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Bachelor Thesis

Introduction to Lie Group Theory and Graphical Tensor Product Reduction Scheme for the Lie Algebras $su(3)$ and $g(2)$

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Abstract

In this thesis an introduction to Lie group and Lie algebra theory is given with regard to physical applications. The notions of weight diagrams and landscapes are described in order to graphically reduce tensor products of irreducible representations of compact simple Lie algebras of rank 2, following the scheme devised by Antoine and Speiser. Additionally, web applets using this method for both the Lie algebras $su(3)$ and $g(2)$ are implemented for practical use.

Contents

1 Introduction

In order to describe physical systems, it is useful to consider the sets of transformations under which the Hamiltonian is invariant. These symmetries simplify the process of finding solutions of its dynamical equations. The so-called gauge symmetries, which relate to intrinsic local invariances of the mathematical description, are fundamental. Invariances of systems under global symmetry transformations can be used to simplify finding solutions as well. As it turns out, algebra, or more precisely the theory of Lie groups, is the appropriate mathematical tool in order to describe continuous symmetries. Sophus Lie ($*1842 - \dagger 1899$) was a Norwegian mathematician who largely created the theory of continuous symmetry and applied it to geometry. Lie realized that continuous transformation groups, which are now known as Lie groups, can be linearized by studying their tangent spaces spanned by so-called generators. The generators themselves are subject to a linearization of the group composition law, called the Lie bracket, and the resulting vector space is called a Lie algebra. Both the German mathematician Wilhelm Killing ($*1847 - \dagger 1923$), and the French mathematician Élie Cartan ($*1869 - \dagger 1951$) have worked, among other things, on the classification of Lie algebras. They discovered and characterized the complete list of all compact simple Lie algebras, which are most important in physics. There are the orthogonal algebras $so(n)$, the unitary algebras $su(n)$, the symplectic algebras $sp(n)$, and the exceptional algebras $g(2), f(4), e(6), e(7)$, and $e(8)$.

As will be elaborated later, Lie algebra generators become physically meaningful through the theory of representations. Oftentimes, it is of interest to couple multiple representations together. This is mathematically accomplished using the tensor product of the two representations. For instance, when two particles, described by representations of Lie algebra elements, are coupled together to form a new system, the available states can be constructed from the tensor product. As proven by the German mathematicians Hermann Weyl ($*1885 - \dagger 1955$) and Fritz Peter ($*1899 - \dagger 1949$) in 1927, tensor products of so-called irreducible representations can always be decomposed into a direct sum of irreducible representations. This fundamental statement is known as the Peter-Weyl theorem. Since such a decomposition is always possible, it is of interest to reduce any tensor product to a direct sum.
Jean-Pierre Antoine and David Speiser have devised a graphical method [1, 2], in order to reduce tensor products of irreducible rank 2 Lie algebra representations, which is much less cumbersome to compute compared to

analytical methods. This graphical scheme has been applied to all the compact simple rank 2 Lie algebras $so(5) = sp(2)$, $su(3)$, and $g(2)$ in [3].

In this thesis, the main focus lies on the theory of Lie groups and algebras in general, and the Lie algebras $su(3)$ and $g(2)$ in particular, which are more closely analysed and discussed by following [3] and [4]. A web implementation of this graphical tensor product reduction method for both algebras is provided alongside the written part of this thesis.

2 Introduction to Lie group theory

The theoretical fundamentals of Lie group theory are essential in order to study physical symmetries and the corresponding notions of Lie algebras and representations provide the necessary instruments to do that. This chapter contains a brief introduction to the mathematical field of Lie group theory with regard to physical applications and provides the necessary basics in order to understand its relevance and potential in physics.

2.1 Continuous groups

In order to discuss physical problems in terms of group theory, the basic concepts of finite groups are no longer sufficient. The generalization to continuous r-parameter groups is necessary for further applications.
Consider a set G with elements $G(a) \equiv G(a_1, a_2, \dots, a_r)$ dependent on r real parameters. G is called a *continuous group* if the well known group axioms hold:

(G1) $* : G \times G \to G$ is the binary group operation such that:
$$\forall\, G(a),\, G(b) \in G \implies G(a) * G(b) = G(c) \in G.$$

(G2) $\forall\, G(a),\, G(b),\, G(c) \in G \implies G(a) * (G(b) * G(c)) = (G(a) * G(b)) * G(c).$

(G3) $\exists\, G(e) \in G,$ such that: $\forall\, G(a) \in G :\ G(e) * G(a) = G(a) * G(e) = G(a).$

(G4) $\forall\, G(a) \in G\ \exists\, G(a') \in G :\ G(a') * G(a) = G(a) * G(a') = G(e).$

Alternatively the axioms can be imposed on the group parameters themselves, yielding the following corresponding constraints:

(F1) $f : U \times U \to U,\ (a, b) \mapsto c$ is continuous for $U \subset \mathbb{R}^r$.

(F2) $\forall\, a,\, b,\, c \in U \implies f(a, f(b, c)) = f(f(a, b), c).$

(F3) $\exists\, e \in U,$ such that: $\forall\, a \in U :\ f(e, a) = f(a, e) = a.$

(F4) $\forall\, a \in U\ \exists\, a' \in U :\ f(a', a) = f(a, a') = e.$

The general subset U of $\mathbb{R}^r$ is called *parameter space* of the r-parameter group.

Already simple symmetry considerations unveil the necessity of continuous groups. For instance, looking at the well known rotations of two dimensional Euclidean space $\mathbb{R}^2$ given by $R \in SO(2, \mathbb{R})$, the dependence on a continuous variable $\theta \in \mathbb{R}$ is simply given by:

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{2.1}$$

2.2 Lie groups

The concept of a Lie group considers especially "regular" continuous groups. A Lie group G is a continuous group where f, as defined above, is not only continuous but analytical. Consequently, there exists a convergent power series of f within the parameter space of G. Consistently, a group G with such properties is called an r-parameter Lie group.
Analogously, a Lie group defined in this way is also a smooth manifold with a group structure, such that the group operation and group element inversion are differentiable maps. In other words, Lie groups are algebraic as well as differential geometric and topological objects. For the following intents and purposes the underlying group manifolds are smooth sub-manifolds of $\mathbb{R}^N$. Essentially, these are the m-dimensional ($m < N$) generalizations of closed curves and parametric surfaces embedded in $\mathbb{R}^2$ or $\mathbb{R}^3$, as discussed for instance in [5].
The dimension of the Lie group is equal to the dimension of the underlying group manifold. Therefore one would expect the aforementioned continuous group $SO(2, \mathbb{R})$ to correspond to a one dimensional manifold, assuming it is in fact a Lie group. Constructing the set of points closed under $SO(2, \mathbb{R})$ transformations indeed yields the circle $S^1 \subset \mathbb{R}^2$.

An important concept in group theory, that will be relevant later on, is the center of a group. The center $Z(G)$ of G is defined as the set of elements which commute with all other group elements:

$$Z(G) = \{G(a) \in G \,|\, \forall\, G(b) \in G, G(a) * G(b) = G(b) * G(a)\}. \tag{2.2}$$

2.2.1 Important Lie groups

There are infinitely many different Lie groups which are categorized by more subtle properties. In this thesis however, the focus lies on Lie algebras and we restrict ourselves to the classification of these. Nevertheless, it is helpful

to list the most important Lie groups in order to study their corresponding Lie algebras later on. The most frequently used Lie groups in physics are matrix groups, or more precisely closed subgroups of the *general linear group* $GL(n, \mathbb{C})$, which is well known to be defined as:

$$GL(n, \mathbb{C}) = \{A \in Mat(n, \mathbb{C}) \,|\, \det(A) \neq 0\}. \tag{2.3}$$

Notable examples are:

$$\begin{aligned}
&O(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \,|\, A^T A = A\, A^T = \mathbb{1}\},\\
&SO(n, \mathbb{R}) = \{A \in O(n, \mathbb{R}) \,|\, \det(A) = 1\},\\
&U(n, \mathbb{C}) = \{A \in GL(n, \mathbb{C}) \,|\, A^\dagger A = A\, A^\dagger = \mathbb{1}\},\\
&SU(n, \mathbb{C}) = \{A \in U(n, \mathbb{C}) \,|\, \det(A) = 1\},\\
&Sp(2n, \mathbb{C}) = \{A \in Mat(2n, \mathbb{C}) \,|\, A^T \Omega\, A = \Omega\},
\end{aligned} \tag{2.4}$$

where

$$\Omega = \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0 \end{pmatrix}.$$

Additionally there are the so-called exceptional groups $G(2)$, $F(4)$, $E(6)$, $E(7)$, and $E(8)$, of which only $G(2)$ will be defined and examined in this thesis in chapter 4.
This list of Lie groups is not arranged in such a way by chance. The groups $SO(n)$, $SU(n)$, $Sp(2n)$, as well as the exceptional groups give rise to all of the so-called compact simple Lie algebras. These have been discovered and characterized in the works of Wilhelm Killing and Élie Cartan in 1890 and 1894 respectively and are the most important Lie algebras in physics.

2.3 Lie algebras

Instead of examining a generally non-linear manifold that is the Lie group G, the corresponding linear Lie algebra g can be defined and discussed in order to understand the details of a corresponding symmetry. In this chapter the formal definition of Lie algebras and the most important properties are given in close reference to [6] and with additions from [7].

2.3.1 Definition and basic properties

Essentially, a Lie algebra g is a vector space over some field $\mathbb{F}$ with an additional binary operation called the *Lie bracket*:

$$[\cdot, \cdot] : g \times g \to g,$$

which satisfies the following constraints:

(A1) Bilinearity for all $a, b \in \mathbb{F}$ and $x, y, z \in g$:

$$[ax + by, z] = a[x, z] + b[y, z],$$
$$[z, ax + by] = a[z, x] + b[z, y].$$

(A2) Antisymmetry for all $x, y \in g$:

$$[x, y] = -[y, x].$$

(A3) Jacoby identity for all $x, y, z \in g$:

$$[\,[x, y], z] + [\,[z, x], y] + [\,[y, z], x] = 0.$$

Like any vector space, a Lie algebra is spanned by some generating system of vectors, called *generators*. Lie algebras are generated by Hermitian generators T^α, $\alpha \in \{1, ..., n_G\}$, where n_G denotes the number of generators, i.e. the dimension of the Lie algebra. As a consequence, all elements of a Lie algebra are given by a linear combination:

$$\omega^\alpha T^\alpha = z \in g, \tag{2.5}$$

where ω^α are real coefficients.[1]

Obviously these are closed under addition and multiplication by some element in $\mathbb{F}$ but not under multiplication in general. The concept of multiplication is rather replaced by the introduced Lie bracket. Therefore, great interest lies in the Lie bracket relations of the generators, given by:

$$[T^\alpha, T^\beta] = i f^{\alpha\beta\gamma} T^\gamma, \tag{2.6}$$

where $f^{\alpha\beta\gamma} \in \mathbb{R}$ are the so-called *structure constants* of the Lie algebra. In fact, a Lie algebra is completely characterized by these relations and thus by its structure constants which are totally antisymmetric under permutations as a consequence of (A2) and (A3).
Due to the Hermicity of the generators, i.e. $T^{\alpha\dagger} = T^\alpha$, the Lie brackets

$$[T^\alpha, T^\beta]^\dagger = -[T^\alpha, T^\beta], \tag{2.7}$$

[1] Note that Einstein's summation convention is always used in this thesis and no distinction between covariant and contravariant indices is made.

$$i[T^\alpha, T^\beta]^\dagger = [T^\alpha, T^\beta] \tag{2.8}$$

are anti-Hermitian and Hermitian respectively.
Using the generator notation of two Lie algebra elements $z_1 = \omega_1^\alpha T^\alpha$ and $z_2 = \omega_2^\beta T^\beta$, the Lie bracket reads:

$$\begin{aligned} i[z_1, z_2] &= i\omega_1^\alpha \omega_2^\beta [T^\alpha, T^\beta] \\ &= -\omega_1^\alpha \omega_2^\beta f^{\alpha\beta\gamma} T^\gamma \\ &= \omega^\gamma T^\gamma. \end{aligned} \tag{2.9}$$

Evidently, $i[z_1, z_2]$ is again an element of the Lie algebra.

Another useful concept left to define is a *sub-algebra* $\widetilde{g}$ of a Lie algebra g, where $\widetilde{g}$ is spanned by a subset of generators S^α such that the Lie bracket relations hold:

$$[S^\alpha, S^\beta] = if^{\alpha\beta\gamma} S^\gamma \tag{2.10}$$

or equivalently, $i[S^\alpha, S^\beta]$ is closed in $\widetilde{g}$ and, as a result, $\widetilde{g}$ is a sub vector space of g.

An important type of a sub-algebra is a so-called *ideal*, sometimes called invariant sub-algebra. Generators S^α of an ideal $\widehat{g}$ of g additionally satisfy:

$$[S^\alpha, T^\beta] = if^{\alpha\beta\gamma} S^\gamma \tag{2.11}$$

for all generators T^β of g. As before, this means $i[S^\alpha, T^\beta]$ is again an element of $\widehat{g}$.

A Lie algebra is called *simple* if there exist no non-trivial ideals and *semi-simple* if there are no non-trivial Abelian ideals. As usual, a sub-algebra is called Abelian if all generators commute. Note that a semi-simple Lie algebra may include non-Abelian ideals as well.
The number of generators of the maximal Abelian sub-algebra (which may not be an ideal) of a semi-simple Lie algebra is called the *rank* of the Lie algebra. In practice, the maximal set of commuting generators is searched for in order to determine the rank.

In physics, the focus lies on compact Lie groups which have compact group manifolds corresponding to compact semi-simple Lie algebras. These are among others the groups listed in chapter 2.2.1.

2.3.2 Connection to Lie groups

Until now, the correspondence between a Lie algebra and a certain Lie group was not obvious. Geometric considerations prove to be useful in order to illustrate this connection. By definition, the set of vector fields that are *left-invariant* on group elements of G is the corresponding Lie algebra g. *Left-translation* is defined for an element $a \in G$ on any $x \in G$ such that

$$L_a : G \to G, \quad x \mapsto ax. \tag{2.12}$$

A vector $v(x)$, given by the derivative at $x \in G$, is said to be left-invariant if

$$L_a(v(x)) = v(ax). \tag{2.13}$$

It turns out, that the set of left-invariant vector fields on G does indeed always carry the structure of a Lie algebra. Furthermore, the Lie algebra is isomorphic to the tangent space at the neutral element of G. Left-invariance essentially means, that this tangent space is entirely determined by the vector at the neutral element, which can generate the rest of the vector field by left-translation. In conclusion, the corresponding Lie algebra can be constructed by the Lie group, or more precisely, by the immediate neighbourhood of the neutral element.

Conversely, it is of interest to derive the Lie group from a given Lie algebra. Note that it is possible for globally different but locally identical Lie groups to give rise to the same Lie algebras.[2] Nevertheless, it is possible to define the *exponential map* in order to construct the corresponding Lie group. Whether this map is surjective in every case is beyond the scope of this thesis but for the following intents and purposes and the given examples it is a justifiable assumption.
The exponential map is defined on a Lie algebra g of a Lie group G as

$$\exp : g \to G, \qquad z \mapsto \exp(iz). \tag{2.14}$$

Note that $z = \omega^\alpha T^\alpha \in g$ is usually expressed as a linear combination of generators and the exponential is sometimes written as e^{iz} and explicitly given by:

$$\exp(i\omega^\alpha T^\alpha) = \sum_{k=0}^{\infty} \frac{(i\omega^\alpha T^\alpha)^k}{k!}. \tag{2.15}$$

[2] elaborated in chapter 2.4

2.3.3 Construction of $so(2)$ and $su(2)$

To convince oneself of the connection between Lie groups and Lie algebras it is useful to look at a few examples.
Consider, for instance, a matrix $R \in SO(n, \mathbb{R})$ in the neighborhood of the neutral element

$$R = \mathbb{1} + i\epsilon X. \tag{2.16}$$

Using the properties of $R \in SO(n, \mathbb{R})$

$$\det(R) = 1 \qquad \Longrightarrow \qquad 1 + i\epsilon \operatorname{Tr}(X) + \mathcal{O}(\epsilon^2) = 1, \tag{2.17}$$

$$R^T R = \mathbb{1} \qquad \Longrightarrow \qquad \mathbb{1} + i\epsilon(X + X^T) + \mathcal{O}(\epsilon^2) = \mathbb{1}, \tag{2.18}$$

results in traceless[3], skew symmetric ($X^T = -X$) matrices. Therefore, in the case of $R \in SO(2, \mathbb{R})$ the one-dimensional corresponding Lie algebra $so(2)$ is given by the Hermitian generator

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{2.19}$$

identified as the second Pauli matrix.

According to the theory, using the σ_2 generator expression for an algebra element and applying the exponential map should give rise to the Lie group $SO(2)$. In order to verify this, consider equation (2.15) and the even and odd exponents of σ_2

$$\begin{aligned} \exp(i\theta\sigma_2) &= \sum_{k=0}^{\infty} \frac{(i\theta\sigma_2)^k}{k!} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sum_{k=0}^{\infty} \frac{(-1)^k\, \theta^{2k}}{(2k)!} + i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sum_{k=0}^{\infty} \frac{(-1)^k\, \theta^{2k+1}}{(2k+1)!} \\ &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \end{aligned} \tag{2.20}$$

As expected, the exponential yields the already discussed $SO(2)$ matrix.

[3] This follows from the Taylor expansion of the determinant and the fact that the derivative of the determinant is given by $\frac{d}{d\epsilon} \det(\mathbb{1} + \epsilon X) = \operatorname{Tr}(X)$.

A second, physically most relevant, example is the Lie group $SU(2)$. Constructing the corresponding Lie algebra $su(2)$ allows for a deeper understanding of the symmetry. For the sake of completeness, the group manifold of $SU(2)$ is first identified using

$$SU(2) = \left\{ U = \begin{pmatrix} u & -v^* \\ v & u^* \end{pmatrix} : u, v \in \mathbb{C},\ |u|^2 + |v|^2 = 1 \right\}. \tag{2.21}$$

Considering $u = \Re u + i\,\Im u$ and $v = \Re v + i\,\Im v$ the length constraint gives rise to the real equation

$$(\Re u)^2 + (\Im u)^2 + (\Re v)^2 + (\Im v)^2 = 1. \tag{2.22}$$

Consequentially, group elements of $SU(2)$ are points on the S^3 sphere embedded in $\mathbb{R}^4$.

In order to construct the Lie algebra $su(2)$ a general $SU(2)$ element is expressed in terms of the exponential of a general $su(2)$ element

$$U = \exp(i\omega^\alpha T^\alpha), \tag{2.23}$$

where the T^α are the a priori unknown generators of $su(2)$ and $U \in SU(2)$. Keep Einstein's summation convention in mind. Since $SU(2) = S^3$ is three dimensional, $\alpha \in \{1, 2, 3\}$. Also, recall that $\det(e^A) = e^{\operatorname{Tr} A}$ holds for Hermitian (i.e. unitarily diagonalizable) matrices A.[4]

Imposing the properties of $SU(n)$ yields

$$\begin{aligned} \det(U) = 1 \qquad &\Longrightarrow \qquad \det(\exp(i\omega^\alpha T^\alpha)) = \exp(i\omega^\alpha \operatorname{Tr} T^\alpha) = 1 \\ &\Longrightarrow \qquad \operatorname{Tr} T^\alpha = 0, \end{aligned} \tag{2.24}$$

$$\begin{aligned} U^\dagger U = \mathbb{1} \qquad &\Longrightarrow \qquad \exp(-i\omega^\alpha (T^\alpha)^\dagger) \exp(i\omega^\alpha T^\alpha) = \mathbb{1} \\ &\Longrightarrow \qquad (T^\alpha)^\dagger = T^\alpha. \end{aligned} \tag{2.25}$$

Apparently, $su(n)$ is generated by traceless Hermitian matrices T^α. For $su(2)$, three independent generators can be identified:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.26}$$

[4] We write $A = PDP^\dagger$, where P is unitary and D diagonal. Use the series definition of the exponential, the unitarity of P, the fact that e^D is a matrix with diagonal entries e^{d_i} and the invariance of the trace under cyclic permutations:
$\det(e^A) = \det(P\,e^D P^\dagger) = \det(e^D) = \prod_{i=1}^n e^{d_i} = e^{\operatorname{Tr} D} = e^{\operatorname{Tr}(PDP^\dagger)} = e^{\operatorname{Tr} A}$.

Therefore, the well-known *Pauli matrices* are generating the Lie algebra $su(2)$. This further illustrates the fact that the spacial rotational symmetry given by $SU(2)$ governs the spin property of particles.

2.3.4 Representations

Technically, Lie group elements as well as Lie algebra generators and their linear combinations are abstract mathematical objects. Nevertheless, it is common to assign them to matrices in order to execute calculations as it has already been done throughout chapter 2. The theory of representations provides the mathematical justification for this common practice. The notion of a representation can be defined on a variety of mathematical constructs such as Lie groups and Lie algebras. Here, the explicit definition is only given for Lie algebras. Note, however, that the corresponding definition for Lie groups is closely related. A representation of a Lie algebra g is defined as a Lie algebra homomorphism

$$\rho_{\mathbb{V}} : g \to \mathfrak{gl}(\mathbb{V}), \tag{2.27}$$

where $\mathfrak{gl}(\mathbb{V})$ denotes the set of endomorphisms on a vector space $\mathbb{V}$. $\mathfrak{gl}(\mathbb{V})$ can be extended to a Lie algebra by additionally introducing the notion of the Lie bracket. Note that for an n-dimensional $\mathbb{V}$, $\mathfrak{gl}(\mathbb{V})$ can be identified with the set of $n \times n$ matrices $\mathfrak{gl}_n(\mathbb{V})$ over $\mathbb{V}$ and the Lie bracket is the commutator. Essentially, the representation $\rho_{\mathbb{V}}$ maps a Lie algebra element to a square matrix and preserves the Lie algebra structure. Note that many authors refer to the matrices themselves as representations instead of the map to the matrix algebra $\mathfrak{gl}_n(\mathbb{V})$. In physics, $\mathbb{V}$ is typically a Hilbert space such that the representation maps Lie algebra elements onto operators of that Hilbert space. As a consequence, the Lie algebra gains physical meaning through the notion of representations.

Since there are infinitely many representations for any Lie algebra element it is helpful to classify representations into *reducible* and *irreducible* kinds: A subspace $\mathbb{W} \subset \mathbb{V}$ is called invariant if for the representation $\rho_{\mathbb{V}}$ defined as above $\rho_{\mathbb{V}}(x) * w$ is in $\mathbb{W}$ for all $w \in \mathbb{W}$ and $x \in g$. Accordingly, a representation $\rho_{\mathbb{V}}$ is called irreducible if there are no non-trivial invariant subspaces of $\mathbb{V}$.
Irreducible representations are indecomposable, i.e. they cannot be further decomposed into direct sums of representations.

Note that in the case of Abelian Lie groups, only one-dimensional representations are irreducible. Since $SO(2)$ is Abelian, like every one-dimensional group, the already encountered two-dimensional repesentation (2.1):

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

must be reducible. Indeed there is a unitary transformation

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \tag{2.28}$$

such that:

$$U^\dagger R U = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}. \tag{2.29}$$

As expected, R is reduced into two one-dimensional irreducible representations $e^{i\theta}$ and $e^{-i\theta}$ of $SO(2)$.

Interestingly, U also diagonalizes σ_2 into σ_3

$$U^\dagger \sigma_2 U = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.30}$$

This shows that the reducible representation σ_2 can be decomposed into two one dimensional irreducible representations 1 and -1 of $so(2)$.

A more fundamental example of a representation is the so-called *adjoint representation* of a Lie algebra, which maps the Lie algebra onto the *adjoint action* $\mathrm{ad}_x \in \mathfrak{gl}(g)$:

$$\rho_{\mathrm{ad}} : g \to \mathfrak{gl}(g), \qquad x \mapsto \mathrm{ad}_x; \tag{2.31}$$

$$\mathrm{ad}_x : g \to g, \qquad \mathrm{ad}_x(y) = [x, y], \quad \forall\, y \in g. \tag{2.32}$$

The adjoint representation exists for every semi-simple Lie algebra g of dimension n_G and maps the generators to $n_G \times n_G$ matrices. The elements of these matrices are explicitly given by:

$$T^\alpha_{\beta\gamma} = -i f^{\alpha\beta\gamma}, \tag{2.33}$$

where the indices β and γ correspond to rows and columns of the matrix and α enumerates the generators of g. This is consistent since

$$\begin{aligned}[T^\alpha, T^\beta]_{\gamma\delta} &= T^\alpha_{\gamma\epsilon} T^\beta_{\epsilon\delta} - T^\beta_{\gamma\epsilon} T^\alpha_{\epsilon\delta} \\ &= -f^{\alpha\gamma\epsilon} f^{\beta\epsilon\delta} + f^{\beta\gamma\epsilon} f^{\alpha\epsilon\delta},\end{aligned} \tag{2.34}$$

together with the commutation relations (2.6) in components

$$\begin{aligned}[T^\alpha, T^\beta]_{\gamma\delta} &= i f^{\alpha\beta\mu} T^\mu_{\gamma\delta} \\ &= f^{\alpha\beta\mu} f^{\mu\gamma\delta}\end{aligned} \tag{2.35}$$

reproduces the Jacobi identity (A3)

$$-f^{\alpha\gamma\epsilon} f^{\beta\epsilon\delta} + f^{\beta\gamma\epsilon} f^{\alpha\epsilon\delta} = f^{\alpha\beta\mu} f^{\mu\gamma\delta}. \tag{2.36}$$

Furthermore, the conjugate $\widetilde{T}^\alpha$ of a matrix representation T^α is defined, such that

$$\begin{aligned} i\widetilde{T}^\alpha &= (iT^\alpha)^* \\ &= -i(T^\alpha)^T, \\ \Longrightarrow \widetilde{T}^\alpha &= -(T^\alpha)^T. \end{aligned} \tag{2.37}$$

Coherently, the commutation relations (2.6) are preserved

$$\begin{aligned} [\widetilde{T}^\alpha, \widetilde{T}^\beta] &= [(T^\alpha)^T, (T^\beta)^T] &&= (T^\alpha)^T (T^\beta)^T - (T^\beta)^T (T^\alpha)^T \\ &= (T^\beta T^\alpha - T^\alpha T^\beta)^T &&= -[T^\alpha, T^\beta]^T \\ &= -i f^{\alpha\beta\gamma} (T^\gamma)^T &&= i f^{\alpha\beta\gamma} \widetilde{T}^\gamma. \end{aligned} \tag{2.38}$$

As a consequence, representations can be further distinguished. A representation is called *real* if $\widetilde{T}^\alpha = T^\alpha$ and *pseudo-real* if $T^\alpha = U\widetilde{T}^\alpha U^\dagger$ for a single unitary transformation U and for all α. The adjoint representation is an example of a real representation.[5]
Representations for which neither of the above conditions hold are called *complex*.

[5] Use the total antisymmetry of the structure constants:
$\widetilde{T}^\alpha_{\beta\gamma} = -(T^\alpha_{\beta\gamma})^T = -T^\alpha_{\gamma\beta} = i f^{\alpha\gamma\beta} = -i f^{\alpha\beta\gamma} = T^\alpha_{\beta\gamma}$.

2.3.5 Cartan-Weyl basis and weight diagrams

As in any vector space, the basis of a Lie algebra is not unique. It is customary to look for a convenient basis in order to construct the Lie algebra in a transparent manner. A useful choice of basis is the so-called *Cartan-Weyl basis*, which allows for the description of semi-simple Lie algebras in terms of *weight* and *ladder operators*.

Let the Lie algebra g be given by some arbitrary basis of generators $\{T^1, T^2, \dots, T^{n_G}\}$. The maximal Abelian sub-algebra $\mathcal{H}$ of mutually commuting generators, renamed H^i, serves as a starting point. In a semi-simple Lie algebra of rank r and dimension n_G exist, by definition, exactly r such generators ($i \in \{1, \dots, r\}$). $\mathcal{H}$ is called the *Cartan sub-algebra* and the generators H^i are the so-called weight operators.

The remaining $n_G - r$ (not commuting) generators T^β are linearly combined to form the missing generators of the basis E^α, with $\alpha = (\alpha^1, \dots, \alpha^r)$, in such a way that the eigenvalue equation

$$[H^i, E^\alpha] = \alpha^i \, E^\alpha \tag{2.39}$$

holds for all $i \in \{1, 2, ..., r\}$. The eigenvalues α^i are called *roots* and the eigenvectors E^α are known as *root vectors*. In terms of the basis, the generators E^α are the ladder operators and the complete Cartan-Weyl basis is given by $\mathcal{H} \cup \{E^\alpha\}$.
Since the weight operators commute, there exists a simultaneous system of eigenvectors. It is convenient to order the corresponding eigenvalues as points (called *weights*) in Euclidian space $\mathbb{R}^r$ for a Lie algebra of rank r. The commutation relations of the weight and ladder operators restrict the action of the ladder operators on eigenstates of the weight operators. It turns out that the ladder operators translate the weights in Euclidean space, generating the so-called *weight diagram*. Since multiple states can correspond to the same weight, i.e. multiple states can have the same eigenvalues, each weight is given the value of the corresponding degree of degeneracy, called *multiplicity*. In this context, the weight diagram of a given irreducible representation or the representation itself is often called a *multiplet*.

As an explicit example, the generators of $su(2)$ can be transformed into weight and ladder operators. Consider the usual spin basis $\{S^1, S^2, S^3\}$ of $su(2)$, where $S^\alpha = \frac{1}{2}\sigma^\alpha$ with $\hbar = 1$. Since the Pauli matrices do not commute, choose S^3 as the Cartan sub-algebra $\mathcal{H}$, i.e. as the weight operator.

In order to solve the eigenvalue equation (2.39), the appropriate linear combinations of S^1 and S^2 are found using the commutation relations $[S^\alpha, S^\beta] = i\epsilon^{\alpha\beta\gamma}S^\gamma$. The ladder operators are thus given by $S^\pm = S^1 \pm iS^2$ and the Cartan-Weyl basis of $su(2)$ is $\{S^3, S^+, S^-\}$. The Lie algebra $su(2)$ is completely characterized by the Cartan-Weyl basis and the new commutation relations

$$[S^3, S^\pm] = \pm S^\pm \qquad \text{and} \qquad [S^+, S^-] = 2S^3. \tag{2.40}$$

Since $su(2)$ only has one weight operator, i.e. it is a rank 1 algebra, the weight diagram consists of points on a straight line. Consider the eigenvalue equation of S^3 and the total spin $\vec{S}^2$ with eigenstates $|s, m\rangle$

$$S^3|s, m\rangle = m|s, m\rangle, \tag{2.41}$$

$$\vec{S}^2|s, m\rangle = s(s+1)|s, m\rangle, \tag{2.42}$$

and the a priori unknown (but properly normalized) actions of the ladder operators

$$S^+|s, m\rangle = \sqrt{s(s+1) - m(m+1)}|s, m_+\rangle, \tag{2.43}$$

$$S^-|s, m\rangle = \sqrt{s(s+1) - m(m-1)}|s, m_-\rangle. \tag{2.44}$$

Using the commutation relations (2.40), the states $|s, m_\pm\rangle$ are found to be eigenvectors of S^3

$$S^3|s, m_\pm\rangle = m_\pm|s, m_\pm\rangle = (m \pm 1)|s, m_\pm\rangle, \tag{2.45}$$

where $|s, m_\pm\rangle$ is identified with the state $|s, m \pm 1\rangle$.

Graphically, the weights of a general representation of dimension $2s + 1$ lie on a line and the ladder operators $S^\pm$ transform the points into the neighbouring states as depicted in *Fig. 2.1*.

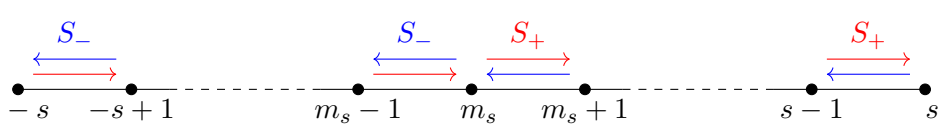

Fig. 2.1: The weight diagram of an $su(2)$ multiplet of dimension $\{2s + 1\}$ and the action of the ladder operators $S^\pm$. By construction, the weight operator S^3 yields the weight m of each state $|s, m\rangle$.

2.4 Universal covering group

Lie algebras are useful to understand the local structure of a Lie group, which can be the same for globally different manifolds. As a consequence, two Lie algebras can be identical while their Lie groups are not. The Lie group of a given Lie algebra containing the largest possible center is called the *universal covering group*. By successive division of subgroups of the center, the universal cover can be reduced up to a group of trivial center. The covering groups of $SO(n)$ are called $Spin(n)$.
The center symmetry is vital in order to compare Lie groups. Consider for example the Lie groups $SU(2)$ which has the center $\mathbb{Z}_2$ and $SO(3)$ which has a trivial center. They are locally identical and therefore the Lie algebras $su(2)$ and $so(3)$ are isomorphic whereas $SU(2)$ and $SO(3)$ are not. There is, however, a close relation between the adjoint representation O of $SO(3)$ and the fundamental representation U of $SU(2)$

$$O^{\alpha\beta} = \frac{1}{2}\,\mathrm{Tr}(U\sigma^\alpha U^\dagger \sigma^\beta). \tag{2.46}$$

Clearly, the unit element $U = \mathbb{1}$ is mapped to

$$O^{\alpha\beta} = \frac{1}{2}\,\mathrm{Tr}(\sigma^\alpha \sigma^\beta) = \delta^{\alpha\beta}, \tag{2.47}$$

which is the unit element $O = \mathbb{1}$ of $SO(3)$. Similarly, the inverse $U^\dagger$ of U is mapped to the inverse O^T of O

$$\frac{1}{2}\,\mathrm{Tr}(U^\dagger \sigma^\alpha U \sigma^\beta) = \frac{1}{2}\,\mathrm{Tr}(U\sigma^\beta U^\dagger \sigma^\alpha) = O^{\beta\alpha}. \tag{2.48}$$

Notice that $-U$ and U are both mapped to the same O in $SO(3)$. As a result, antipodal points on $SU(2) = S^3$ are mapped onto the same point on the group manifold of $SO(3)$. $SO(3)$ is indifferent to the center $\mathbb{Z}_2$ of $SU(2)$. Therefore, the $SO(3)$ group manifold is $S^3/\mathbb{Z}_2$ and the universal covering group of $SO(3) = SU(2)/\mathbb{Z}_2$ is $Spin(3) = SU(2)$.

3 The Lie group $SU(3)$

The Lie group $SU(3)$ describes important physical symmetries. It is the local color gauge symmetry of the strong interaction, making it an essential component of the standard model of particle physics. In addition, $SU(3)$ appears as an approximate global quark flavor symmetry, which extends the $SU(2)$ isospin symmetry of up and down quarks to up, down, and strange quarks. As defined in subsection 2.2.1, the group consists of 3×3 unitary matrices U with $\det U = 1$ and is 8-dimensional. The group manifold of $SU(3)$ is $S^3 \times S^5$ and, in fact, the whole sequence is given by

$$SU(n) = S^3 \times S^5 \times ... \times S^{2n-1}. \tag{3.1}$$

The center of $SU(n)$ is $\mathbb{Z}_n$ and gives rise to the notion of *triality* in the case of $SU(3)$ with center $\mathbb{Z}_3$. This means that there are three types of irreducible representations of $su(3)$, each associated with a center element of $SU(3)$.

3.1 The Lie algebra $su(3)$

The corresponding Lie algebra $su(3)$ is generated by eight Hermitian (2.25) and traceless (2.24) matrices T^α which are related to $U \in SU(3)$ by

$$U = e^{i\omega^\alpha T^\alpha}, \tag{3.2}$$

where ω^α are again real coefficients. The generators $T^\alpha = \frac{1}{2}\lambda^\alpha$ are explicitly given by the so-called Gell-Mann matrices:

$$\begin{aligned}
\lambda^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{3.3}$$

The generators T^α are normalized such that $\mathrm{Tr}(T^\alpha T^\beta) = \frac{1}{2}\delta^{\alpha\beta}$. The commutation relations are given by (2.6) with non-zero values of the totally

antisymmetric structure constants

$$f^{123} = 1, \tag{3.4}$$

$$f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2}, \tag{3.5}$$

$$f^{458} = f^{678} = \frac{\sqrt{3}}{2}. \tag{3.6}$$

It is easily verified that T^3 and T^8 are the only commuting generators and that $su(3)$ is hence of rank 2. Indeed the Cartan sub-algebra is constructed as $\mathcal{H} = \{T^3, T^8\}$, where T^3 and T^8 are the weight operators. The remaining generators are combined according to (2.39)

$$T^{\pm} = T^1 \pm iT^2, \qquad V^{\pm} = T^4 \pm iT^5, \qquad U^{\pm} = T^6 \pm iT^7. \tag{3.7}$$

In order to fully characterize $su(3)$ in terms of the Cartan-Weyl basis, the commutation relations have to be considered

$$\begin{aligned}
&[T^3, T^{\pm}] = \pm T^{\pm}, && [T^3, V^{\pm}] = \pm\frac{1}{2}V^{\pm}, && [T^3, U^{\pm}] = \mp\frac{1}{2}U^{\pm}, \\
&[T^8, T^{\pm}] = 0, && [T^8, V^{\pm}] = \pm\frac{\sqrt{3}}{2}\,V^{\pm}, && [T^8, U^{\pm}] = \pm\frac{\sqrt{3}}{2}\,U^{\pm}, \\
&[T^+, T^-] = 2T^3, && [V^+, V^-] = \sqrt{3}\,T^8 + T^3, && [U^+, U^-] = \sqrt{3}\,T^8 - T^3, \\
&[T^+, V^-] = -U^-, && [T^+, U^+] = V^+, && [U^+, V^-] = T^-, \\
&[T^+, V^+] = 0, && [T^+, U^-] = 0, && [U^+, V^+] = 0,
\end{aligned} \tag{3.8}$$

and as already discussed: $[T^3, T^8] = 0$.

Note that $su(3)$ has both real and complex representations. The weight diagrams are constructed according to subsection 2.3.5. The weights are now realized as points in a plane, since $su(3)$ is of rank 2. The actions of the ladder operators on the eigenvalues t^3 and t^8 of T^3 and T^8 respectively, are specified by the commutation relations (3.8) and are visualized in *Fig. 3.1*, together with the resulting three axes of symmetry of $su(3)$ multiplets.

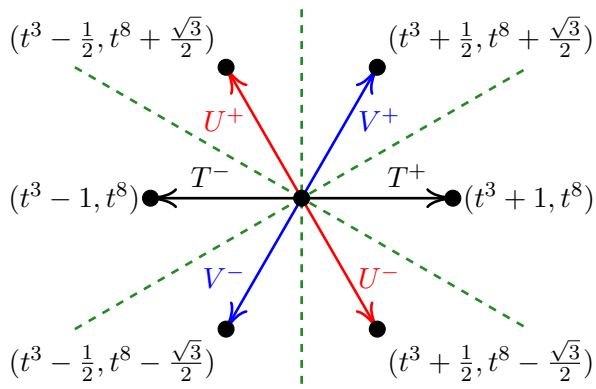

Fig. 3.1: The actions of the ladder operators $T^{\pm}, V^{\pm}$ and $U^{\pm}$ on a weight with the eigenvalues (t^3, t^8) of the weight operators T^3 and T^8. The dashed green lines represent the reflection symmetries of $su(3)$ weight diagrams.

Every multiplet of $su(3)$ can be generated by acting with the ladder operators on a specific initial state. Note that this state does not have to be in the center of the multiplet and that not all ladder operators generate normalizable states. As a result of the symmetries, triangles and hexagons are the only possible shapes of irreducible $su(3)$ multiplets. The weight diagrams are completely characterized by the length p of the top side, along the axis defined by $T^{\pm}$, and the length q of the adjacent sides, along the axes defined by $U^{\pm}$ and $V^{\pm}$ respectively. Therefore, the representations are denoted as (p, q). Alternatively, an irreducible representation can be denoted by its dimension as $(p, q) = \{D(p, q)\}$. The dimension of an irreducible representation (p, q) of $su(3)$ is given by the formula

$$D(p, q) = \frac{1}{2}(p + 1)(q + 1)(p + q + 2). \tag{3.9}$$

However, the dimension $D(p, q)$ does not uniquely determine a representation (p, q), since equation (3.9) is invariant under exchange of p and q. In fact, the conjugate representation of $(p, q) = \{D(p, q)\}$ is given by $(q, p) = \{\overline{D(p, q)}\}$ and representations of the form (p, p) are real.
Furthermore, there are other ambiguous cases like for instance $D(2, 1) = D(4, 0) = 15$. These are distinguished by adding primes in the form of $\{D(2, 1)\} = \{15\}$ and $\{D(4, 0)\} = \{15'\}$. As found in [3], for representations of dimensions lower than 10^8, these degeneracies range up to a factor of $d = 22$ (denoted by up to 21 primes).[6] .

[6] including a factor 2 for the conjugate representations.

The aforementioned notion of triality corresponding to the center $\mathbb{Z}_3$ manifests itself in the weight diagrams. Three different types of multiplets are identified and depicted in *Figs 3.2, 3.3*, and *3.4*, in close analogy to the illustrations in [3]. Note that the corresponding multiplicity is denoted with an integer value above each weight and that since the dimension D of a representation $\{D\}$ is given by the number of states in the weight diagram, the sum over all multiplicities reproduces the dimension.

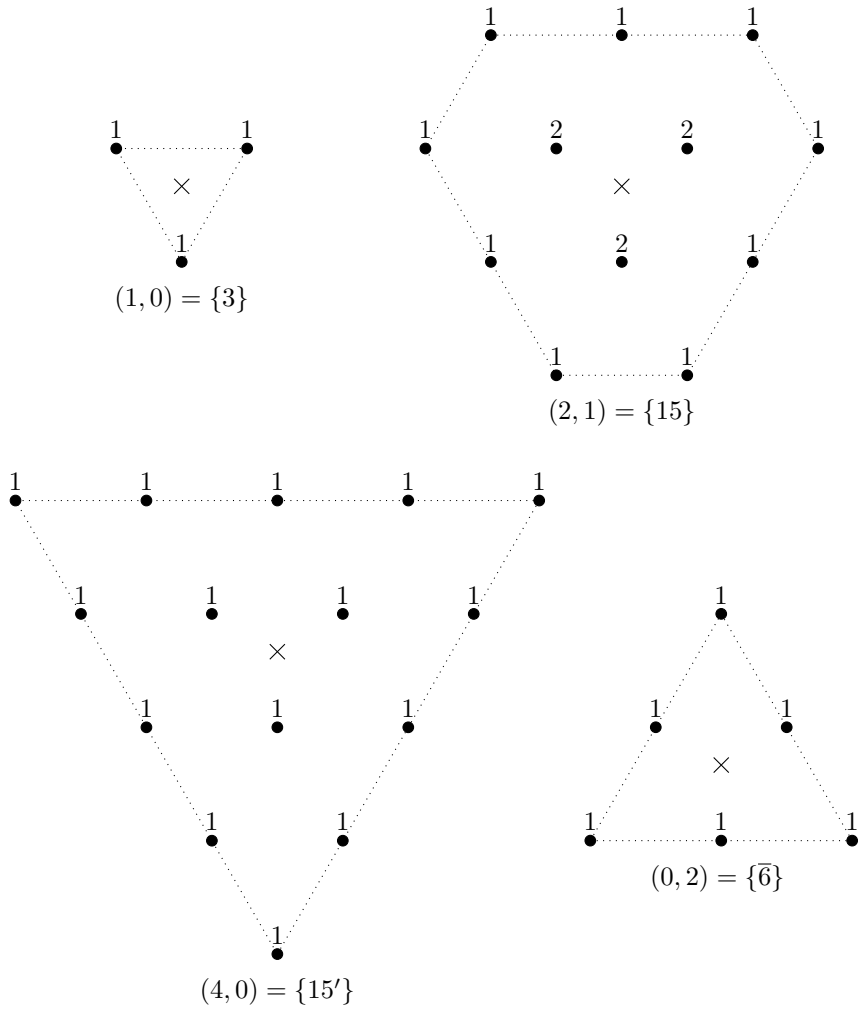

Fig. 3.2: The weight diagrams of a selection of complex $su(3)$ representations (p,q) alternatively denoted as $\{D(p,q)\}$ with the same non-trivial triality. The axes spanning the plane are given by the weight operators T^3 and T^8 of $su(3)$. The superscripts denote the multiplicities of the states of a multiplet.

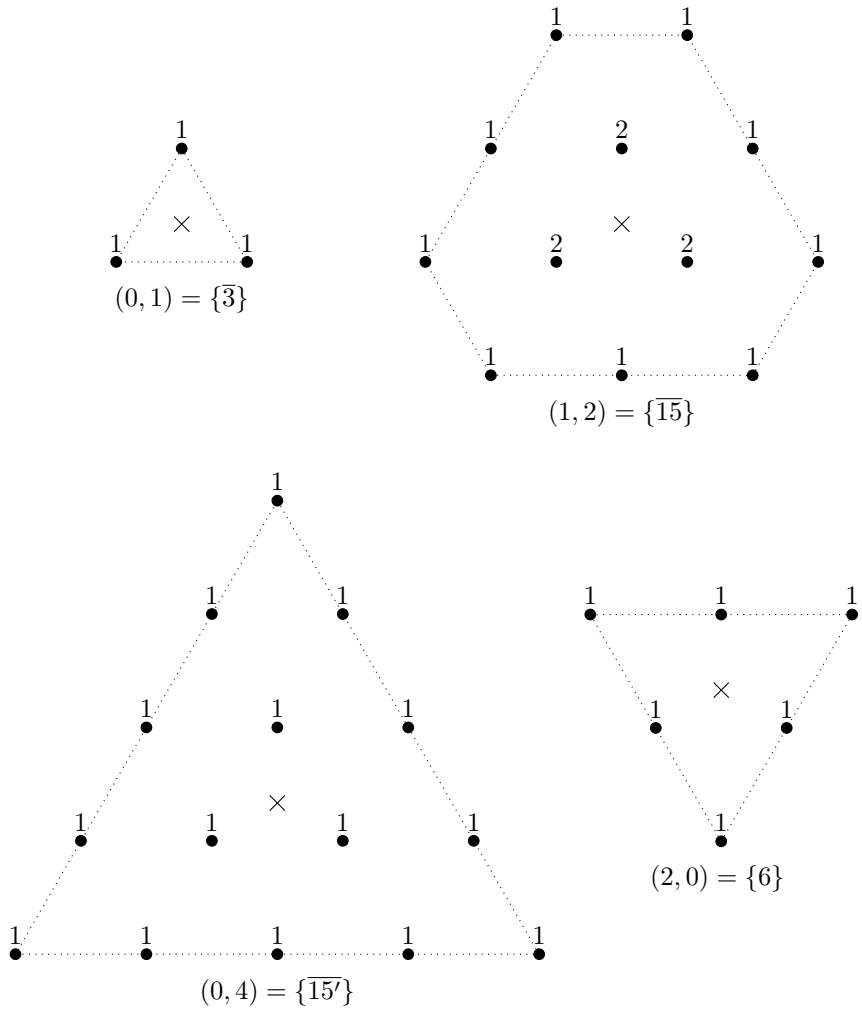

Fig. 3.3: The weight diagrams of the complex $su(3)$ representations conjugate to those in Fig. 3.2. These multiplets are of opposite non-trivial triality to their conjugate representations.

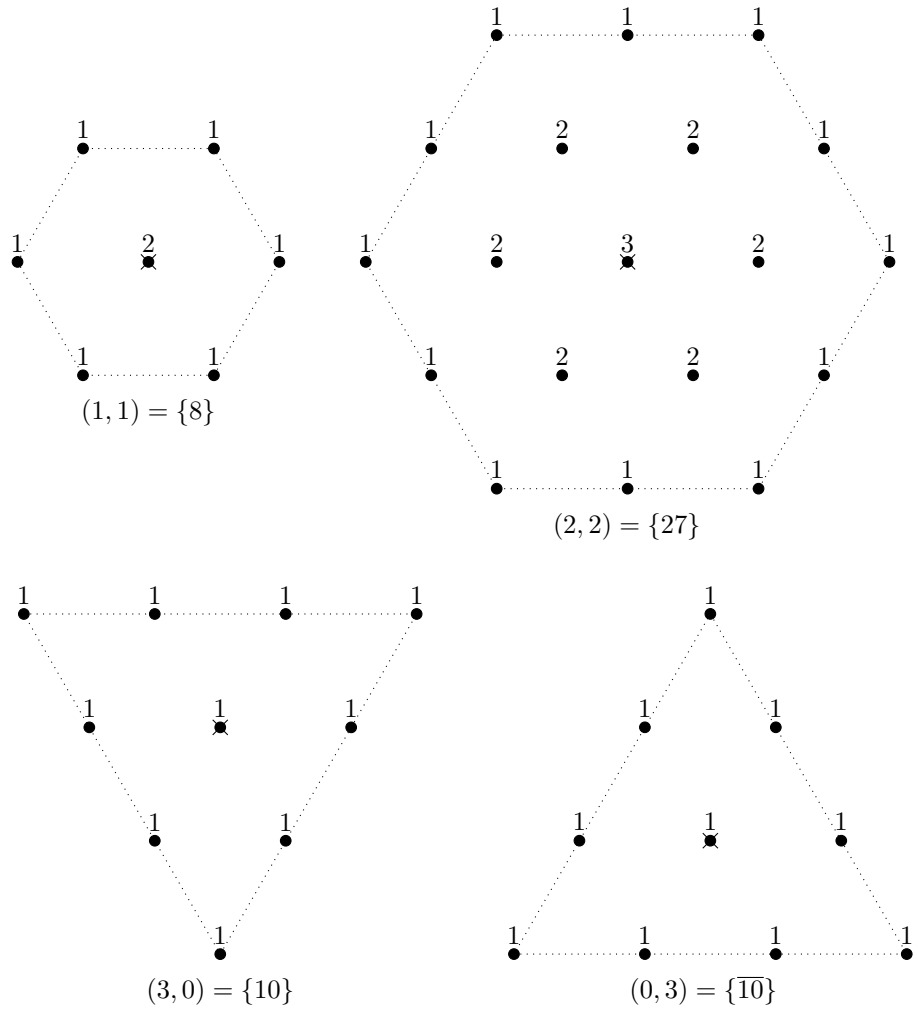

Fig. 3.4: The weight diagrams of several $su(3)$ representations of trivial triality, which all contain states occupying the origin. The adjoint representation $\{8\}$ as well as $\{27\}$ are real representations of $su(3)$, while $\{10\}$ and its conjugate $\{\overline{10}\}$ are obviously complex but still belong to the same triality.

It is apparent, that the multiplicities of the weights of $su(3)$ multiplets are structured in a shell-like manner. More precisely, they increase by one with each layer closer to the center until a triangular shape is reached upon which the multiplicities remain constant. Note that this is by no means a general behavior of multiplicities but rather a unique property of $su(3)$ weight diagrams.

4 The Lie group $G(2)$

The Lie group $G(2)$ is the simplest of the so-called exceptional groups. It is of interest in theoretical physics mostly because it has $SU(3)$ as a subgroup. Therefore, one can use $G(2)$ to describe the same physics as $SU(3)$ and examine the various additional implications of such a symmetry, as is discussed, for example, in [4] and [8]. The following approach to $G(2)$ and its algebra $g(2)$ is in close reference to [4].
$G(2)$ is commonly defined as a subgroup of the 21 dimensional Lie group $SO(7)$, the real 7×7 orthogonal matrices with determinant 1. In components, the orthogonality restriction is given by the Kronecker delta

$$O^{\alpha\beta} O^{\alpha\gamma} = \delta^{\beta\gamma}. \tag{4.1}$$

Elements of the subgroup $G(2)$ additionally satisfy the cubic constraint

$$F^{\alpha\beta\gamma} = F^{\delta\epsilon\mu} O^{\delta\alpha} O^{\epsilon\beta} O^{\mu\gamma}, \tag{4.2}$$

where F is a totally antisymmetric tensor. The non-zero elements follow from

$$F^{127} = F^{154} = F^{163} = F^{235} = F^{264} = F^{374} = F^{576} = 1 \tag{4.3}$$

by using the antisymmetric property.
As a result of equation (4.3) the constraint (4.2) represents 7 non-trivial restrictions to elements of $G(2)$. As a consequence, the 21 degrees of freedom of the Lie group $SO(7)$ are reduced to 14 independent parameters of the subgroup $G(2)$. Additionally, the reality of $SO(7)$ extends to $G(2)$, which implies that all representations of $G(2)$ are real.

The group manifold of $G(2)$ is given by the product of spheres

$$G(2) = S^3 \times S^5 \times S^6, \tag{4.4}$$

where $S^3 \times S^5$ is identified with $SU(3)$ according to (3.1)

$$G(2) = SU(3) \times S^6. \tag{4.5}$$

Using equation (4.4), $G(2)$ is related to $SO(7)$ by

$$\begin{aligned} SO(7) &= S^1 \times S^2 \times S^3 \times S^4 \times S^5 \times S^6 \\ &= G(2) \times S^1 \times S^2 \times S^4 \end{aligned}$$

$$
\begin{aligned}
&= G(2) \times S^3 \times S^4 \\
&= G(2) \times S^7. \qquad (4.6)
\end{aligned}
$$

The center of $G(2)$ is trivial, which indicates that the $SU(3)$ concept of triality does not extend to $G(2)$. Therefore, one would intuitively expect to find some enveloping universal covering group of $G(2)$ with non-trivial center. Due to the trivial center, $G(2)$ is its own universal cover. This is an interesting special feature of $G(2)$ with implications, for instance, for gauge theory applications (see for example [4] and [8]).

4.1 The Lie algebra $g(2)$

The corresponding 14-dimensional Lie algebra $g(2)$ is generated by 14 Hermitian generators. A convenient choice for the first eight generators in the 7-dimensional representation is

$$
\Lambda^\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} \lambda^\alpha & 0 & 0 \\ 0 & -\lambda^{\alpha^*} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (4.7)
$$

where λ^α are the 3×3 Gell-Mann matrices used in $su(3)$. The definition ensures the extension of the normalization $\mathrm{Tr}(\Lambda^\alpha \Lambda^\beta) = \mathrm{Tr}(\lambda^\alpha \lambda^\beta) = 2\delta^{\alpha\beta}$. Note that all representations of $g(2)$ are real and that, even though (4.7) involves complex numbers, every Λ^α is unitarily equivalent to an entirely real generator.

In analogy to the $su(3)$ case, the operators Λ^3 and Λ^8 commute and are therefore used as weight operators. Up to a scaling factor of $\frac{1}{\sqrt{2}}$ the remaining generators are recombined to the ladder operators $T^\pm$, $V^\pm$, and $U^\pm$ as before

$$
\begin{aligned}
T^+ &= \frac{1}{\sqrt{2}}(\Lambda^1 + i\Lambda^2) = |1\rangle\langle 2| - |5\rangle\langle 4|, \\
T^- &= \frac{1}{\sqrt{2}}(\Lambda^1 - i\Lambda^2) = |2\rangle\langle 1| - |4\rangle\langle 5|, \\
V^+ &= \frac{1}{\sqrt{2}}(\Lambda^4 + i\Lambda^5) = |1\rangle\langle 3| - |6\rangle\langle 4|, \\
V^- &= \frac{1}{\sqrt{2}}(\Lambda^4 - i\Lambda^5) = |3\rangle\langle 1| - |4\rangle\langle 6|, \\
U^+ &= \frac{1}{\sqrt{2}}(\Lambda^6 + i\Lambda^7) = |2\rangle\langle 3| - |6\rangle\langle 5|,
\end{aligned}
$$

$$U^- = \frac{1}{\sqrt{2}}(\Lambda^6 - i\Lambda^7) = |3\rangle\langle 2| - |5\rangle\langle 6|. \tag{4.8}$$

The ladder operators act on the states $|1\rangle, |2\rangle, \dots, |7\rangle$ of the fundamental representation $\{7\}$ of $g(2)$ according to *Fig. 4.1*, which is a particularly compact notation for the same concept as in *Fig. 3.1*.

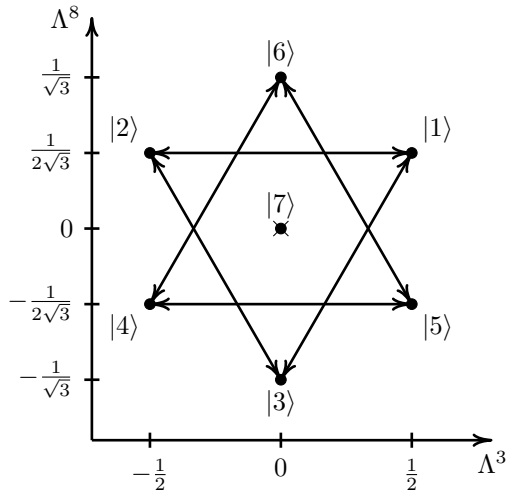

Fig. 4.1: The weight diagram of the fundamental representation $(0,1) = \{7\}$ *of* $g(2)$ *without multiplicities. The arrows illustrate the actions of the ladder operators* $T^\pm$, $V^\pm$, *and* $U^\pm$, *as specified in the relations* (4.8) *on the eigenstates of* Λ^3 *and* Λ^8 *abstractly denoted by* $|1\rangle, |2\rangle, \dots, |7\rangle$.

The remaining 6 generators define shifts as well, which do not commute with the weight operators. Therefore, $g(2)$ is indeed a rank 2 Lie algebra. The actions of the additional ladder operators on the states of $\{7\}$ are given by

$$\begin{aligned}
X^+ &= \frac{1}{\sqrt{2}}(\Lambda^9 + i\Lambda^{10}) = |2\rangle\langle 4| - |1\rangle\langle 5| - \sqrt{2}|7\rangle\langle 3| - \sqrt{2}|6\rangle\langle 7|,\\
X^- &= \frac{1}{\sqrt{2}}(\Lambda^9 - i\Lambda^{10}) = |4\rangle\langle 2| - |5\rangle\langle 1| - \sqrt{2}|3\rangle\langle 7| - \sqrt{2}|7\rangle\langle 6|,\\
Y^+ &= \frac{1}{\sqrt{2}}(\Lambda^{11} + i\Lambda^{12}) = |6\rangle\langle 1| - |4\rangle\langle 3| - \sqrt{2}|2\rangle\langle 7| - \sqrt{2}|7\rangle\langle 5|,\\
Y^- &= \frac{1}{\sqrt{2}}(\Lambda^{11} - i\Lambda^{12}) = |1\rangle\langle 6| - |3\rangle\langle 4| - \sqrt{2}|3\rangle\langle 7| - \sqrt{2}|7\rangle\langle 6|,\\
Z^+ &= \frac{1}{\sqrt{2}}(\Lambda^{13} + i\Lambda^{14}) = |3\rangle\langle 5| - |2\rangle\langle 6| - \sqrt{2}|7\rangle\langle 1| - \sqrt{2}|4\rangle\langle 7|,
\end{aligned}$$

$$Z^- = \frac{1}{\sqrt{2}}(\Lambda^{13} - i\Lambda^{14}) = |5\rangle\langle 3| - |6\rangle\langle 2| - \sqrt{2}|1\rangle\langle 7| - \sqrt{2}|7\rangle\langle 4|. \qquad (4.9)$$

These relations recursively extend the eight Gell-Mann matrices to a complete set of 14 generators Λ^α ($\alpha \in \{1, \dots, 14\}$) of $g(2)$.
The actions of the new ladder operators $X^\pm, Y^\pm$, and $Z^\pm$ are again illustrated on the fundamental representation $\{7\}$ of $g(2)$ in *Fig. 4.2*:

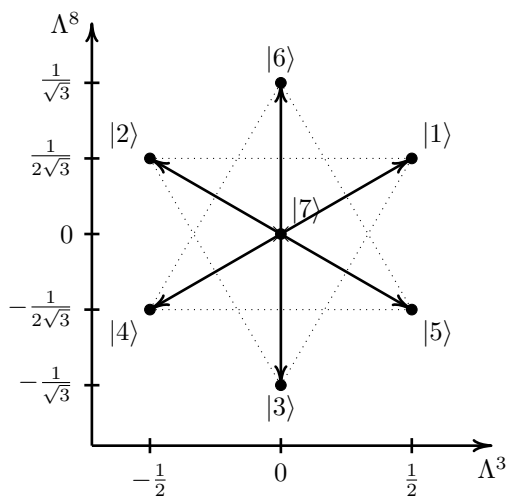

Fig. 4.2: The weight diagram of the fundamental representation $(0,1) = \{7\}$ *of* $g(2)$ *without multiplicities. The arrows illustrate the actions of the ladder operators* $X^\pm$, $Y^\pm$, *and* $Z^\pm$, *as specified in the relations* (4.9) *on the eigenstates of* Λ^3 *and* Λ^8 *abstractly denoted by* $|1\rangle, |2\rangle, \dots, |7\rangle$. *The dotted lines indicate the decomposition of* $\{7\}$ *under* $su(3)$ *sub-algebra transformations (Fig. 4.4).*

Finally, the action of all ladder operators are compactly illustrated on the adjoint representation $\{14\}$ of $g(2)$ in *Fig. 4.3*:

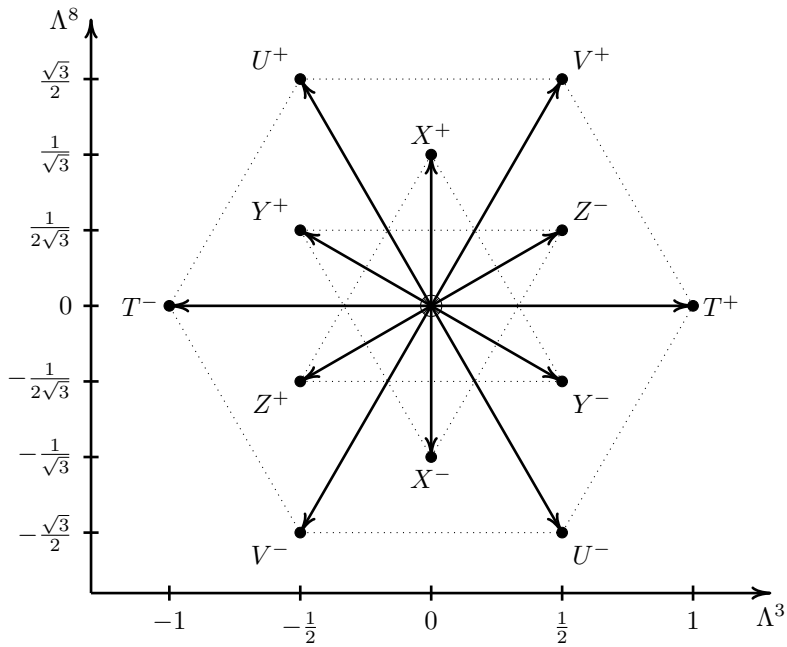

Fig. 4.3: The weight diagram of the adjoint representation $\{14\}$ *of* $g(2)$ *without weight multiplicities. The action of all ladder operators on the central state, are highlighted by arrows connecting the weights. The dotted lines indicate the decomposition under* $su(3)$ *sub-algebra transformations (Fig. 4.5).*

From definition (4.7) it is manifest that the fundamental representation $\{7\}$ of $g(2)$ decomposes under $su(3)$ sub-algebra transformations into the triplet, anti-triplet, and singlet representation of $su(3)$:

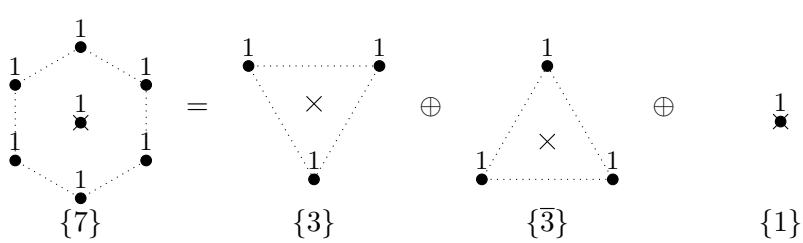

Fig. 4.4: The fundamental representation $\{7\}$ *of* $g(2)$ *decomposes into the direct sum of the triplet* $\{3\}$*, the anti-triplet* $\{\overline{3}\}$*, and the singlet representation* $\{1\}$ *of the sub-algebra* $su(3)$*.*

Similarly, the adjoint representation $\{14\}$ decomposes into the $su(3)$ octet, triplet, and anti-triplet:

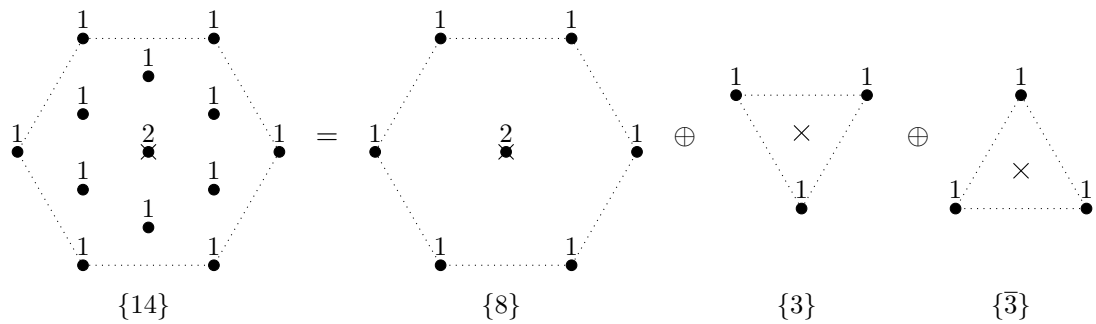

Fig. 4.5: The adjoint representation $\{14\}$ *of* $g(2)$ *decomposes into a direct sum of the octet* $\{8\}$*, the triplet* $\{3\}$*, and the anti-triplet* $\{\overline{3}\}$ *representation of the subalgebra* $su(3)$*.*

Note that in both cases the $g(2)$ multiplets decompose into $su(3)$ representations of different triality, which is consistent with the fact that $G(2)$ has a trivial center.

Just as in the case of $su(3)$, the multiplets of $g(2)$ (and any other rank 2 Lie algebra) are completely characterized by their side lengths (p, q), where p is the length of the top side, along the axis defined by $T^{\pm}$, and q is the length of the adjacent sides, along the axes defined by $Y^{\pm}$ and $Z^{\pm}$ respectively. The symmetry properties of $su(3)$ multiplets are extended to $g(2)$ and due to the six additional generators there are three additional symmetry axes of $g(2)$ multiplets. Every $g(2)$ weight diagram is symmetric about all the axes defined by the shift operators depicted in *Fig. 4.3*. As a consequence, the weight diagrams of general irreducible representations of $g(2)$ are dodecagon-shaped and the sector enclosed by, for instance, T^{+} and Z^{-} from the origin ($\theta \in \{0, \frac{\pi}{6}\}$) contains all information. The other weights including their multiplicities are then generated by reflections.

The dimension D of a $g(2)$ representation is given by

$$D(p, q) = \frac{1}{120}(p+1)(q+1)(p+q+2)(2p+q+3)(3p+q+4)(3p+2q+5). \tag{4.10}$$

Note that a representation (p, q) is again not uniquely described by the dimension $\{D(p, q)\}$ as illustrated by, for instance, $D(0, 3) = D(2, 0) = 77$. Such ambiguities are distinguished by adding primes as before ($(0, 3) = \{77\}$,

$(2,0) = \{77'\}$) but, in contrast to $su(3)$, there are no degeneracies of a factor larger than $d = 2$ to be found for dimensions $D(p,q)$ up to 10^7, as presented in [3]. Note that the degeneracies are already halved compared to $su(3)$ due to the reality of $g(2)$ representations.

Apart from the already shown fundamental $(0,1) = \{7\}$ and adjoint representation $(1,0) = \{14\}$, some more weight diagrams of $g(2)$ representations are depicted in *Fig. 4.6*:

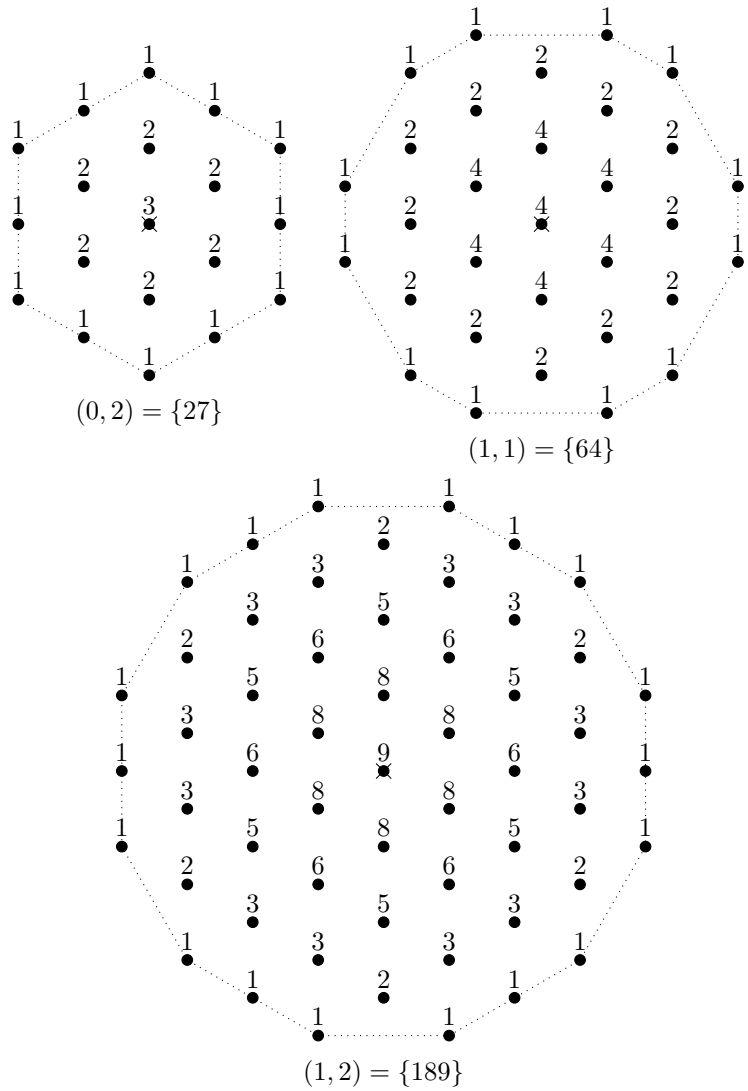

Fig. 4.6: The weight diagrams of several irreducible $g(2)$ representations (p, q), alternatively denoted $\{D(p, q)\}$ as illustrated in [3]. The plane is spanned by the commuting weight operators T^3 and T^8 of the sub-algebra $su(3)$. The superscripts denote the degeneracies of each state of the multiplet.

It is apparent, that $g(2)$ weight diagrams are more densely packed and feature larger multiplicities than the $su(3)$ sub-algebra diagrams. Furthermore, there is no obvious structure of the multiplicities apart from the universal symmetries of $g(2)$ multiplets. In chapter 5 a general recursive algorithm is

presented in order to obtain the correct values of the weight multiplicities of both $su(3)$ and $g(2)$ multiplets.

5 Graphical tensor product reduction

Up until now, different multiplets have been treated individually. In practice though, interest lies in coupling together multiple irreducible representations of Lie algebras. In *Fig. 4.4* and *4.5*, for instance, the fact that a direct sum of representations again yields a representation has been used. This also holds true for the notion of the *tensor product* $\otimes$. Note that the tensor product as well as the direct sum of irreducible representations of simple Lie algebras commute. Since the resulting representation of a tensor product of irreducible representations is not necessarily irreducible, it is convenient to reduce the tensor product into a direct sum of irreducible representations. The so-called Peter-Weyl theorem (1927) proves that such a decomposition is always possible in the case of compact simple Lie algebras. As an example, the triplet and anti-triplet representations of $su(3)$ can be coupled, such that

$$\{3\} \otimes \{\overline{3}\} = \{1\} \oplus \{8\}. \tag{5.1}$$

The reduction of tensor products into direct sums of irreducible representations is a difficult algebraic problem, which is tedious to solve. Weight diagrams, on the other hand, offer a graphical way of tensor product reduction by superimposing one diagram onto every point of the other as illustrated in *Fig. 5.1*:

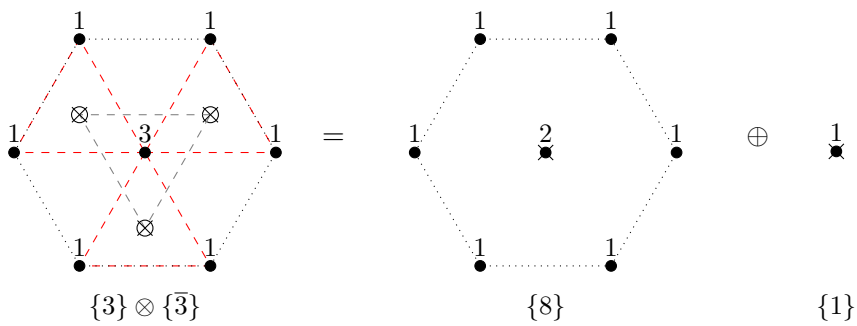

Fig. 5.1: The centres of three anti-triplets $\{\overline{3}\}$ *are superimposed onto the weights of the triplet* $\{3\}$ *in order to graphically evaluate the tensor product* $\{3\} \otimes \{\overline{3}\}$*. The resulting weights are identified with the direct sum of the octet* $\{8\}$ *and the singlet* $\{1\}$ *representation of* $su(3)$*. Due to the commutativity of the tensor product, superimposing three triplets onto an anti-triplet yields the same result.*

This method works for both $su(3)$ and $g(2)$ representations but one can imagine that for higher-dimensional representations the calculations get increasingly cumbersome, especially for the quickly extending representations of $g(2)$. Consider for instance the reduction of the $g(2)$ tensor product $\{1000000\} \otimes \{7\}$ or equivalently $(9,9) \otimes (0,1)$:

$$\begin{aligned} \{1000000\} \otimes \{7\} \quad &= \quad \{773604\} \oplus \{839762\} \oplus \{890967\} \oplus \{1000000\} \\ &\quad \oplus \{1095633\} \oplus \{1127763\} \oplus \{1272271\}. \end{aligned} \tag{5.2}$$

The corresponding illustration is obviously highly complicated and requires knowledge of the shape of all the involved high-dimensional weight diagrams. Furthermore, for higher-rank algebras with higher-dimensional weight diagrams this method naturally becomes more complex.

For this reason the concept, which is referred to as *landscape* in [3], devised by Antoine and Speiser in [1] and [2] in 1964 greatly simplifies the graphical tensor product reduction of irreducible rank 2 algebra representations to a simple task of addition and subtraction. Instead of superimposing two weight diagrams, one diagram is superimposed on the landscape and the reduction can be read off from the overlapping representations, as elaborated in section 5.2.

5.1 Landscapes of $su(3)$ and $g(2)$

The following approach to the explicit cases of $su(3)$ and $g(2)$ is inspired by [3].
A landscape consists of a grid of irreducible Lie algebra representations and reflects all the symmetries of the corresponding weight diagrams. The landscapes, just like the multiplets, of rank 2 Lie algebras are realized in a plane. The positions of representations on the grid are described by grid coordinates (p, q). These coincide with the side lengths of the weight diagram of the representation (p, q) but extend to negative values for both p and q. The grid coordinates of $su(3)$ relate to the Cartesian coordinates by

$$\begin{aligned} x &= \frac{\sqrt{3}}{2}(p+q+2), \\ y &= \frac{1}{2}(p-q). \end{aligned} \tag{5.3}$$

The corresponding relation in the case of $g(2)$ reads

$$\begin{aligned} x &= \frac{\sqrt{3}}{2}(2p+q+3), \\ y &= \frac{1}{2}(q+1). \end{aligned} \tag{5.4}$$

Note that the Cartesian origin corresponds to the grid coordinates $(-1, -1)$ in both cases. The representations are labelled with their dimension $\{D(p, q)\}$ according to equations (3.9) and (4.10). Since the landscape is generated by (p, q) in $\mathbb{Z} \times \mathbb{Z}$ the dimensions of representations in alternating sectors are negative. The sectors are separated by representations with dimension 0, which are connected to straight lines coinciding with the symmetry axes of the multiplets.
The resulting landscapes of $su(3)$ and $g(2)$ are shown in *Fig. 5.2* and *5.3*, respectively

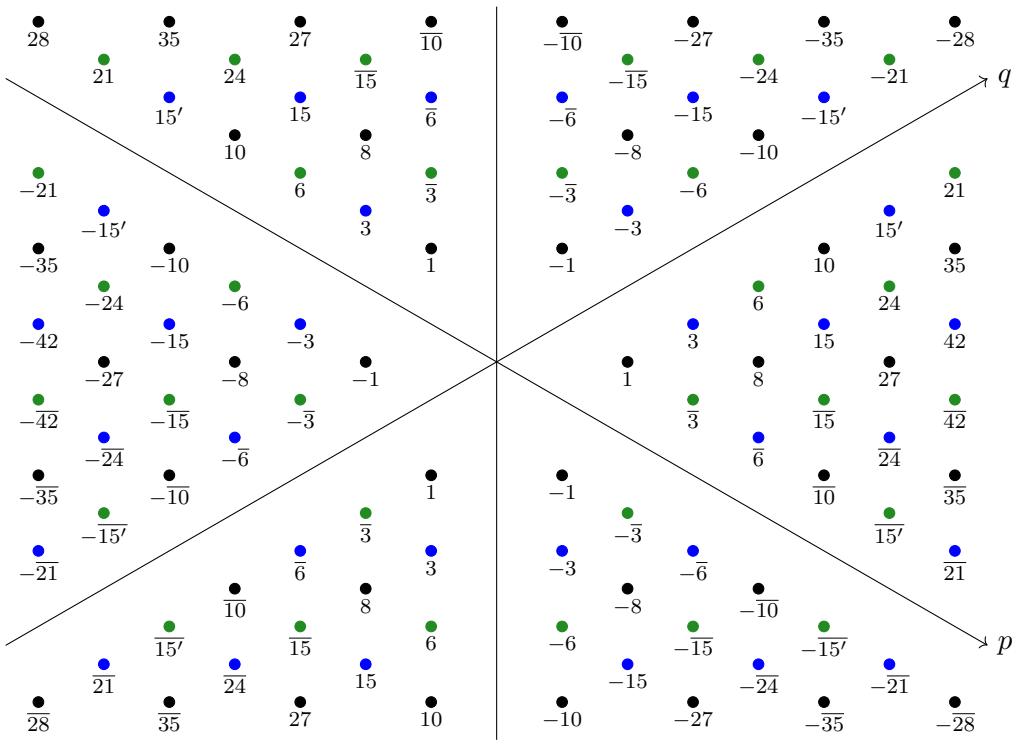

Fig. 5.2: The landscape of irreducible su(3) representations (p, q), labelled below with their dimension $\{D(p, q)\}$ given by (3.9). The landscape consists of three sublattices illustrated by different colors, each corresponding to a different element of the center $\mathbb{Z}_3$ of $SU(3)$. It is divided into six sectors by the axes constituted of the representations of dimension 0 which coincide with the symmetry axes of the hexagonal weight diagrams. The positions of the irreducible representations are given by (5.3) (using the indicated grid coordinates p and q) and the alternating signs $\pm$ follow directly from (3.9).

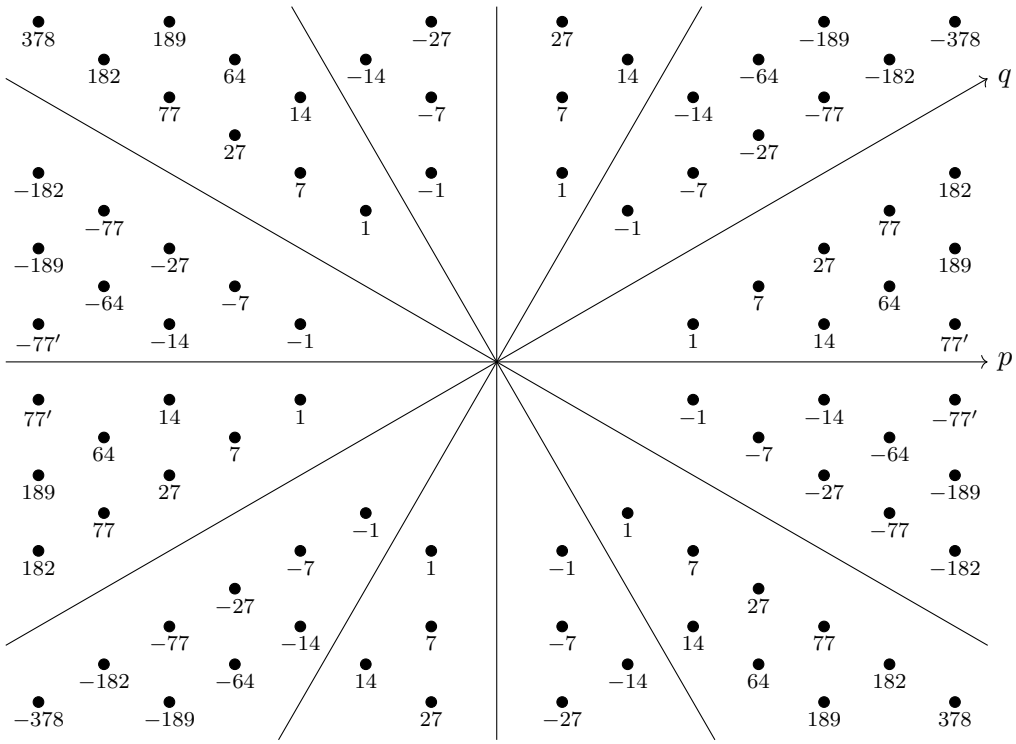

Fig. 5.3: The landscape of irreduclible $g(2)$ representations (p, q), labelled below with their dimension $\{D(p, q)\}$ given by (4.10). Since the center of $g(2)$ is trivial, the landscape cannot be decomposed into sub-lattices. The twelve alternating sectors are again separated by representations of dimension 0, which coincide with the six reflection symmetries of the dodecagon-shaped weight diagrams. The Cartesian positions are identical to the landscape of $su(3)$ but the indicated grid coordinates p and q are chosen in order to satisfy (5.4).

5.2 Antoine-Speiser reduction scheme

As first described by Antoine and Speiser, a given tensor product of irreducible representations $\{A\} \otimes \{B\}$ can now be reduced by superimposing the weight diagram of $\{A\}$ or $\{B\}$ with the representation $\{B\}$ or $\{A\}$ in the first positive sector of the landscape. The direct sum over all now overlapping representations multiplied with the respective multiplicity of the weight exactly yields the decomposition of $\{A\} \otimes \{B\}$ into the direct sum of irreducible representations. The publication [3], on which this thesis is based, describes this method applied to the simple compact Lie algebras $so(5)$, $su(3)$, and $g(2)$.

Consider as an example the graphical reduction of the $su(3)$ tensor product $\{3\} \otimes \{\overline{3}\}$ from (5.1):

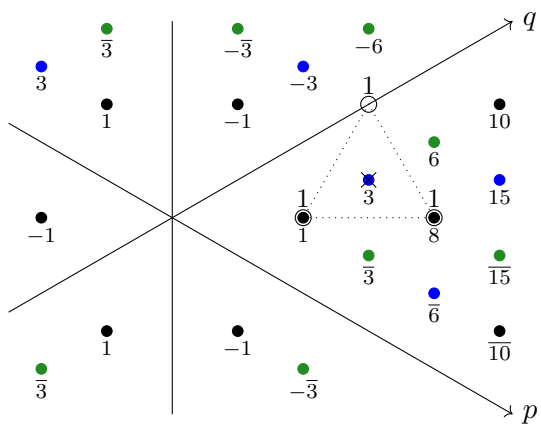

Fig. 5.4: The $su(3)$ anti-triplet representation $\{\overline{3}\}$ of $su(3)$ superimposed onto the triplet representation $\{3\}$ in the $su(3)$ landscape, following the Antoine-Speiser method. The covered representations are multiplied with the multiplicities of the overlapping weights (labelled as superscripts) and the reduction reads $\{3\} \otimes \{\overline{3}\} = 1\{0\} \oplus 1\{1\} \oplus 1\{8\} = \{1\} \oplus \{8\}$.

The graphical Antoine-Speiser method yields the same result as the standard graphical superposition of weight diagrams as shown in *Fig. 5.1*. Note that, in the $su(3)$ case, the superimposed multiplets of a given triality only cover representations of the same color since the generating ladder operators of $su(3)$ only operate within one triangular sub-lattice.

Especially the coupling of higher-dimensional representations according to the Antoine-Speiser scheme is greatly simplified. Consider the larger $g(2)$ tensor product reduction $\{14\} \otimes \{14\} = \{1\} \oplus \{14\} \oplus \{27\} \oplus \{77\} \oplus \{77'\}$, for instance, as graphically determined in *Fig. 5.5*:

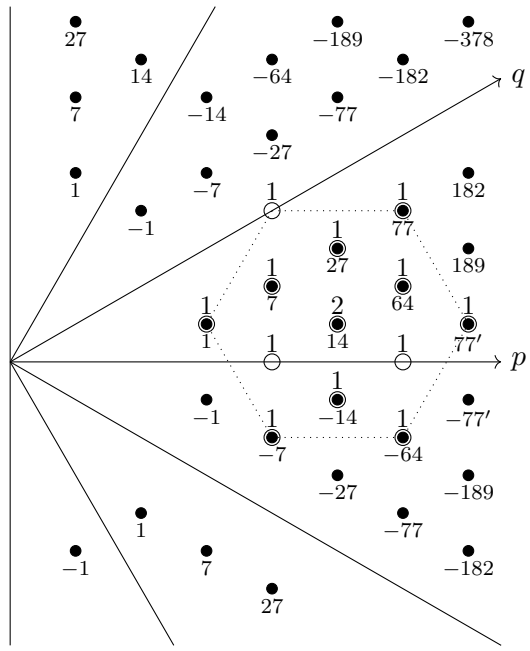

Fig. 5.5: The graphical reduction of the tensor product of two adjoint representations $\{14\}$ *of* $g(2)$ *by superimposing one weight diagram onto the representation in the landscape. The reduction reads* $\{14\} \otimes \{14\} = 3\{0\} \oplus 1\{1\} \oplus (1-1)\{7\} \oplus (2-1)\{14\} \oplus 1\{27\} \oplus (1-1)\{64\} \oplus 1\{77\} \oplus 1\{77'\} = \{1\} \oplus \{14\} \oplus \{27\} \oplus \{77\} \oplus \{77'\}$.

For even larger tensor products this method gets too complicated to perform by hand but a computer program has been implemented in order to reduce tensor products using the Antoine-Speiser scheme, as described in the following section.

5.3 Implementation of an Antoine-Speiser algorithm

The following description of an algorithm for tensor product reduction is devised in such a way that it can be applied to every compact simple Lie algebra of rank 2, namely $so(5)$, $su(3)$, and $g(2)$. As part of this thesis JavaScript programs for both $su(3)$ and $g(2)$ have been implemented explicitly. For this reason, the focus lies on these two cases but the respective formulas for $so(5)$ can be found in [3] and could also be implemented in the algorithm. An $so(5)$ specific algorithm is devised and implemented in [9].
The description of the algorithm is of conceptual nature and aims at describing the idea and method that has been implemented. The explicit realization of the algorithm is less interesting from a physics point of view

and is therefore not included in this thesis.

The only input for the program are two irreducible representations (p_1, q_1), (p_2, q_2) of which the tensor product is to be reduced. These are ordered such that the smaller multiplet is always superimposed onto the larger, since the construction of a large multiplet is more complicated than superimposing a smaller multiplet with a high-dimensional representation in the landscape. In order to do this, the area (number of weights) of a weight diagram (p, q) is calculated as

$$A_{su(3)}(p, q) = \frac{1}{2}(p^2 + 4pq + q^2 + 3p + 3q + 2), \tag{5.5}$$

$$A_{g(2)}(p, q) = 9p^2 + 12pq + 3q^2 + 3p + 3q + 1. \tag{5.6}$$

Now, a sufficiently large segment of the landscape can be constructed around the origin, based on the area $A(p, q)$ of the smaller weight diagram and the position in the landscape of the larger multiplet (p, q). For each landscape point with grid coordinates (p, q), the Cartesian coordinates (x, y) as well as the dimension of the representation $D(p, q)$ are calculated and stored in an array list.[7]

Next, the weight diagram of the smaller representation (p, q) is constructed. In order to do this, it is convenient to define a reflection function for each symmetry axis of the weight diagram, which reflects given grid coordinates about this axis of symmetry. Using these reflection functions it is sufficient to iterate over each point of the weight diagram in the first sector (where both p and q are positive), generating the remaining weights by reflecting about the appropriate axes.
The weight diagram is constructed such that its center has grid coordinates $(0, 0)$ which will be relevant later. In order to achieve this, the iteration of the weight diagram of the representation (p, q) begins at the grid coordinates (p, q). This weight is guaranteed to be a part of the weight diagram and always lies in the corner that is first, the most to the right and second, the most to the top of the diagram. The reflection functions of this weight already yield all the corners of the weight diagram. In the following steps, the iteration passes through the grid coordinates to the left in steps given by T^- in the case of $su(3)$ and Z^+ in the case of $g(2)$, generating the symmetry partners of each point reached. This keeps going as long as the weights still lie in the first sector.

[7] Consider equations (5.3), (5.4), (3.9), and (4.10) for $su(3)$ and $g(2)$, respectively.

This process is repeated for each starting point above and below the initial starting weight given by (p, q) until the sector and therefore the whole weight diagram is completely constructed. The new upper starting weights are reached by operating onto the corner state with U^+ in the case of $su(3)$ and with Y^+ and U^+ in an alternating fashion in the case of $g(2)$. The starting points below are reached by V^- and X^-, respectively. All generated weights are stored in an array list, so that they can be compared with the landscape list.

Examples of generating the positions of the weights in the weight diagrams of $su(3)$ and $g(2)$ are illustrated in *Fig. 5.6* and *5.7*, respectively.

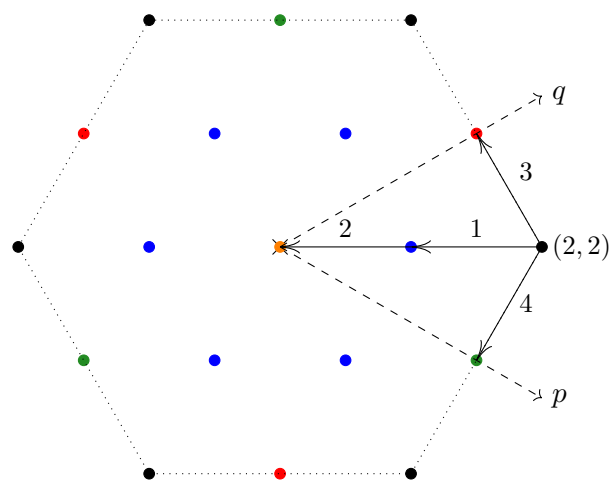

Fig. 5.6: The construction of the weight diagram of the $(2, 2)$ *representation of* $su(3)$*. Each point in the indicated first sector, where* p *and* q *are both positive, has a distinct color. The weights obtained by reflection symmetries are indicated by the same color as their generating weight. The arrows indicate the iteration in the weight diagram. The first and second steps are reached by the ladder operator* T^-*, the third by* U^+*, and the fourth by the operator* V^-*.*

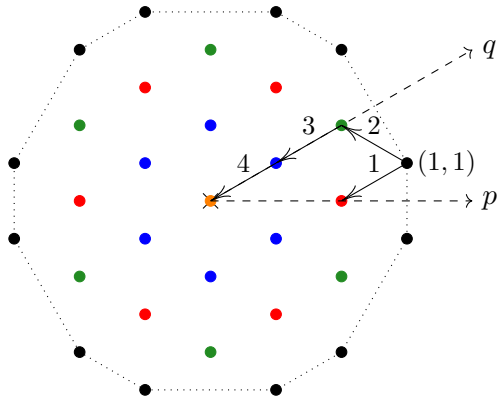

Fig. 5.7: The construction of the weight diagram of the $(1,1)$ *representation of* $g(2)$*. Every point in the indicated first sector, where* p *and* q *are both positive, has a distinct color. The weights obtained by reflection symmetries are indicated by the same color as their generating weight. The arrows indicate the iteration in the weight diagram. The first, third, and fourth steps are reached by the ladder operator* Z^+ *and the second step by the operator* Y^+*.*

In the next step, the multiplicities of the weights are constructed to completely characterize the weight diagram of a representation.
In order to find the multiplicities of a given representation $\{D(p,q)\}$, the tensor product with the trivial representation is considered:

$$\{D(p,q)\} \otimes \{1\} = \{D(p,q)\}. \tag{5.7}$$

This well-known result must be reproduced by the Antoine-Speiser method. Consequently, any other overlapping representation in the landscape than $\{D(p,q)\}$ must sum up to zero and the multiplicity of the weight covering $\{D(p,q)\}$ must be one. Furthermore, the multiplicities comply with the reflection symmetries of the weight diagram as well. These are sufficiently many constraints to devise a general algorithm in order to find the multiplicities.

By comparing the weight diagram list coordinate pairs (p,q) to the coordinates in the landscape list, the matching landscape representations are copied to the corresponding entry in the weight diagram list. In this way, the weight diagram array list contains all necessary information in order to compute the multiplicities of each weight.
As a starting point, the multiplicity of the weight with coordinates (p,q), which is always the one covering the representation $\{D(p,q)\}$, is set to one. The symmetry functions copy this value to all the corresponding weights.

As it turns out, an appropriate way to pass through the weight diagram is by columns from top to bottom. Therefore, the next multiplicity to be determined is of the weight reached by action of U^+. Whenever a weight is reached which has no assigned multiplicity yet, the representation it is covering is compared to all previously determined weights and their multiplicities. Since the algorithm guarantees that this representation will not appear at a later step again and since these multiplicities have to sum up to zero, the multiplicity of the current weight is uniquely determined. As before, the multiplicity extends to all corresponding weights by symmetry.
On the other hand, whenever a current weight's multiplicity has already been calculated, the algorithm proceeds directly downwards to the next weight until there are again multiplicities to be calculated. This continues until the bottom of the diagram is reached. After the current column has been covered the next column is passed through in the same fashion until the multiplicities of the center column of the weight diagram are determined. At this point, the weight diagram is completely characterized and the multiplet array list contains all positions and multiplicities.

As an example, consider the process of determining the multiplicities of the $(2,2) = \{27\}$ representation of $su(3)$ illustrated in *Fig. 5.8*.

After the first trivial multiplicity is set to 1, the next weight (step 1) is covering the representation $\{10\}$. Checking with the already determined corner weights, this representation appears once with multiplicity 1 in the negative sector. Therefore, the multiplicity of this weight must be 1 in order to make sure that the $\{10\}$ representation disappears from the reduction. As a consequence, all weights related by symmetry are weighed with the same multiplicity.
The next weight (step 2) is covering the representation $\{\overline{10}\}$ which is covered once with multiplicity 1 in a negative sector. Consequently this weight's and all symmetry partner's multiplicity must be 1 as well.
The next non-trivial iteration (step 4) reaches a weight covering the representation $\{8\}$. This representation is already covered twice in negative sectors with multiplicity 1. The multiplicity has to be 2 in order to erase this representation from the tensor product reduction, which is again extended to the corresponding weights by symmetry.
The last weight that is left is reached in step 11 and covers the $\{1\}$ representation as it is in the center of the weight diagram. This representation is reached multiple times with multiplicities 1 and 2, which imposes the

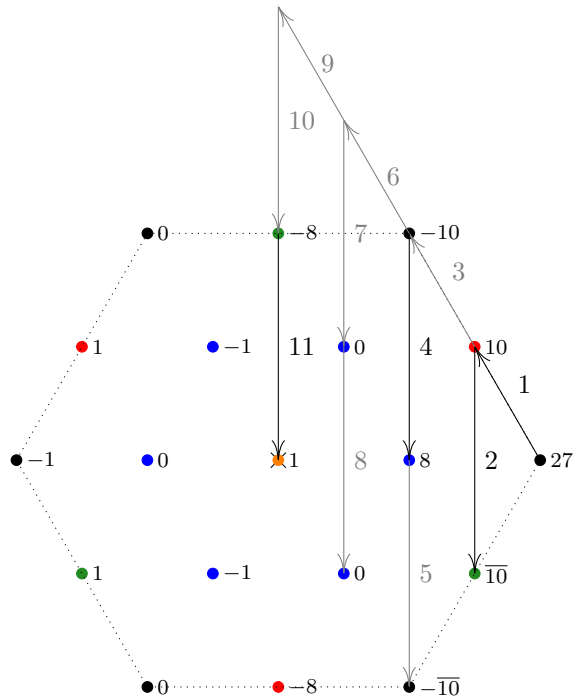

Fig. 5.8: Illustration of the iteration through the weight diagram of the $(2,2) = \{27\}$ *representation of* $su(3)$*. Weights that are connected by the reflection symmetries are indicated by the same color. The representations that are covered by the weights are given by small dimension values to the right considering the sector sign. Gray arrows indicate iterations that do not reach weights or weights with already determined multiplicities. Black arrows illustrate steps which contain multiplicity calculations. Each arrow is numbered to clarify the path taken through the multiplet.*

following equation for the multiplicity m of the last state

$$(-2 - 2 + 1 + 1 - 1 + m) \stackrel{!}{=} 0. \tag{5.8}$$

The last weight is therefore found to be 3-fold degenerate and the shell-like structure from *Fig. 3.4* is indeed reproduced.
This method of constructing multiplicities is general for rank 2 algebras and therefore works accordingly in the case of $g(2)$.

In the last step, the multiplet coordinates are shifted according to the input such that the center now coincides with the coordinates of the representation with which the tensor product is to be taken. The covered representations are now simply weighed by the sum over all corresponding multiplicities, by

iterating over the multiplet array list, and displayed as a direct sum.

The programs devised for both the Lie algebras $su(3)$ and $g(2)$ as part of this thesis are implemented such that the reduction is graphically illustrated as well. Tensor products requiring large parts of the landscape are displayed such that the window is centered around the middle of the multiplet. The rest of the landscape can always be explored by using the scroll bars, which appear if necessary. The exterior design of the programs was kindly provided by Patrick Bühlmann, who implemented a similar algorithm for the Lie algebra $so(5)$ as a part of [9], in order to present uniformly appearing programs for all the rank 2 algebras. The final web applets can be found at *http://www.wiese.itp.unibe.ch/techniques.html* and are presented in *Fig. 5.9* and *5.10*:

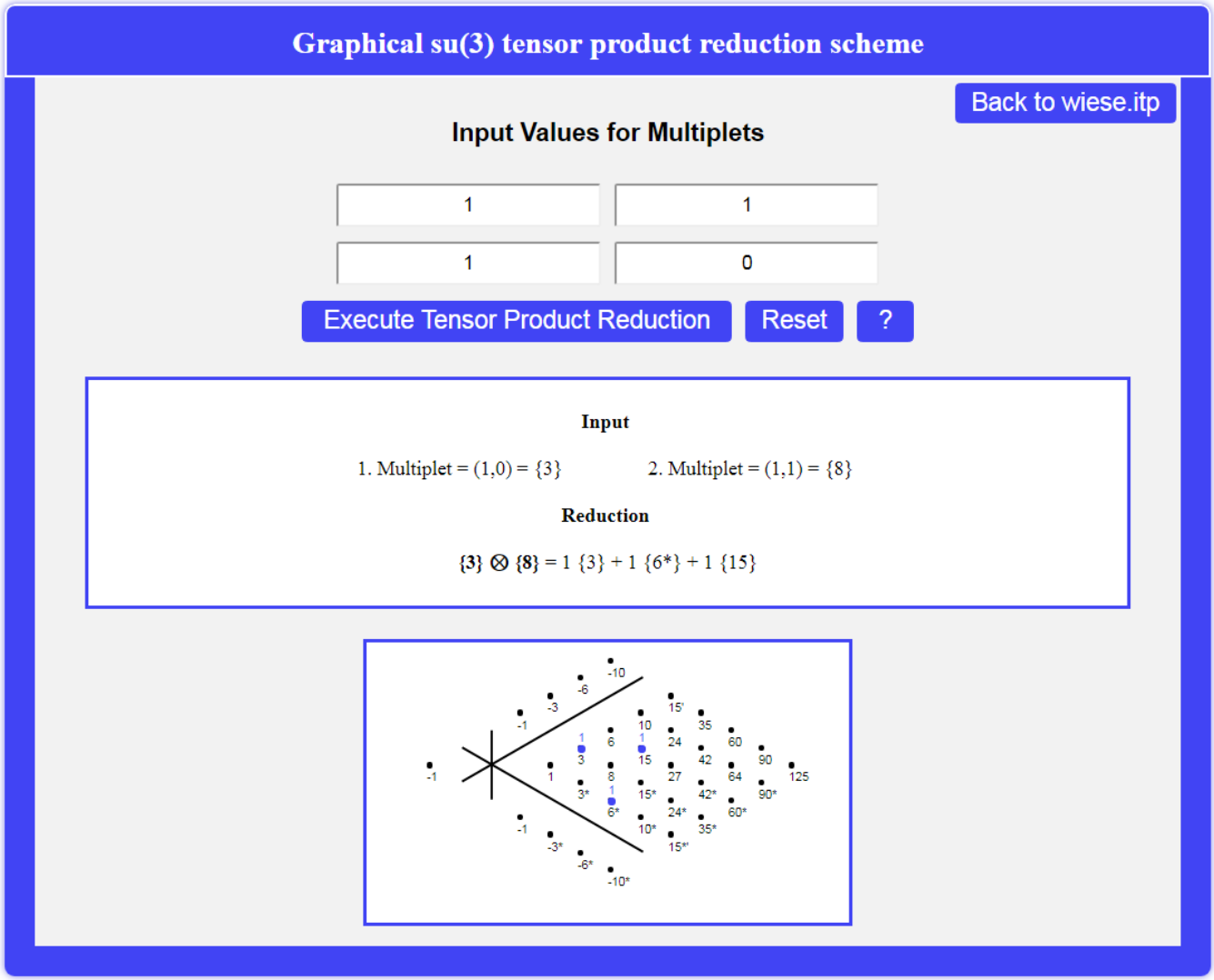

Fig. 5.9: Screenshot of the su(3) web applet reducing the tensor product $\{3\} \otimes \{8\}$ *to the direct sum* $\{3\} \oplus \{6^*\} \oplus \{15\}$.

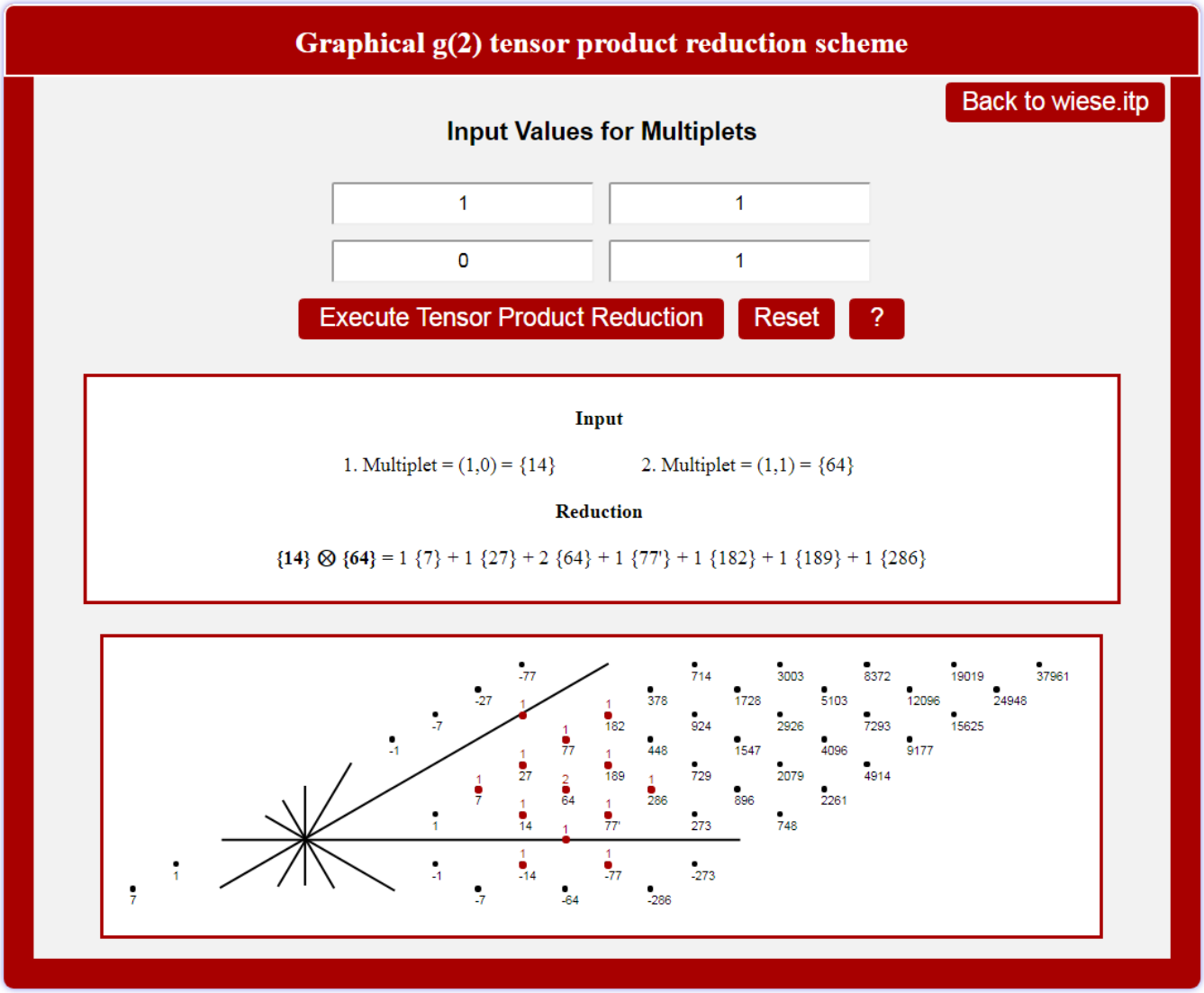

Fig. 5.10: Screenshot of the g(2) web applet reducing the tensor product $\{14\}\otimes\{64\}$ *to the direct sum* $\{7\}\oplus\{27\}\oplus 2\{64\}\oplus\{77'\}\oplus\{182\}\oplus\{189\}\oplus\{286\}$.

The web applets are free to use and the readers are encouraged to make use of these tools for any future more difficult tensor product reductions.

6 Conclusion and outlook

The aim of this thesis was to become familiar with the Lie group theory and its relevance in physics. Furthermore, the graphical tensor product reduction method, devised by Jean-Pierre Antoine and David Speiser, was to be studied and a working algorithm for the $g(2)$ Lie algebra had to be implemented.
In the end, the theoretical introduction to Lie group theory took up more space in the thesis and consumed more time than initially expected, which has definitely resulted in a deeper understanding of the subject. It turned out to be more perceptive to examine the sub-algebra $su(3)$ first in order to finally extend the concepts to $g(2)$. As a result of this approach, the thesis has been structured accordingly and the parts of the program were first constructed for the $su(3)$ algebra and then extended to the $g(2)$ algebra. At the end, web applets for both $su(3)$ and $g(2)$ have been implemented.
I have learned very much about Lie groups, Lie algebras, the implementation of algorithms in general and web applets in particular. Hopefully, this thesis is helpful for the reader as well and serves as a good introduction to the subject.

Since this graphical method has now been implemented for all the compact simple Lie algebras of rank 2, one can easily imagine that this scheme is applicable to rank 3 algebras as well. The resulting 3-dimensional weight diagrams and landscapes could be a further interesting generalization of this tensor product reduction method.

6.1 Acknowledgement

A special thanks goes to Prof. Uwe-Jens Wiese for the supervision of the thesis in an uncommon semester and for many interesting and productive conversations. Furthermore, I would like to thank Patrick Bühlmann for the design of the programs, Naomi Tokuda for many helpful conversations about mathematical intricacies, and Nicolas Kessler for his invaluable advice on coding in general.

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