behaviour for two odd states with energy E and E'.

$$\left\langle \psi_E^{odd} \middle| \psi_{E'}^{odd} \right\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\pi}{i} (\delta(p - \sqrt{E^2 - m^2}) - \delta(p + \sqrt{E^2 - m^2})) \right)^* \left(\frac{\pi}{i} (\delta(p - \sqrt{E'^2 - m^2}) - \delta(p + \sqrt{E'^2 - m^2})) \right) dp \tag{2.134}$$

Analogous to the non-relativistic case, the above expression yields

$$\pi(\delta(\sqrt{E'^2 - m^2} - \sqrt{E^2 - m^2}) - \delta(\sqrt{E'^2 - m^2} + \sqrt{E^2 - m^2})) = 0, \tag{2.135}$$

where we used that $E' \neq E$. Therefore, orthogonality holds in the case of two odd functions too. How about the case where we have two even scattering states?

$$\left\langle \psi_E^{even} \middle| \psi_{E'}^{even} \right\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\pi(\delta(p - \sqrt{E^2 - m^2}) + \delta(p + \sqrt{E^2 - m^2})) + \tilde{\phi}_E(p) \right)^* \left(\pi(\delta(p - \sqrt{E'^2 - m^2}) + \delta(p + \sqrt{E'^2 - m^2})) + \tilde{\phi}_{E'}(p) \right) dp. \tag{2.136}$$

Expanding the latter expression and simplifying all generated δ-dependent terms the same way as above, leads to

$$\frac{c_{s'}}{E'-E} - \frac{c_s}{E'-E} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{c_{s'}c_s}{(E-\sqrt{p^2+m^2})(E'-\sqrt{p^2+m^2})}dp. \tag{2.137}$$

One more time the integral can be handled with a partial fraction expansion. After the procedure the above expression becomes

$$\frac{c_{s'}}{E'-E} - \frac{c_s}{E'-E} + \frac{c_s c_{s'}}{E'-E}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{E-\sqrt{p^2+m^2}}dp - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{E'-\sqrt{p^2+m^2}}dp\right). \tag{2.138}$$

Recalling (2.63) yields

$$\left\langle \psi_E^{even} \middle| \psi_{E'}^{even} \right\rangle = \frac{c_{s'}}{E'-E} - \frac{c_s}{E'-E} + \frac{1}{E'-E}((\lambda+\lambda\phi_{E'}(0))\phi_E(0) - (\lambda+\lambda\phi_E(0))\phi_{E'}(0)). \tag{2.139}$$

After further processing, we get

$$\left\langle \psi_E^{even} \middle| \psi_{E'}^{even} \right\rangle = \frac{c_{s'}}{E'-E} - \frac{c_s}{E'-E} + \frac{1}{E'-E}(\lambda\phi_E(0) - \lambda\phi_{E'}(0)). \tag{2.140}$$

With help of (2.63) we can rewrite the expression above as

$$\left\langle \psi_E^{even} \middle| \psi_{E'}^{even} \right\rangle = \frac{c_{s'}}{E'-E} - \frac{c_s}{E'-E} + \frac{1}{E'-E}(c_s - \lambda - (c_{s'} - \lambda)). \tag{2.141}$$

This leads us to

$$\left\langle \psi_E^{even} \middle| \psi_{E'}^{even} \right\rangle = \frac{1}{E'-E}(c_{s'} - c_s) + \frac{1}{E'-E}(c_s - c_{s'}) = 0. \tag{2.142}$$

Hence, orthogonality holds. What remains to be checked is the orthogonality between odd and even scattering states. Since the integration interval is symmetric ($p \in]-\infty, \infty[$) and the product of even and odd functions is odd, its integration delivers zero, which means that orthogonality holds. For the orthogonality in position space, we can argue the same way as in chapter 1 (see page 20).

Chapter 3

Recapitulation, Remarks, and Conclusions

3.1 Chapter 1

In chapter 1 we investigated the behaviour of a non-relativistically moving particle in the δ-potential. In the investigation progress we found that for the attractive potential there exists one bound state and infinitely many scattering states (this scattering can be thought of as a one-dimensional contact interaction between a particle with mass m and an infinitely massive and point-like entity represented by the δ-well). Further, we discovered that the repulsive potential yields scattering states only. From the identical form of reflection and transmission probabilities in both settings (attractive and repulsive potential) we could deduce that there is no way to distinguish a well from a barrier (as long as we just consider the reflection and transmission probabilities). Since the δ-potential is spatially symmetric, we could exploit this property to describe scattering states with the help of even and odd functions. In this description we introduced the phase factor $\phi(\lambda)$. After some computations we concluded that the sign of the coupling strength λ affects directly this phase factor which then determines if the even wave function fits the attractive scenario (the particles probability for sitting right on the δ-peak is higher than for its surrounding) or the repulsive setting (the particle's probability for sitting on the peak is lower in comparison to the surrounding). Amazingly in both cases the particle's probability for being at $x = 0$ has the same value. We explored the same setting in momentum space. This exploration lead us to the gap equation for the bound state and to the formulation of sine and cosine in momentum space as a sum of δ functions. Further we confirmed full compatibility between the formulations of the problem in momentum and position space.

3.2 Chapter 2

Chapter 2 was dedicated to the exploration of the behaviour of a relativistically moving particle in an attractive δ-potential. After formulating the relativistic Schrödinger equation we had to remark that it demands "by default" further treatment in momentum space. In a first step we took a look at the bound state. Analogous to the momentum space considerations in chapter 1, a gap equation was formulated. In it we hit on an integral which was logarithmically divergent. Despite this problem we were able to characterize the bound state by introducing and applying a method known as dimensional regularisation. After applying an inverse Fourier transformation to the momentum space bound state, we investigated the properties of a term of the position space bound state wave function. The fact that it contained the energy of the bound state in combination with its exponential dependence of the modulus of x lead us to the conclusion that it represents a non-locality, which only occurs in the relativistic setting. Consideration of the non-relativistic limit revealed that our results from chapter 1 are packed in the relativistic setting. Afterwards we set our focus to the relativistic scattering states. Once more we encountered an integral with the same divergent behaviour as in the bound state case. Applying again dimensional regularization lead us to an expression quite similar to the one for the bound state. Amazingly this lead to a reformulation of the coupling for the scattering states. Where before a divergence would blow up, a finite part of the regularized expressions remained. This coupling $\lambda(E, E_b)$ depends on the bound state energy and the scattering state energy. Since it has the properties of a running coupling, it is named a running coupling constant. Since considerations of the limit $E \to \infty$ revealed that the coupling was only then disappearing (i.e. becoming zero), we concluded that the system shows the behaviour known as asymptotic

freedom. In a next step we constructed the plane wave approach (as mentioned in chapter 1) out of even and odd scattering states. As a final comment we state that it is amazing how much physics can be packed in a quite simple set up.

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