Relativistic Quantum Mechanics of Interacting Particles

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Abstract

Acknowledging Leutwyler's no-interaction theorem for classical Hamiltonian particle mechanics, we show that in one spatial dimension there actually is an interaction, a linear confining potential, that is at least at the classical level Poincaré invariant. Not being able to prove Poincaré invariance at the quantum level explicitly, we go on to solve a relativistic version of the Schrödinger equation in order to show that the spectrum is Lorentz invariant. However, we arrive at the conclusion that the spectrum is either not relativistically invariant, or that the Hamiltonian may not be extended to be self-adjoint. Both possible interpretations of our results suggest that in the usual Cartesian coordinates relativistic invariant particle quantum mechanics does not exist. Finally, we show that in light cone coordinates the Poincaré algebra also closes at the quantum level.

Die Einsichten des Wissenschaftlers
sind in aller Regel grösser
als seine Wirkungsmöglichkeiten.
Beim Politiker ist es meistens umgekehrt.

Wolfgang Engelhardt

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Introduction

The principles of special relativity and quantum mechanics are naturally incorporated in relativistic quantum field theories such as QED, QCD, or the standard model, which provide accurate descriptions of elementary particle interactions. Since field theories are systems with infinitely many degrees of freedom — a given number per space point — their quantization is not straightforward. In particular, for non-perturbative questions, e.g. concerning bound states, it is a highly non-trivial step from the quantum mechanics of a few interacting non-relativistic particles to quantum field theory. For example, the QED description of positronium using the Bethe-Salpeter equation is already much more complicated than the Schrödinger equation for the hydrogen atom, and understanding the binding of quarks and gluons inside hadrons remains one of the biggest challenges in QCD. Relativistic mechanics — the theory of systems with a finite number of relativistic degrees of freedom — should be a lot easier to quantize than a field theory and seems to provide a natural intermediate step between non-relativistic quantum mechanics and relativistic field theory. Relativistic quantum mechanics is usually discussed in the context of the Klein-Gordon and the Dirac equation, which one considers either for free particles or for particles in an external field. Of course, the single particle interpretation of these equations is problematical, since they really belong to quantum field theory and not to relativistic quantum mechanics. Consequently, bound states of interacting relativistic particles are usually discussed in the context of quantum field theory.

Of course, there is a good reason for that. The interaction potentials that are familiar from non-relativistic quantum mechanics are unacceptable in relativistic theories, because they are instantaneous. In relativistic theories interactions are naturally mediated by particle exchange, for example by the exchange of photons between the electron and positron forming positronium. As a consequence of causality, an electron that has just emitted a virtual photon cannot be prevented from emitting additional photons before the first one has been absorbed by the positron. As a result, the number of exchanged particles seems to involve infinitely many degrees of freedom and its treatment seems to require quantum field theory and not just relativistic quantum mechanics.

Dirac was first to point out that the Poincaré algebra can be realized in different forms [1]. In the most familiar "instant" form (usual Cartesian space-time coordinates) of the dynamics the initial conditions are imposed at a given instant of time. In the "front" form (light cone coordinates) of the dynamics the initial conditions are specified on a light front. It has been shown by Currie, Jordan, and Sudarshan that — due to the problems with instantaneous potentials — the Poincaré algebra in instant form cannot be realized for two interacting particles [2, 3]. This statement was extended to three particles by Cannon and Jordan [4]. Finally, Leutwyler proved a no-interaction theorem for any number of particles in the context of relativistic classical Hamiltonian mechanics [5]. However, as Currie already pointed out [3], this theorem does not hold in one space plus one time dimension. In this case there is just one potential for which the classical Poincaré algebra closes, namely a linear confining potential that results from a string which, in one dimension, has no degrees of freedom in addition to the position of its endpoints. Hence two relativistic particles in one dimension confined by a linear rising potential also interact locally and thus respect causality.

Unfortunately, it turns out that in the usual Cartesian coordinates it is not possible to quantize this system without running into serious problems. In light cone coordinates, however, the Poincaré algebra also closes at the quantum level. In the first three chapters of this thesis we derive the Poincaré algebra, review Leutwyler's no-interaction theorem, and show that in one dimension the linear confining potential is relativistically invariant, at least classically. In the fourth chapter we try to solve the resulting relativistic form of the Schrödinger equation, first in a semi-classical Bohr-Sommerfeld approximation and then fully quantum mechanically. Finally, Chapter 5 deals with the linear confining potential in light cone coordinates.

Chapter 1

Poincaré Algebra

1.1 Notation and Commutation Relations

The Poincaré group in d dimensions includes the following transformations of $d+1$–dimensional space–time:

- rotations of the d space dimensions,
- boosts on space–time,
- space translations,
- time translations.

A generic transformation belonging to the Poincaré group may be written as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}. \tag{1.1}$$

Performing two such transformations,

$$\begin{aligned} x'^{\mu} &= \Lambda_1{}^{\mu}{}_{\nu} x^{\nu} + a_1{}^{\mu}, \\ x''^{\mu} &= \Lambda_2{}^{\mu}{}_{\nu} x'^{\nu} + a_2{}^{\mu}, \end{aligned} \tag{1.2}$$

in a row yields

$$x''^{\mu} = \Lambda_2{}^{\mu}{}_{\nu}(\Lambda_1{}^{\nu}{}_{\rho} x^{\rho} + a_1{}^{\nu}) + a_2{}^{\mu} = \Lambda_2{}^{\mu}{}_{\nu}\Lambda_1{}^{\nu}{}_{\rho} x^{\rho} + \Lambda_2{}^{\mu}{}_{\nu} a_1{}^{\nu} + a_2{}^{\mu}. \tag{1.3}$$

So we obtain the following composition law:

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2). \tag{1.4}$$

We denote by

- H the generator of time translations,
- $P_1, \ldots, P_d$ the generators of space translations,
- $J_1, \ldots, J_{\frac{d(d-1)}{2}}$ the generators of rotations,

- $K_1, \ldots, K_d$ the generators of boosts.

Using the composition law for an infinitesimal transformation one obtains the following commutation relations (see Appendix A) for $d = 3$:

$$\begin{aligned}&[P_i, H] = 0,\ [J_i, H] = 0,\ [K_i, H] = iP_i,\ [P_i, P_j] = 0,\ [J_i, P_j] = i\epsilon_{ijk}P_k,\\&[K_i, P_j] = i\delta_{ij}H,\ [J_i, J_j] = i\epsilon_{ijk}J_k,\ [J_i, K_j] = i\epsilon_{ijk}K_k,\ [K_i, K_j] = -i\epsilon_{ijk}J_k.\end{aligned} \tag{1.5}$$

In one spatial dimension this reduces to

$$[P, H] = 0,\ [K, H] = iP,\ [K, P] = iH. \tag{1.6}$$

The ten (three) generators together with the commutation relations form the Poincaré algebra. A compact way to write the commutation relations is

$$[X_i, X_k] = \sum_l c_{ik}^l X_l, \tag{1.7}$$

where the c_{ik}^l are the structure constants and the X_i the generators.

1.2 Casimir Operators

Of special interest are those quantities that are invariant under symmetry operations. Such quantities may be described by operators that commute with all the generators of the group and are called Casimir operators. It is an interesting question how many linearly independent Casimir operators there are. The answer for non-semi-simple Lie algebras as the Poincaré algebra is (see Appendix B for details)

$$N_c = r - \text{Rank}\left(\sum_{l=1}^{r} c_{ik}^l \alpha_l\right), \tag{1.8}$$

where r is the dimension of the Lie algebra, i.e. the number of generators. The α_l are to be treated as independent variables. Putting $X_1 = H, X_2 = P, X_3 = K$, the only structure constants different from zero are $C_{13}^2 = -C_{31}^2 = -i, C_{23}^1 = -C_{32}^1 = -i$. Thus we have

$$\text{Rank}\left(\sum_{l=1}^{r} c_{ik}^l \alpha_l\right) = \text{Rank}\begin{pmatrix} 0 & 0 & -i\alpha_2 \\ 0 & 0 & -i\alpha_1 \\ i\alpha_2 & i\alpha_1 & 0 \end{pmatrix} = 2. \tag{1.9}$$

So in one dimension there is only one Casimir operator. We of course know an invariant physical quantity which is the rest mass squared. Hence $M^2 = H^2 - P^2$ should be a Casimir operator and it is easy to show that M^2 commutes with all the three generators.

In three dimensions we have to deal with a 10 by 10 matrix, but things are just as simple as in the one-dimensional case. One finds that the rank of the matrix is 8 and thus there must be two linearly independent Casimir operators. One is again the rest mass squared

$$M^2 = H^2 - \vec{P}^2. \tag{1.10}$$

The other one is not that easy to find. The idea is to construct a translation invariant four vector and square it, such that it becomes Lorentz invariant. The result is

$$W^2 = (\vec{P} \cdot \vec{J})^2 - (\vec{P} \times \vec{K} + H\vec{J})^2. \tag{1.11}$$

The eigenvalues of this Casimir operator are [6] $-m^2 s(s+1)$, where s may be interpreted as the spin. All the irreducible representations of the Poincaré group may be classified through the eigenvalues of the two Casimir operators.

For a system of N free spinless relativistic particles of rest mass m_a with positions $\vec{x}_a$ and momenta $\vec{p}_a, (a \in \{1, 2, \ldots, N\})$ the operators of the Poincaré algebra may be represented as

$$\begin{aligned} H &= \sum_{a=1}^{N} \sqrt{\vec{p}_a^{\,2} + m_a^2}, \; \vec{P} = \sum_{a=1}^{N} \vec{p}_a, \; \vec{J} = \sum_{a=1}^{N} \vec{x}_a \times \vec{p}_a, \\ \vec{K} &= \sum_{a=1}^{N} \frac{1}{2} \left(\vec{x}_a \sqrt{\vec{p}_a^{\,2} + m_a^2} + \sqrt{\vec{p}_a^{\,2} + m_a^2}\, \vec{x}_a \right). \end{aligned} \tag{1.12}$$

Using the canonical commutation relations

$$[x_{ai}, p_{bj}] = i \delta_{ab} \delta_{ij}, \tag{1.13}$$

it is straightforward to show that the operators above indeed obey the commutation relations of the Poincaré algebra.

Chapter 2

Leutwyler's No-Interaction Theorem in Three Dimensions

In this chapter a no-interaction theorem in classical relativistic Hamiltonian particle mechanics is stated and part of the proof is reviewed.
This theorem was presented by H. Leutwyler in [5] and most of this chapter follows the description therein. A proof of this theorem for two particles has already been established earlier on by Currie, Jordan and Sudarshan [2], while Cannon and Jordan extended the proof to three particles [4].

We consider the classical Poincaré algebra in this chapter. Thus we deal with Poisson brackets rather than with commutators as in Chapter 1. The Poisson bracket is defined as

$$\{A, B\} = \sum_{a=1}^{N} \sum_{i=1}^{3} \left(\frac{\partial A}{\partial x_i^a} \frac{\partial B}{\partial p_i^a} - \frac{\partial A}{\partial p_i^a} \frac{\partial B}{\partial x_i^a} \right). \tag{2.1}$$

Here N denotes the number of particles and a labels them, whereas x_i^a and p_i^a are the components of the position and the momentum of particle a. The Poisson brackets for the Poincaré algebra read

$$\begin{aligned} &\{P_i, H\} = 0,\ \{J_i, H\} = 0,\ \{K_i, H\} = P_i,\ \{P_i, P_j\} = 0,\ \{J_i, P_j\} = \epsilon_{ijk} P_k, \\ &\{K_i, P_j\} = \delta_{ij} H,\ \{J_i, J_j\} = \epsilon_{ijk} J_k,\ \{J_i, K_j\} = \epsilon_{ijk} K_k,\ \{K_i, K_j\} = -\epsilon_{ijk} J_k. \end{aligned} \tag{2.2}$$

Further the no-interaction theorem requires manifest Lorentz invariance, i.e. the particle coordinates transform correctly under Lorentz transformations. This requirement is equivalent to

$$\{x_i^a, P_k\} = \delta_{ik},\ \{x_i^a, J_k\} = \epsilon_{ikl} x_l^a,\ \{x_i^a, K_k\} = x_k^a \{x_i^a, H\}. \tag{2.3}$$

Theorem: *If the set of ten functions H, P_i, J_i, K_i satisfies the bracket relations (2.2) and (2.3) and if the equations of motion are not degenerate, i.e.*

$$\det \frac{\partial^2 H}{\partial p_i^a \partial p_k^a} \neq 0, \tag{2.4}$$

then the acceleration of each particle ($a = 1, \ldots, N$) vanishes

$$\{\{x_i^a, H\}, H\} = 0. \tag{2.5}$$

The condition that the equations of motion are not degenerate excludes Hamilton functions that only depend linearly on the momenta. If there is a kinetic term, i.e. a term quadratic in the momenta, the condition is fulfilled. Let's go on to give part of the proof for this theorem. First we show that the six generators P_i, J_i may be brought to their free particle form by a suitable canonical transformation. The bracket

$$\{x_i^a, P_k\} = \delta_{ik} \tag{2.6}$$

implies

$$\frac{\partial P_k}{\partial p_i^a} = \delta_{ik}. \tag{2.7}$$

Thus P_i is of the form

$$P_i = \sum_{a=1}^{N} p_i^a + W_i(x). \tag{2.8}$$

Further

$$\{x_i^a, J_k\} = \epsilon_{ikl} x_l^a \tag{2.9}$$

implies

$$\frac{\partial J_k}{\partial p_i^a} = \epsilon_{ikl} x_l^a, \tag{2.10}$$

and thus we have

$$J_i = \sum_{a=1}^{N} \epsilon_{ikl} x_k^a p_l^a + F_i(x). \tag{2.11}$$

The Poisson brackets (2.2) impose non-trivial restrictions on the function W_i and F_i which imply [5]

$$F_i = \{J_i, F\},\ W_i = \{P_i, F\}, \tag{2.12}$$

where F is some function of the x variables only. The above brackets are

$$F_i = -\sum_{l,a} \frac{\partial J_i}{\partial p_l^a} \frac{\partial F}{\partial x_l^a},\ W_i = -\sum_{l,a} \frac{\partial P_i}{\partial p_l^a} \frac{\partial F}{\partial x_l^a}. \tag{2.13}$$

This leads to

$$P_i = P_i^0 - \sum_a \frac{\partial F}{\partial x_i^a},\ J_i = J_i^0 - \sum_a \epsilon_{ikl} x_k^a \frac{\partial F}{\partial x_l^a}. \tag{2.14}$$

Here J_i^0 and P_i^0 denote the free particle form of the generators. We now perform the canonical transformation

$$x_i^{a\prime} = x_i^a,\ p_i^{a\prime} = p_i^a - \frac{\partial F}{\partial x_i^a} \tag{2.15}$$

and obtain for J_i and P_i in the new variables

$$\begin{aligned} P_i &= \sum_a (p_i^{a\prime} + \frac{\partial F}{\partial x_i^a}) - \sum_a \frac{\partial F}{\partial x_i^a} = \sum_a p_i^{a\prime}, \\ J_i &= \sum_a \epsilon_{ikl} x_k^a (p_l^{a\prime} + \frac{\partial F}{\partial x_l^a}) - \sum_a \epsilon_{ikl} x_k^a \frac{\partial F}{\partial x_l^a} = \sum_a \epsilon_{ikl} x_k^{a\prime} p_l^{a\prime}. \end{aligned} \tag{2.16}$$

So we have indeed brought all the J_i and P_i to their free particle form. The next step is now to bring H and K_i to their free particle form while leaving J_i and P_i unmodified. We drop the primes and work with the new variables only. We start from the last equation of (2.3),

$$\{x_l^a, K_k\} = \frac{\partial K_k}{\partial p_l^a} = x_k^a \frac{\partial H}{\partial p_l^a}. \tag{2.17}$$

Taking the partial derivative with respect to p_i^b we obtain

$$\frac{\partial^2 K_k}{\partial p_i^b \partial p_l^a} = x_k^a \frac{\partial^2 H}{\partial p_l^a \partial p_i^b}.$$

Of course, one may do the same for a and b interchanged which leads to

$$\frac{\partial^2 K_k}{\partial p_i^b \partial p_l^a} = x_k^a \frac{\partial^2 H}{\partial p_l^a \partial p_i^b} = x_k^b \frac{\partial^2 H}{\partial p_i^b \partial p_l^a}.$$

So we have

$$(x_k^b - x_k^a) \frac{\partial^2 H}{\partial p_l^a \partial p_i^b} = 0. \tag{2.18}$$

For $a \neq b$ we thus have

$$\frac{\partial^2 H}{\partial p_l^a \partial p_i^b} = 0.$$

Therefore we may write H as

$$H = \sum_a h^a(p^a, x). \tag{2.19}$$

So the Hamiltonian is a sum of terms of which each may depend on the position of every particle but only on the momentum of one of them. The brackets $\{P_i, H\} = 0$ and $\{J_i, H\} = 0$ imply that H must be invariant both under rotations and translations. One can show [5] that

$$H = \sum_a \bar{h}^a(p^a, x), \tag{2.20}$$

with each of the $\bar{h}^a$ being invariant under both rotations and translations. We drop the bar and insert this expression for H into (2.17)

$$\frac{\partial K_k}{\partial p_l^a} = x_k^a \frac{\partial h^a}{\partial p_l^a},$$

which leads to

$$K_i = \sum_a x_i^a h^a(p^a, x) + k_i(x). \tag{2.21}$$

Evaluation of $\{K_i, P_j\} = 0$ leads to $\{k_i, P_j\} = 0$, so the term $k_i(x)$ is also translation invariant.

Until now we have used all the Poisson brackets of (2.2) and (2.3) except $\{K_i, H\} = P_i$. So let's evaluate this bracket using the expressions (2.20) for H and (2.21) for K_i

$$\begin{aligned}
\{K_i, H\} &= \sum_{b,k} \left\{ \frac{\partial}{\partial x_k^b} \left(\sum_a x_i^a h^a(p^a, x) + k_i(x) \right) \frac{\partial}{\partial p_k^b} \left(\sum_c h^c(p^c, x) \right) \right. \\
&\quad \left. - \frac{\partial}{\partial p_k^b} \left(\sum_a x_i^a h^a(p^a, x) + k_i(x) \right) \frac{\partial}{\partial x_k^b} \left(\sum_c h^c(p^c, x) \right) \right\} \\
&= \sum_a h^a(p^a, x) \frac{\partial h^a}{\partial p_i^a} + \sum_{a,k} \frac{\partial k_i}{\partial x_k^a} \frac{\partial h^a}{\partial p_k^a} + \sum_{a,b,k} \frac{\partial h^b}{\partial x_k^a} \frac{\partial h^a}{\partial p_k^a} (x_i^b - x_i^a) = \sum_a p_i^a.
\end{aligned}$$

We may simplify this to

$$0 = \frac{1}{2}\sum_a \frac{\partial}{\partial p_i^a}\left(h^{a2} - p^{a2}\right) + \sum_{a,k}\frac{\partial k_i}{\partial x_k^a}\frac{\partial h^a}{\partial p_k^a} + \sum_{a,b,k}\frac{\partial h^b}{\partial x_k^a}\frac{\partial h^a}{\partial p_k^a}\left(x_i^b - x_i^a\right). \tag{2.22}$$

In section 4 of [5] it is shown, using the non-degeneracy condition of the theorem, that this simplifies to

$$\frac{\partial h^a}{\partial x_i^b} = \sum_k \frac{\partial^2 L}{\partial x_i^b \partial x_k^a}\frac{\partial h^a}{\partial p_k^a} + \frac{\partial M^a}{\partial x_i^b}, \tag{2.23}$$

where L and M^a are functions of the positions only, invariant with respect to translations and rotations. It remains to solve this differential equation. In order to do this we perform the following canonical transformation

$$x_i^{a\prime} = x_i^a,\ p^{a\prime} = p_i^a + \frac{\partial L}{\partial x_i^a}. \tag{2.24}$$

Let's first show that this transformation indeed leaves P_i and J_i invariant. This uses the fact that L is invariant under translations and rotations, i.e. from $\{J_i, L\} = 0$ follows

$$\sum_a \epsilon_{ikl} x_k^a \frac{\partial L}{\partial x_l^a} = 0, \tag{2.25}$$

and from $\{P_i, L\} = 0$ we obtain

$$\sum_a \frac{\partial L}{\partial x_i^a} = 0, \tag{2.26}$$

and thus

$$\begin{aligned} P_i &= \sum_a p_i^a = \sum_a \left(p_i^{a\prime} - \frac{\partial L}{\partial x_i^a}\right) = \sum_a p_i^{a\prime} = P_i^0, \\ J_i &= \sum_a \epsilon_{ikl} x_k^a p_l^a = \sum_a \epsilon_{ikl} x_k^a \left(p_l^{a\prime} - \frac{\partial L}{\partial x_l^a}\right) = \sum_a \epsilon_{ikl} x_k^a p_l^{a\prime} = J_i^0. \end{aligned} \tag{2.27}$$

With

$$h^{a\prime}(p', x') = h^a(p^a, x),\ M^{a\prime}(x') = M^a(x) \tag{2.28}$$

and

$$\begin{aligned} \frac{\partial}{\partial x_i^a} &= \frac{\partial}{\partial x_k^{b\prime}}\frac{\partial x_k^{b\prime}}{\partial x_i^a} + \frac{\partial}{\partial p_k^{b\prime}}\frac{\partial p_k^{b\prime}}{\partial x_i^a} = \frac{\partial}{\partial x_i^{a\prime}} + \frac{\partial^2 L}{\partial x_i^a \partial x_k^b}\frac{\partial}{\partial p_k^{b\prime}}, \\ \frac{\partial}{\partial p_i^a} &= \frac{\partial}{\partial x_k^{b\prime}}\frac{\partial x_k^{b\prime}}{\partial p_i^a} + \frac{\partial}{\partial p_k^{b\prime}}\frac{\partial p_k^{b\prime}}{\partial p_i^a} = \frac{\partial}{\partial p_i^{a\prime}}, \end{aligned} \tag{2.29}$$

equation (2.23) reduces to

$$\frac{\partial h^{a\prime}}{\partial x_i^{b\prime}} = \frac{\partial M^{a\prime}}{\partial x_i^{b\prime}}. \tag{2.30}$$

This implies

$$h^{a\prime}(x', p') = M^{a\prime}(x') + N^{a\prime}(p^{a\prime}), \tag{2.31}$$

i.e. in terms of the new variables h^a decomposes into a function M^a of the position only and a function N^a of the momentum p^a only. We shall now work with the new variable only and drop the primes. Equation (2.22) is still valid in the new variables since we only used the Poisson bracket $\{K_i, H\}$ and the decomposition of H and K_i in terms of h^a to derive it. Both the bracket and the decomposition are still valid after the transformation. Inserting (2.31) into (2.22) gives

$$\begin{aligned}
0 &= \frac{1}{2}\sum_a \frac{\partial}{\partial p_i^a}\left((M^a+N^a)^2 - p^{a2}\right) + \sum_{a,k}\frac{\partial N^a}{\partial p_k^a}\left(\frac{\partial k_i}{\partial x_k^a} + \sum_b \frac{\partial M^b}{\partial x_k^a}(x_i^b - x_i^a)\right) \\
&= \frac{1}{2}\sum_a \frac{\partial}{\partial p_i^a}\left(N^{a2} - p^{a2}\right) + \sum_{a,k}\frac{\partial N^a}{\partial p_k^a}\left(\frac{\partial k_i}{\partial x_k^a} + M^a\delta_{ik} + \sum_b \frac{\partial M^b}{\partial x_k^a}(x_i^b - x_i^a)\right) \\
&= \frac{1}{2}\sum_a \frac{\partial}{\partial p_i^a}\left(N^{a2} - p^{a2}\right) + \sum_{a,k}\frac{\partial N^a}{\partial p_k^a}\underbrace{\left(\frac{\partial}{\partial x_k^a}\left(\sum_b M^b x_i^b + k_i\right) - x_i^a\frac{\partial}{\partial x_k^a}\sum_b M^b\right)}_{C_{ki}^a} \qquad (2.32)
\end{aligned}$$

As $k_i(x)$ and $M^b(x)$ are functions of the position only, so is C_{ik}^a. Taking the derivative of the above equation with respect to p_j^c and x_l^b we obtain (using the fact that N^a only depends on p^a)

$$0 = \frac{\partial}{\partial x_l^b}\sum_{a,k}\frac{\partial^2 N^a}{\partial p_j^c \partial p_k^a}C_{ki}^a = \frac{\partial}{\partial x_l^b}\sum_k \frac{\partial^2 N^a}{\partial p_j^a \partial p_k^a}C_{ki}^a,$$

where we have also used (2.18). From the non-degeneracy condition in the theorem follows $\det \frac{\partial^2 N^a}{\partial p_j^a \partial p_k^a} \neq 0$ and thus the inverse exists. Multiplying the above equation from the left with the inverse (which is of course independent of x since N^a is) we obtain

$$\frac{\partial}{\partial x_l^b}C_{ki}^a = 0, \qquad (2.33)$$

so C_{ki}^a is a constant but also transforms as a second rank tensor. Thus it follows that

$$C_{ki}^a = C^a\delta_{ki}, \qquad (2.34)$$

and therefore

$$\begin{aligned}
0 &= \frac{1}{2}\sum_a \frac{\partial}{\partial p_i^a}\left(N^{a2} - p^{a2}\right) + \sum_{a,k}\frac{\partial N^a}{\partial p_k^a}C^a\delta_{ki} \\
&= \frac{1}{2}\sum_a \frac{\partial}{\partial p_i^a}\left(N^{a2} - p^{a2} + 2C^a N^a\right) \\
&= \frac{1}{2}\sum_a \frac{\partial}{\partial p_i^a}\left((N^a + C^a)^2 - p^{a2}\right), \qquad (2.35)
\end{aligned}$$

since $\frac{\partial C^a}{\partial p_i^a} = 0$. Taking the derivative with respect to p_k^b yields

$$\frac{\partial^2}{\partial p_i^a \partial p_k^a}\left((N^a + C^a)^2 - p^{a2}\right) = 0, \qquad (2.36)$$

which has the solution

$$(N^a + c^a)^2 - p^{a2} = \sum_k d_k^a p_k^a + d^a. \tag{2.37}$$

Since h^a and M^a are rotational invariant, so is N^a. Thus the whole left-hand side of the above equation is invariant under rotations and thus d_k^a must transform as a vector, but there is no numerically invariant vector, so d_k^a must be zero and we have

$$(N^a + C^a) = \sqrt{p^{a2} + d^a}. \tag{2.38}$$

Now this looks rather familiar. In [5] it is shown that in fact

$$k_i = \sum_a (C^a - M^a) x_i^a, \ \sum_a (C^a - M^a) = 0. \tag{2.39}$$

So we obtain

$$\begin{aligned} H &= \sum_a (N^a + M^a) = \sum_a \sqrt{p^{a2} + d^a}, \\ K_i &= \sum_a h^a x_i^a + k_i = \sum_a x_i^a \sqrt{p^{a2} + d^a}. \end{aligned} \tag{2.40}$$

The requirement that the velocity of the particles is smaller than the velocity of light further requires that $d^a > 0$, such that we may put $d^a = m_a^2$. Hence we have

$$H = \sum_a \sqrt{p^{a2} + m_a^2}, \ K_i = \sum_a x_i^a \sqrt{p^{a2} + m_a^2} \tag{2.41}$$

and all ten generators have been brought to their usual free particle form.

Chapter 3

Interacting Particles in One Dimension

The proof presented in the preceding chapter relies crucially on the dimension of space-time. As we will show below, there exists a representation of the Hamiltonian form in one space and one time dimension that satisfies the requirements of relativistic invariance and that has a non-vanishing interaction potential.

Let's first write down the classical Poincaré algebra in one dimension:

$$\{P, H\} = 0,\ \{K, P\} = H,\ \{K, H\} = P. \tag{3.1}$$

Explicit invariance of the position x requires

$$\{x^a, P\} = 1,\ \{x^a, K\} = x^a\{x^a, H\}. \tag{3.2}$$

As in the three-dimensional case one can find a suitable canonical transformation which brings P to its free particle form. The bracket $\{x^a, P\}$ implies $\frac{\partial P}{\partial p^a} = 1$ and thus

$$P(p, x) = \sum_{a=1}^{N} p^a + F(x). \tag{3.3}$$

Making the canonical transformation $x^{a\prime} = x^a, p^{a\prime} = p^a + \frac{1}{N}F(x)$ brings P to its free particle form. The second requirement for explicit invariance $\{x^a, K\} = x^a\{x^a, H\}$ leads to

$$\frac{\partial K}{\partial p^a} = x^a \frac{\partial H}{\partial p^a},\ (x^a - x^b)\frac{\partial^2 H}{\partial p^a \partial p^b} = 0, \tag{3.4}$$

and thus

$$H = \sum_{a=1}^{N} h^a(p^a, x),\ K = \sum_{a=1}^{N} x^a h^a(p^a, x) + k(x). \tag{3.5}$$

From now on we restrict ourselves to the simplest case, namely two interacting particles. We make the following ansatz for the generators which satisfies the requirements (3.2) for explicit Lorentz invariance:

$$H = \sqrt{p_1^2 + m_1^2} + \sqrt{p_2^2 + m_2^2} + V(x_1, x_2), \tag{3.6}$$

$$P = p_1 + p_2, \tag{3.7}$$

$$K = x_1\sqrt{p_1^2 + m_1^2} + x_2\sqrt{p_2^2 + m_2^2} + W(x_1, x_2). \tag{3.8}$$

The first restriction on $V(x_1, x_2)$ comes from the bracket $\{P, H\} = 0$

$$\frac{\partial V}{\partial x_1} + \frac{\partial V}{\partial x_2} = 0, \tag{3.9}$$

which implies $V(x_1, x_2) = V(x_1 - x_2)$. From $\{K, P\} = H$ we obtain

$$\frac{\partial W}{\partial x_1} + \frac{\partial W}{\partial x_2} = V(x_1 - x_2). \tag{3.10}$$

The last bracket $\{K, H\} = P$ yields

$$-x_1 \frac{p_1}{\sqrt{p_1^2 + m_1^2}} \frac{\partial V}{\partial x_1} - x_2 \frac{p_2}{\sqrt{p_2^2 + m_2^2}} \frac{\partial V}{\partial x_2} = p_1 + p_2. \tag{3.11}$$

We may write this as

$$\frac{p_1}{\sqrt{p_1^2 + m_1^2}} \left(\frac{\partial W}{\partial x_1} - x_1 \frac{\partial V}{\partial x_1} \right) + \frac{p_2}{\sqrt{p_2^2 + m_2^2}} \left(\frac{\partial W}{\partial x_2} - x_2 \frac{\partial v}{\partial x_2} \right) = 0. \tag{3.12}$$

Since this must be true for any value of p_1 and p_2, we obtain

$$\frac{\partial W}{\partial x_1} = x_1 \frac{\partial V}{\partial x_1}, \quad \frac{\partial W}{\partial x_2} = x_2 \frac{\partial V}{\partial x_2}. \tag{3.13}$$

Inserting (3.13) in (3.10) and using (3.9) yields

$$V(x_1 - x_2) = x_1 \frac{\partial V}{\partial x_1} + x_2 \frac{\partial V}{\partial x_2} = (x_1 - x_2) \frac{\partial V(x_1 - x_2)}{\partial x_1}. \tag{3.14}$$

Introducing $x = x_1 - x_2$ we may write

$$V(x) = x \frac{\partial V}{\partial x}, \tag{3.15}$$

which has as solution

$$V(x) = \sigma |x|. \tag{3.16}$$

In order to obtain $W(x_1, x_2)$ we make the ansatz $W(x_1, x_2) = (a_1 x_1 + a_2 x_2) V(x)$. Inserting into eq. (3.13) and adding the two equations one obtains $(a_1 + a_2) V(x) = x \frac{\partial V}{\partial x}$, i.e. $a_1 + a_2 = 1$. We subtract the two equations one gets from eq. (3.13) by inserting the ansatz and obtain

$$2(a_1 x_1 + a_2 x_2) = (x_1 + x_2). \tag{3.17}$$

Since this must be true for any value of x_1, x_2 we finally get $a_1 = a_2 = \frac{1}{2}$ and thus

$$W(x_1, x_2) = \frac{x_1 + x_2}{2} \sigma |x_1 - x_2|. \tag{3.18}$$

Now this has all been done in classical Hamiltonian particle mechanics. The question is though, whether this system preserves the Poincaré symmetry through the process of quantization. What can be shown [7] is that for the simplest symmetrisation of the boost operator

$$K = \sum_{i=1}^{2} \frac{1}{2} \left(x_i \sqrt{p_i^2 + m_i^2} + \sqrt{p_i^2 + m_i^2} x_i \right) + W(x_1, x_2), \tag{3.19}$$

the Poincaré algebra does not close at the quantum level. However, this does not allow for the conclusion that hence we have a Poincaré anomaly here, as is done in [7]. There might actually exist some complicated ordering that allows the Poincaré algebra to close at the quantum level and that is classically equivalent with the above representation. As will be shown in Chapter 5, the Poincaré algebra does close at the quantum level in light cone coordinates. Our strategy in the next chapter will be to solve the modified Schrödinger equation arising from the Hamiltonian

$$H = \sqrt{p_1^2 + m_1^2} + \sqrt{p_2^2 + m_2^2} + \sigma|x_1 - x_2|, \tag{3.20}$$

and to explicitly check whether the spectrum satisfies the usual relativistic energy-momentum relation $E^2 = M^2 + P^2$.

It is easy to see that the above considerations may be repeated for N particles. We then have the following expressions for the generators

$$H = \sum_{a=1}^{N} \sqrt{p_a^2 + m_a^2} + \sum_{pairs} \sigma|x_a - x_b|, \tag{3.21}$$

$$P = \sum_{a=1} p_a, \tag{3.22}$$

$$K = \sum_{a=1} x_a \sqrt{p_a^2 + m_a^2} + \sum_{pairs} \frac{x_a + x_b}{2} \sigma|x_a - x_b|. \tag{3.23}$$

In what follows we will only consider the two-particle case.

Chapter 4

Linear Confining Potential

In this chapter we try to solve the time independent Schrödinger equation with the relativistic Hamiltonian written down at the end of last chapter. Before we do this, we first solve the time independent Schrödinger equation for this potential with the usual non-relativistic Hamiltonian, since there we may illustrate a method we will use later on in the relativistic case. We then go on to find the semi-classical spectrum for the relativistic Hamiltonian before we eventually tackle the problem fully quantum mechanically.

4.1 Non-Relativistic Schrödinger Equation

In this section we solve the non-relativistic Schrödinger equation for two particles interacting through a linear confining potential. The two-particle Hamiltonian reads

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \sigma|x_1 - x_2|. \tag{4.1}$$

Separating the center of mass motion, as is commonly done when solving the two-particle problem, one arrives at the following time independent Schrödinger equation for the relative coordinate $x = x_1 - x_2$ (we have also set $m_1 = m_2 = m$)

$$H\psi(x) = \left(-\frac{1}{m}\partial_x^2 + \sigma|x|\right)\psi(x) = E\psi(x). \tag{4.2}$$

The usual way to solve this problem is to solve the problem for $x > 0$ and $x < 0$ separately and put it back together using some boundary condition. Denoting with $\psi_+(x)$ the solution for $x > 0$ and with $\psi_-(x)$ the one for $x < 0$ we may write

$$\partial_x^2\psi_\pm(x) = (\sigma x \mp E)\, m\psi_\pm(x). \tag{4.3}$$

Introducing $y = (\pm\sigma x - E)m(\sigma m)^{-\frac{2}{3}}$ we have $dy = \pm(\sigma m)^{\frac{1}{3}}dx$ and thus

$$\partial_y^2\psi_\pm(y) - y\psi_\pm(y) = 0. \tag{4.4}$$

The solutions of this differential equation are the so called Airy functions $\mathrm{Ai}(y)$ and $\mathrm{Bi}(y)$ of which only $\mathrm{Ai}(y)$ is normalizable. So we have the following solutions in the two regions:

$$\psi_\pm(x) = C\mathrm{Ai}\left[(\sigma m)^{\frac{1}{3}}\left(\pm x - \frac{E}{\sigma}\right)\right], \tag{4.5}$$

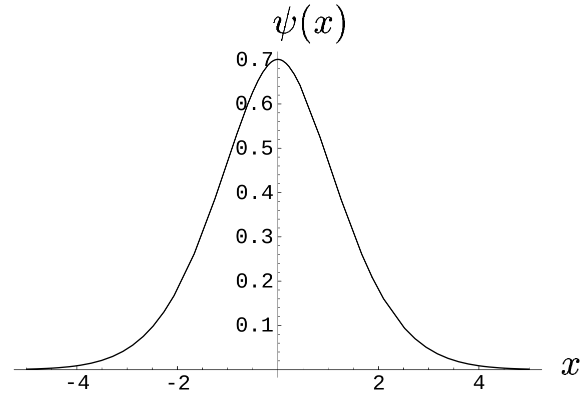

Figure 4.1: **Ground state:** *wave function for* $\sigma = m = 1$.

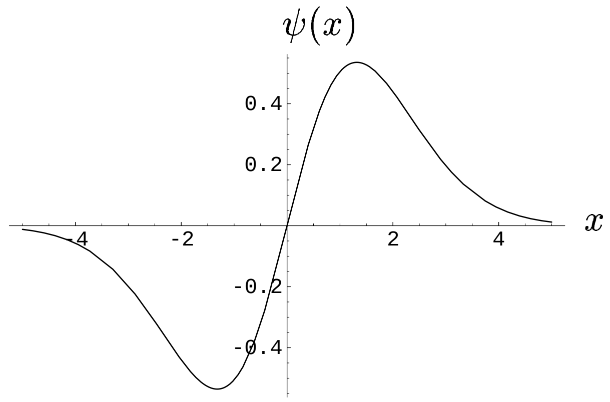

Figure 4.2: **First excited state:** *wave function for* $\sigma = m = 1$.

where normalization fixes the factor C. The energy spectrum is determined through the boundary condition at $x = 0$. One sees from the original equation (4.2) that the solution must be twice differentiable where the second derivative has a cusp at $x = 0$. Parity symmetry only allows even or odd solutions so we have the boundary condition $\psi(x)|_{x=0} = 0$ for the odd solutions and $\psi'(x)|_{x=0}$ for the even solutions. For $\sigma = m = 1$ the first few values following from these boundary conditions are: (1.01879, 2.33811,3.2482,4.08795,4.8201). The normalized wave functions for the ground state and the first excited state are shown in Fig. 4.1 and Fig. 4.2, respectively.

Since we will later on use this method, let's show that in this case one may solve the problem in a different way. We again consider the equation separately for $x > 0$ and $x < 0$:

$$\left(\frac{p^2}{m} \pm \sigma x \right) \psi_{\pm}(x) = E\psi_{\pm}(x). \tag{4.6}$$

Instead of solving these two equations as before, we first go into momentum space where they read

$$\left(\frac{p^2}{m} \pm i\sigma\partial_p \right) \tilde{\psi}_{\pm}(p) = E\tilde{\psi}_{\pm}(p), \tag{4.7}$$

and lead to the following first order differential equation:

$$\partial_p \tilde{\psi}_{\pm}(p) = \mp \frac{i}{\sigma}\left(E - \frac{p^2}{m}\right)\tilde{\psi}_{\pm}(p), \tag{4.8}$$

which has the unique solution

$$\tilde{\psi}_{\pm}(p) = \psi_0 \exp\left(\mp \frac{i}{\sigma}\left(Ep - \frac{p^3}{3m}\right)\right). \tag{4.9}$$

Transforming back to coordinate space we obtain

$$\begin{aligned} \psi_{\pm}(x) &= \frac{\psi_0}{\pi}\int_0^{\infty} \cos\left(\frac{p^3}{3\sigma m} \pm p\left(x \mp \frac{E}{\sigma}\right)\right) dp \\ &= \psi_0 (\sigma m)^{\frac{1}{3}} \mathrm{Ai}\left[\pm(\sigma m)^{\frac{1}{3}}\left(x \mp \frac{E}{\sigma}\right)\right], \end{aligned} \tag{4.10}$$

where ψ_0 is determined through the normalization. This is the same solution as above.

4.2 Semi-Classical Non-Relativistic Calculation

The Bohr-Sommerfeld quantization condition is given by

$$J = \oint p\ dx = \int_T p\dot{x}\ dt = 2\pi n, \tag{4.11}$$

where n is a positive integer and T is one period of motion. In this section we do the calculation for the non-relativistic Schrödinger equation so that we afterwards may compare with the non-relativistic limit of the result obtained in the relativistic calculation, which we do in the next section. For simplicity we put $m_1 = m_2$, but the calculation may also be carried out for $m_1 \neq m_2$. The Hamiltonian in relative coordinates is given by ($m_1 = m_2$)

$$H = \frac{p^2}{m} + \sigma|x|. \tag{4.12}$$

The equations of motion are

$$\begin{aligned} \dot{x} &= \{x, H\} = \frac{\partial H}{\partial p} = \frac{2p}{m}, \\ \dot{p} &= \{p, H\} = -\frac{\partial H}{\partial x} = -\sigma \mathrm{sgn}(x), \end{aligned} \tag{4.13}$$

where sgn(x) denotes the sign function. For $x > 0, x < 0$ respectively, we have the following solutions:

$$\begin{aligned} x_{\pm}(t) &= \frac{1}{m}\left(\mp\sigma t^2 + 2p(0)t\right) + x(0), \\ p_{\pm}(t) &= \mp\sigma t + p(0). \end{aligned} \tag{4.14}$$

Choosing the initial conditions such that the two particles are furthest apart, one period of motion has four parts. During the first part the two particles move towards each other until they meet and we have $x(T/4) = 0$. In the second part we have the first part reversed: the

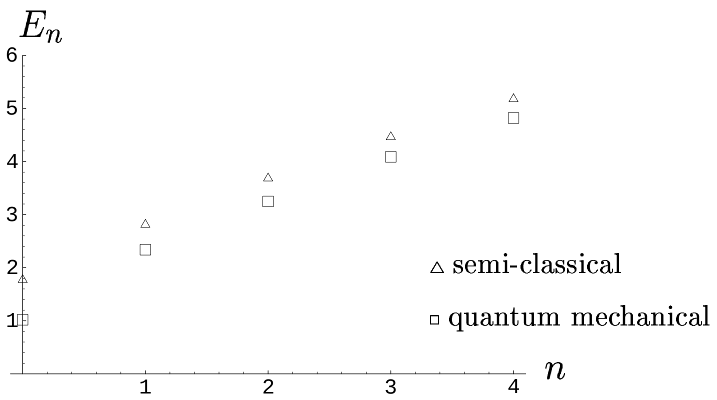

Figure 4.3: **Semi-classical vs. quantum mechanical calculation:** *spectrum from the quantum mechanical calculation compared to the result of the semi-classical Bohr-Sommerfeld approximation for $\sigma = m = 1$.*

particles start at the same point and move away from each other until they are furthest apart and we have the same situation as at the beginning only with the two particles interchanged. The third part then corresponds to the first, and the fourth to the second only with particles interchanged. The action integral J, however, is the same in all parts, since comparing first and second part p and $\dot{x}$ both change sign, which does not alter the integrand. Thus we only need to integrate over the first quarter of a period and multiply it by four:

$$J = \int_T p\dot{x}\; dt = 4\int_0^{\frac{T}{4}} p\dot{x}\; dt. \tag{4.15}$$

Starting with the two particles furthest apart means $x(0) = x_{\max} = \frac{H}{\sigma}$ and $p(0) = 0$. Inserting into $x_+(t)$ and setting $x_+(t) = 0$ we have $\frac{T}{4} = \frac{\sqrt{Hm}}{\sigma}$. Thus we obtain

$$J = 4\int_0^{\frac{\sqrt{Hm}}{\sigma}} (-\sigma t)\left(-\frac{2\sigma t}{m}\right) dt = 4\int_0^{\frac{\sqrt{Hm}}{\sigma}} \frac{2\sigma^2 t^2}{m} dt = \frac{8}{3\sigma m}(Hm)^{\frac{3}{2}}. \tag{4.16}$$

Applying the quantization condition we obtain the following spectrum:

$$E_{n-1} = \left(\frac{9}{16m}\pi^2\sigma^2 n^2\right)^{\frac{1}{3}}, \quad n = 1, 2, \ldots \quad . \tag{4.17}$$

We put E_{n-1} because we denote the ground state energy by E_0 and in the semi-classical approximation $n = 1, 2, \ldots$. In Fig. 4.3 the semi-classical spectrum is plotted against the quantum mechanical spectrum we obtained in Section (4.1).

4.3 Semi-Classical Relativistic Calculation

For two relativistic particles interacting through a linear potential, we have the following Hamiltonian as derived in Chapter 3:

$$H = \sqrt{p_1^2 + m_1^2} + \sqrt{p_2^2 + m_2^2} + \sigma|x_1 - x_2|. \tag{4.18}$$

Introducing relative and "center of mass" coordinates $x = x_1 - x_2, p = \frac{p_1 - p_2}{2}, P = p_1 + p_2$ we have

$$H = \frac{1}{2}\sqrt{(P+2p)^2 + 4m^2} + \frac{1}{2}\sqrt{(P-2p)^2 + 4m^2} + \sigma|x|. \tag{4.19}$$

The equations of motion for the relative coordinates are

$$\begin{aligned} \dot{x} &= \{x, H\} = \frac{P+2p}{\sqrt{(P+2p)^2+4m^2}} - \frac{P-2p}{\sqrt{(P-2p)^2+4m^2}}, \\ \dot{p} &= \{p, H\} = -\sigma \mathrm{sgn}(x). \end{aligned} \tag{4.20}$$

For $x > 0$ we have the following solution

$$\begin{aligned} p(t) &= -\sigma t, \\ x(t) &= -\frac{1}{2\sigma}\left(\sqrt{(P-2\sigma t)^2+4m^2} + \sqrt{(P+2\sigma t)^2+4m^2}\right) + \text{const.} \end{aligned} \tag{4.21}$$

We again choose the initial condition such that $x(0) = x_{\max}$, i.e. at the start the particles are furthest apart. This sets the constant in $x(t)$ to $\frac{H}{\sigma}$. After a quarter of a period they meet each other, after two quarters they have swapped places comparing to the start and after three quarters they meet again. As in the non-relativistic case, all quarters give again the same contribution. The value of $\frac{T}{4}$ is determined through $x(t) = 0$ and is given by

$$\frac{T}{4} = \frac{H}{2\sigma}\sqrt{\frac{H^2 - P^2 - 4m^2}{H^2 - P^2}}. \tag{4.22}$$

We thus obtain

$$J = 4\int_0^{\frac{T}{4}} p\dot{x}\; dt = 4\int_0^{\frac{T}{4}} \left(\frac{\sigma t\,(P+2\sigma t)}{\sqrt{(P+2\sigma t)^2+4m^2}} - \frac{\sigma t\,(P-2\sigma t)}{\sqrt{(P-2\sigma t)^2+4m^2}}\right) dt. \tag{4.23}$$

Introducing the rescaled variable $s = 2\sigma t$ and putting $s_0 = 2\sigma\frac{T}{4} = H\sqrt{\frac{H^2-P^2-4m^2}{H^2-P^2}}$ we obtain

$$\begin{aligned} J &= \frac{1}{\sigma}\left\{\left(s\sqrt{(P+s)^2+4m^2} + s\sqrt{(P-s)^2+4m^2}\right)\Big|_0^{s_0}\right. \\ &\quad \left. - \int_0^{s_0}\left(\sqrt{(P+s)^2+4m^2} + \sqrt{(P-s)^2+4m^2}\right) ds\right\} \\ &= \frac{1}{\sigma}\left\{\frac{(P+s)}{2}\sqrt{(P-s)^2+4m^2} - \frac{(P-s)}{2}\sqrt{(P+s)^2+4m^2}\right. \\ &\quad \left. +2m^2\mathrm{Arsh}\left[\frac{P-s}{2m}\right] - 2m^2\mathrm{Arsh}\left[\frac{P+s}{2m}\right]\right\}\Big|_0^{s_0} \\ &= \frac{1}{\sigma}\left\{\frac{(P+s_0)}{2}\sqrt{(P-s_0)^2+4m^2} - \frac{(P-s_0)}{2}\sqrt{(P+s_0)^2+4m^2}\right. \\ &\quad \left. +2m^2\mathrm{Arsh}\left[\frac{P-s_0}{2m}\right] - 2m^2\mathrm{Arsh}\left[\frac{P+s_0}{2m}\right]\right\} \\ &= 2\pi n. \end{aligned} \tag{4.24}$$

In the limit $m \to 0$ we have $s_0 = H$ and the above expression simplifies to

$$J = \frac{1}{2\sigma}\left((P+H)|P-H| - (P-H)|P+H|\right). \tag{4.25}$$

Since $H > 0$ and $|H| > |P|$ we may write

$$J = \frac{1}{\sigma}\left(H^2 - P^2\right) = \frac{M^2}{\sigma}, \tag{4.26}$$

which yields the following spectrum for the ultra-relativistic case

$$M_{n-1} = \sqrt{2\pi\sigma n}, \quad n = 1, 2, \dots \quad . \tag{4.27}$$

As above we put M_{n-1} because we want to denote the ground state energy by M_0. In general we have not been able to show analytically that the quantization condition (4.24) is only on M, i.e. that when inserting $H^2 = M^2 + P^2$, J is independent of P. However, up to machine precision Mathematica tells us that this is indeed the case. Hence we may put P to zero. We then have $s_0 = \sqrt{M^2 - 4m^2}$ and

$$J = \frac{1}{\sigma}\left(M\sqrt{M^2-4m^2} - 4m^2 \mathrm{Arsh}\left[\frac{\sqrt{M^2-4m^2}}{2m}\right]\right) = 2\pi n. \tag{4.28}$$

In general this equation can only be solved numerically. We may though have a look at the non-relativistic limit. In order to do that we first need to know the expansion parameter. For $P = 0$ we have $p_1 = \frac{P}{2} + p = p$ and therefore

$$|\dot{x}_1| = \left|\frac{\partial H}{\partial p_1}\right| = \left|\frac{p_1}{\sqrt{p_1^2 + m^2}}\right| = \left|\frac{p}{\sqrt{p^2+m^2}}\right|. \tag{4.29}$$

The particle velocity arrives at its maximum when the two particles meet, i.e. we have

$$|\dot{x}_1|_{\max} - \frac{M^2 - 4m^2}{2M}. \tag{4.30}$$

We may thus put $M = 2m + \epsilon$ and expand in terms of $\frac{\epsilon}{m}$. Keeping only terms up to $\left(\frac{\epsilon}{m}\right)^{\frac{3}{2}}$ we obtain

$$J = \frac{8m^2}{3\sigma}\left(\frac{\epsilon}{m}\right)^{\frac{3}{2}} = 2\pi n. \tag{4.31}$$

So in the non-relativistic limit we have the following spectrum:

$$M_{n-1} = 2m + \left(\frac{9}{16m}\pi^2\sigma^2 n^2\right)^{\frac{1}{3}}, \quad n = 1, 2, \dots \quad , \tag{4.32}$$

which is the same as we obtained in the semi-classical approximation for the non-relativistic Schrödinger equation in Section (4.2).

4.4 Hermite Function Expansion

After the semi-classical approximation we now want to tackle the problem fully quantum mechanically. The difficulty of this problem arises from the kinetic terms with the square root of $p^2 + m^2$. It is not at all clear how to treat this at the operator level. We concentrate on the ultra-relativistic case of massless particles for which the Hamiltonian reads

$$H = |p_1| + |p_2| + \sigma|x_1 - x_2|. \tag{4.33}$$

Again, we introduce relative and "center of mass" coordinates $x = x_1 - x_2$, $P = p_1 + p_2$, $p = \frac{p_1 - p_2}{2}$ such that the Hamiltonian becomes

$$H = \left|\frac{P}{2} + p\right| + \left|\frac{P}{2} - p\right| + \sigma|x|. \tag{4.34}$$

Going to the frame with zero total momentum we obtain the following equation

$$H|\psi\rangle = (2|p| + \sigma|x|)|\psi\rangle = E|\psi\rangle. \tag{4.35}$$

Introducing $\epsilon = \frac{E}{\sqrt{2\sigma}}$, $q = \sqrt{\frac{2}{\sigma}}p$, $y = \sqrt{\frac{\sigma}{2}}x$ the above equation simplifies to

$$H|\psi\rangle = (|q| + |y|)|\psi\rangle = \epsilon|\psi\rangle. \tag{4.36}$$

This Hamiltonian is symmetric under exchange of the rescaled position y and the rescaled momentum q. So the solutions of this equation must be the same in position space and in momentum space, i.e. eigenfunctions under Fourier transformation. A complete set of eigenfunctions of the Fourier transformation are the so called Hermite functions (for the proof see Appendix C). The Fourier transform of

$$\phi_n(y) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(y) \exp(-\frac{y^2}{2}) \tag{4.37}$$

is given by

$$\tilde{\phi}_n(q) = \frac{\sqrt{2\pi}}{\sqrt{2^n n! \sqrt{\pi}}} (-i)^n H_n(q) \exp(-\frac{q^2}{2}), \tag{4.38}$$

where H_n are the Hermite polynomials. Due to the symmetry of the Hamiltonian its eigenfunctions are a linear combination of Hermite functions to the same eigenvalue. Our strategy is thus to calculate matrix elements of the form

$$T_{nm} = \langle\phi_n|H|\phi_m\rangle = \langle\phi_n||q||\phi_m\rangle + \langle\phi_n||y||\phi_m\rangle \tag{4.39}$$

up to a certain order and to then diagonalize the matrix. The first term may be calculated in position space and the second in momentum space if one takes into account the eigenvalues, i.e. we have

$$\begin{aligned} T_{nm} &= \int_{-\infty}^{\infty} \phi_n(y)(|q| + |y|)\phi_m(y)dy \\ &= \int_{-\infty}^{\infty} \phi_n(y)|y|\phi_m(y)dy + \frac{1}{2\pi}\int_{-\infty}^{\infty} (i)^n \tilde{\phi}_n(q)|q|(-i)^m \tilde{\phi}_m(q)dq. \end{aligned} \tag{4.40}$$

So we need to calculate integrals of the form

$$I_{nm} = \int_{-\infty}^{\infty} |x| H_n(x) H_m(x) \exp(-x^2) dx, \tag{4.41}$$

which are different from zero only if n and m are both even or odd. In the even case we obtain

$$\begin{aligned} I_{2n2m} &= \int_0^{\infty} 2x H_{2n}(x) H_{2m}(x) \exp(-x^2) dx = -\int_0^{\infty} (\partial_x \exp(-x^2)) H_{2n}(x) H_{2m}(x) dx \\ &= -\exp(-x^2) H_{2n}(x) H_{2m}(x)\Big|_0^{\infty} \\ &\quad + \int_0^{\infty} \exp(-x^2) \left((\partial_x H_{2n}(x)) H_{2m}(x) + H_{2n}(x)(\partial_x H_{2m}(x))\right) dx. \end{aligned} \tag{4.42}$$

Since $\partial_x H_n(x) = 2n H_{n-1}(x)$ we obtain

$$\begin{aligned} I_{2n2m} &= \int_0^{\infty} \exp(-x^2) \left(4n H_{2n-1}(x) H_{2m}(x) + 4m H_{2n}(x) H_{2m-1}(x)\right) dx \\ &\quad + H_{2n}(0) H_{2m}(0). \end{aligned} \tag{4.43}$$

Using $H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{-\frac{1}{2}}(x^2)$ and $H_{2m-1}(x) = (-1)^{m-1} 2^{2(m-1)+1} (m-1)! x L_{m-1}^{\frac{1}{2}}(x^2)$, where $L_n^{\alpha}(x)$ are the generalized Laguerre polynomials, and putting $z = x^2$ leads to

$$\begin{aligned} I_{2n2m} &= (-1)^{m+n-1} 2^{2n+2m} m! n! \int_0^{\infty} \exp(-z) \left(L_{n-1}^{\frac{1}{2}}(z) L_m^{-\frac{1}{2}}(z) + L_n^{-\frac{1}{2}}(z) L_{m-1}^{\frac{1}{2}}(z)\right) dz \\ &\quad + H_{2n}(0) H_{2m}(0). \end{aligned} \tag{4.44}$$

This integration may be carried out to give

$$I_{2n2m} = H_{2n}(0) H_{2m}(0) + 2^{2n+2m} m! n! \left(\binom{n-\frac{1}{2}}{m} \binom{m-\frac{1}{2}}{n-1} + \binom{n-\frac{1}{2}}{m-1} \binom{m-\frac{1}{2}}{n} \right), \tag{4.45}$$

which may be simplified to

$$\begin{aligned} I_{2n2m} &= 2^{2m+2n} \Gamma\left(\frac{2n}{2} + \frac{1}{2}\right) \Gamma\left(\frac{2m}{2} + \frac{1}{2}\right) \frac{\cos\left(\frac{\pi}{2}(2n-2m)\right)}{\pi\left(1 - (2m-2n)^2\right)} \left((2m-2n)^2 + 2n + 2m\right) \\ &\quad + H_{2n}(0) H_{2m}(0). \end{aligned} \tag{4.46}$$

The same calculation in the odd case leads to

$$\begin{aligned} I_{(2n+1)(2m+1)} &= 2^{(2n+1)+(2m+1)} \frac{\cos\left(\pi\left(\frac{(2n+1)-(2m+1)}{2}\right)\right)}{\pi\left(1 - ((2n+1)-(2m+1))^2\right)} (2n+1)(2m+1) \\ &\quad \Gamma\left(n + \frac{1}{2}\right) \Gamma\left(m + \frac{1}{2}\right). \end{aligned} \tag{4.47}$$

Taking into account the eigenvalue of the Hermite functions under Fourier transformation and the normalization factor, we get the following results for the matrix elements in the even

Table 4.1: **Eigenvalues for $P = 0$ and $\sigma = 1$.** *The order is given by the degree of the Hermite function up to which matrix elements were calculated.*

Order	E_0	E_1	E_2	E_3	E_4
20	1.562065	3.157103	3.922112	4.709790	5.307030
50	1.561470	3.156936	3.921421	4.709372	5.305028
80	1.561426	3.156931	3.921371	4.709357	5.304983
100	1.561415	3.156930	3.921362	4.709356	5.304974
300	1.561399	3.156930	3.921351	4.709354	5.304964
500	1.561398	3.156930	3.921350	4.709354	5.304964
1000	1.561397	3.156930	3.921350	4.709354	5.304963

and odd case:

$$\begin{aligned} T_{2n2m} &= \frac{1+(i)^{2n}(-i)^{2m}}{\sqrt{2^{2n+2m}(2n)!(2m)!\pi}} \left(H_{2n}(0)H_{2m}(0) + 2^{2n+2m}\Gamma\left(n+\frac{1}{2}\right) \right. \\ &\quad \left. \Gamma\left(m+\frac{1}{2}\right) \frac{\cos\left(\pi(n-m)\right)}{\pi\left(1-(2n-2m)^2\right)} \left((2n-2m)^2+2n+2m\right) \right), \\ T_{(2n+1)(2m+1)} &= \frac{1+(i)^{2n+1}(-i)^{2m+1}}{\sqrt{2^{(2n+1)+(2m+1)}(2n+1)!(2m+1)!\pi}} \left(\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right) \right. \\ &\quad \left. 2^{(2n+1)+(2m+1)} \frac{\cos\left(\pi(n-m)\right)}{\pi\left(1-(2n-2m)^2\right)}(2n+1)(2m+1) \right). \end{aligned} \tag{4.48}$$

Using these formulae one may generate a large matrix rather efficiently. The more difficult problem is to diagonalize the matrix. For this we used the NAG routine F02FCF. The results for the energy $E = \sqrt{2\sigma}\epsilon$ in the case of $\sigma = 1$ are listed in Table 4.1. The eigenvalues converge surprisingly quickly.

The question if they are Lorentz invariant, however, remains since we have not been able to show that the Poincaré algebra also closes at the quantum level. Thus we also have to do this calculation for non-zero total momentum, i.e. for the Hamiltonian

$$H = \left|\frac{P}{2}+p\right| + \left|\frac{P}{2}-p\right| + \sigma|x|. \tag{4.49}$$

Doing the same rescaling as above ($\epsilon = \frac{E}{\sqrt{2\sigma}}$, $q = \sqrt{\frac{2}{\sigma}}p$, $y = \sqrt{\frac{\sigma}{2}}x$) we obtain

$$H|\psi\rangle = \left(\left|\frac{P}{2\sqrt{2\sigma}} + \frac{q}{2}\right| + \left|\frac{P}{2\sqrt{2\sigma}} - \frac{q}{2}\right| + |y| \right)|\psi\rangle = \epsilon|\psi\rangle \tag{4.50}$$

This Hamiltonian is of course not invariant under Fourier transformation anymore. Since, however, the Hermite functions form a complete basis of square integrable functions we may still expand in them and hope that convergence is reasonable. The problem is that now the integrals may not be solved analytically as easily as before. So to calculate each matrix element we have to integrate. We did this using a numerical integration routine from the

Table 4.2: **Dependence of E_0 on the total momentum P.** *The last row gives the values calculated from the ground state energy through the relativistic energy-momentum dependence. As ground state energy we took the one from the order 80 calculation.*

Order	$P=0$	$P=1$	$P=2$	$P=3$	$P=4$
20	1.56207	1.92513	2.65264	3.50520	4.41298
50	1.56147	1.92481	2.65216	3.50440	4.41178
80	1.56143	1.92478	2.65212	3.50434	4.41170
$\sqrt{M^2+P^2}$	1.56143	1.85420	2.53733	3.38202	4.29396

NAG library (D01AMF) and to determine the eigenvalues we used the same routine as above. Unfortunately round-off errors in the integration routine prevent us from going to such high order as in the zero momentum frame, but convergence is still good enough so that we can already be certain at this order about the main result, namely, that the spectrum is not Lorentz invariant. In Table 4.2 we have the results for E (again in the case $\sigma = 1$) for different values of the total momentum P and different orders of calculation. We see that convergence is pretty good. The discrepancy between the values obtained through the matrix element calculation for non-zero P (first three rows) and the values obtained by the relativistic energy-momentum relation $E^2 = M^2+P^2$ (last row) is too big that convergence could account for it.

This expansion in terms of Hermite functions requires no boundary condition, the eigenvalues are real, and the eigenfunctions orthogonal. So one point of view is that this solves the problem completely and proves that we have a Poincaré anomaly here. The other point of view is that this description is somehow too rigid, i.e. that there is some degree of freedom which determines the boundary condition at the origin and which is set to some particular value with this method (i.e. $\psi(0) = 0, \psi'(0) = 0$, respectively).

4.5 Auxiliary Fresnel Functions

In Section 4.1 we solved the non-relativistic Schrödinger equation in two different ways. Let's now apply the second method also to the relativistic case, i.e. we consider the equation for $x > 0$ and $x < 0$ separately, solve it in momentum space, and then Fourier transform back to position space to obtain the solution in position space. The main problem is then to find the right boundary condition.

In relative coordinates we have the following Hamiltonian

$$H = \left|\frac{P}{2} + p\right| + \left|\frac{P}{2} - p\right| + \sigma|x|. \tag{4.51}$$

We consider the time independent Schrödinger equation for $x > 0$ and $x < 0$ separately, i.e. we have

$$H\psi_\pm(x) = \left(\left|\frac{P}{2} + p\right| + \left|\frac{P}{2} - p\right| \pm \sigma x\right)\psi_\pm(x). \tag{4.52}$$

In momentum space this is

$$H\tilde{\psi}_{\pm}(p) = \left(\left| \frac{P}{2} + p \right| + \left| \frac{P}{2} - p \right| \pm i\sigma\partial_p \right) \tilde{\psi}_{\pm}(p) = E\tilde{\psi}_{\pm}(p), \tag{4.53}$$

which leads to the following differential equation

$$\frac{\partial \tilde{\psi}_{\pm}(p)}{\partial p} = \mp \frac{i}{\sigma} \left(E - \left| \frac{P}{2} + p \right| - \left| \frac{P}{2} - p \right| \right) \tilde{\psi}_{\pm}(p). \tag{4.54}$$

This equation has of course the following solution

$$\tilde{\psi}_{\pm}(p) = \psi_0 \exp\left(\mp \frac{i}{\sigma} \int_0^p \left(E - \left| \frac{P}{2} + p' \right| - \left| \frac{P}{2} - p' \right| \right) dp' \right). \tag{4.55}$$

In order to obtain the explicit solution we must perform the integration in the exponent. This depends on the value of $\frac{P}{2}$:

$$\tilde{\psi}_{\pm}(p) = \begin{cases} \psi_0 \exp\left(\mp \frac{i}{\sigma} \left(Ep - p^2 - \frac{P^2}{4} \right) \right) & : p \quad > \quad \frac{P}{2} \\ \psi_0 \exp\left(\mp \frac{i}{\sigma} \left(E - P \right) p \right) & : |p| \quad \leq \quad \frac{P}{2} \\ \psi_0 \exp\left(\mp \frac{i}{\sigma} \left(Ep + p^2 + \frac{P^2}{4} \right) \right) & : p \quad < \quad -\frac{P}{2} \end{cases}. \tag{4.56}$$

We now obtain $\psi_{\pm}(x)$ through Fourier transformation

$$\begin{aligned} \psi_{\pm}(x) &= \frac{\psi_0}{2\pi} \left(\int_{-\infty}^{-\frac{P}{2}} \exp\left(\mp \frac{i}{\sigma} \left(Ep + p^2 + \frac{P^2}{4} \right) + ipx \right) dp \right. \\ &\quad + \int_{-\frac{P}{2}}^{\frac{P}{2}} \exp\left(\mp \frac{i}{\sigma} \left(E - P \right) p + ipx \right) dp \\ &\quad \left. + \int_{\frac{P}{2}}^{\infty} \exp\left(\mp \frac{i}{\sigma} \left(Ep - p^2 - \frac{P^2}{4} \right) + ipx \right) dp \right) \\ &= \frac{\psi_0}{2\pi} \left(\int_{\frac{P}{2}}^{\infty} \exp\left(\mp \frac{i}{\sigma} \left(-Ep + p^2 + \frac{P^2}{4} \right) - ipx \right) dp \right. \\ &\quad + 2 \int_0^{\frac{P}{2}} \cos\left(\mp \frac{1}{\sigma} \left(E - P \right) p + px \right) dp \\ &\quad \left. + \int_{\frac{P}{2}}^{\infty} \exp\left(\mp \frac{i}{\sigma} \left(Ep - p^2 - \frac{P^2}{4} \right) + ipx \right) dp \right) \\ &= \frac{\psi_0}{2\pi} \left(2 \int_{\frac{P}{2}}^{\infty} \cos\left(\mp \frac{1}{\sigma} \left(Ep - p^2 - \frac{P^2}{4} \right) + px \right) dp \right. \\ &\quad \left. + 2 \int_0^{\frac{P}{2}} \cos\left(\mp \frac{1}{\sigma} \left(E - P \right) p + px \right) dp \right). \end{aligned} \tag{4.57}$$

Using

$$\begin{aligned} \int \cos\left(ax^2 + 2bx + c \right) dx &= \sqrt{\frac{\pi}{2a}} \left(\cos\left(\frac{b^2 - ac}{a} \right) \mathrm{C}\left[\sqrt{\frac{2}{a\pi}} \left(ax + b \right) \right] \right. \\ &\quad \left. + \sin\left(\frac{b^2 - ac}{a} \right) \mathrm{S}\left[\sqrt{\frac{2}{a\pi}} \left(ax + b \right) \right] \right), \end{aligned} \tag{4.58}$$

where C and S are the so called Fresnel functions

$$\mathrm{C}(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt, \quad \mathrm{S}(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt, \tag{4.59}$$

we obtain for $\psi_\pm(x)$

$$\begin{aligned}\psi_\pm(x) = \frac{\psi_0}{\sqrt{\pi}}\sqrt{\frac{\sigma}{2}}\Bigg\{&\cos\left(\frac{\sigma}{4}\left(\frac{E}{\sigma}\mp x\right)^2 - \frac{P^2}{4\sigma}\right)\left(\frac{1}{2} - \mathrm{C}\left[\sqrt{\frac{2\sigma}{\pi}}\left(\frac{P}{2\sigma} - \frac{1}{2}\left(\frac{E}{\sigma}\mp x\right)\right)\right]\right) \\ &+ \sin\left(\frac{\sigma}{4}\left(\frac{E}{\sigma}\mp x\right)^2 - \frac{P^2}{4\sigma}\right)\left(\frac{1}{2} - \mathrm{S}\left[\sqrt{\frac{2\sigma}{\pi}}\left(\frac{P}{2\sigma} - \frac{1}{2}\left(\frac{E}{\sigma}\mp x\right)\right)\right]\right) \\ &+ \sqrt{\frac{2}{\pi\sigma}}\frac{\sin\left(\frac{P}{2\sigma}(E-P) \mp \frac{Px}{2}\right)}{\frac{1}{\sigma}(E-P)\mp x}\Bigg\} \end{aligned}\tag{4.60}$$

For $P = 0$ the energy eigenvalue E equals the rest mass M because of $E^2 = M^2 + P^2$. So in order to remind us that we are in the rest frame of the relativistic calculation we write M instead of E whenever we put $P = 0$. We do this now and obtain

$$\begin{aligned}\psi_\pm(x) = \frac{\psi_0}{\sqrt{\pi}}\sqrt{\frac{\sigma}{2}}\Bigg\{&\cos\left(\frac{\sigma}{4}\left(\frac{M}{\sigma}\mp x\right)^2\right)\left(\frac{1}{2} - \mathrm{C}\left[\sqrt{\frac{2\sigma}{\pi}}\left(-\frac{1}{2}\left(\frac{M}{\sigma}\mp x\right)\right)\right]\right) \\ &+ \sin\left(\frac{\sigma}{4}\left(\frac{M}{\sigma}\mp x\right)^2\right)\left(\frac{1}{2} - \mathrm{S}\left[\sqrt{\frac{2\sigma}{\pi}}\left(-\frac{1}{2}\left(\frac{M}{\sigma}\mp x\right)\right)\right]\right)\Bigg\}. \end{aligned}\tag{4.61}$$

Introducing $z_\pm = -\sqrt{\frac{\sigma}{2\pi}}\left(\frac{M}{\sigma}\mp x\right)$ we may write (again for $P = 0$)

$$\begin{aligned}\psi_\pm(x) &= \frac{\psi_0}{\sqrt{\pi}}\sqrt{\frac{\sigma}{2}}\left\{\cos\left(\frac{\pi}{2}z_\pm^2\right)\left(\frac{1}{2} - \mathrm{C}[z_\pm]\right) + \sin\left(\frac{\pi}{2}z_\pm^2\right)\left(\frac{1}{2} - \mathrm{S}[z_\pm]\right)\right\} \\ &= \frac{\psi_0}{\sqrt{\pi}}\sqrt{\frac{\sigma}{2}}g(z_\pm), \end{aligned}\tag{4.62}$$

where $g(z)$ is one of the two auxiliary Fresnel functions, with the other being

$$f(z) = \cos\left(\frac{\pi}{2}z^2\right)\left(\frac{1}{2} - \mathrm{S}[z]\right) - \sin\left(\frac{\pi}{2}z^2\right)\left(\frac{1}{2} - \mathrm{C}[z]\right). \tag{4.63}$$

The two auxiliary Fresnel functions $g(z)$ and $f(z)$ are connected through their derivatives

$$\frac{dg(z)}{dz} = \pi z f(z) - 1, \quad \frac{df(z)}{dz} = -\pi z g(z). \tag{4.64}$$

The difficult question now is what boundary condition one has to impose at $x = 0$ and how this boundary condition depends on the total momentum P. The Hamiltonian has parity symmetry, so the wave functions must be either even or odd. Since $\psi_-(-x) = \psi_+(x)$, one may easily construct even and odd functions. One possible route to follow is to use the symmetry under Fourier transformation for $P = 0$ that we described in the last section. One can show (see Appendix D) that symmetry under Fourier transformation implies $\psi(0) = 0$ for odd functions and $\psi'(0) = 0$ for even functions. This is true as long as $\psi(x)$ is a square integrable function, which is the case for the auxiliary Fresnel functions. So we have the following boundary condition for odd functions:

$$\cos\left(\frac{M^2}{4\sigma}\right)\left(\frac{1}{2}-\mathrm{C}\left[-\frac{M}{\sqrt{2\pi\sigma}}\right]\right)+\sin\left(\frac{M^2}{4\sigma}\right)\left(\frac{1}{2}-\mathrm{S}\left[-\frac{M}{\sqrt{2\pi\sigma}}\right]\right)=0. \tag{4.65}$$

For large values of M the Fresnel functions go to $\frac{1}{2}$, so the condition simplifies to

$$\tan\left(\frac{M^2}{4\sigma}\right)=-1, \tag{4.66}$$

which leads to the following spectrum

$$M=\sqrt{2\pi\sigma\left(2n+\frac{3}{2}\right)},\quad n=0,1,2,\ldots\quad . \tag{4.67}$$

For the even functions the condition is $\psi'(0)=0$ which implies

$$-\frac{M}{2}\cos\left(\frac{M^2}{4\sigma}\right)\left(\frac{1}{2}+\mathrm{S}\left[\frac{M}{\sqrt{2\pi\sigma}}\right]\right)+\frac{M}{2}\sin\left(\frac{M^2}{4\sigma}\right)\left(\frac{1}{2}+\mathrm{C}\left[\frac{M}{\sqrt{2\pi\sigma}}\right]\right)-\sqrt{\frac{\sigma}{2\pi}}=0. \tag{4.68}$$

For large M this simplifies to

$$\tan\left(\frac{M^2}{4\sigma}\right)=1, \tag{4.69}$$

which leads to the following spectrum

$$M=\sqrt{2\pi\sigma\left(2n+\frac{1}{2}\right)},\quad n=0,1,2,\ldots\quad . \tag{4.70}$$

So altogether we obtain for large values of M the following spectrum

$$M_n=\sqrt{2\pi\sigma\left(n+\frac{1}{2}\right)},\quad n=0,1,2,\ldots\quad . \tag{4.71}$$

So at least in the classical limit this spectrum is consistent with the semi-classical approximation. To obtain the low lying part of the spectrum one needs to numerically solve the equations above. In the case of $\sigma=1$ we get (1.84644,3.05106,3.96079,4.69371,5.31788). One important check whether this route is correct is to see whether the eigenfunctions for these eigenvalues are orthogonal and unfortunately this is not the case. The scalar products are all of the order of $10^{-2}-10^{-3}$. So assuming symmetry under Fourier transformation for $P=0$ does not lead to an orthogonal set of eigenfunctions, hence the Hamiltonian with this boundary condition in the frame with $P=0$ is not self-adjoint. When defined on $x\in(-\infty,\infty)$, the operator is at least hermitian as one sees from the following calculation:

$$\begin{aligned}
\langle H\psi_1|\psi_2\rangle &= \langle 2|p|\psi_1|\psi_2\rangle+\langle\sigma|x|\psi_1|\psi_2\rangle\\
&= \int_{-\infty}^{\infty}(2|p|\psi_1(p))^*\psi_2(p)dp+\int_{-\infty}^{\infty}(\sigma|x|\psi_1(x))^*\psi_2(x)dx\\
&= \int_{-\infty}^{\infty}\psi_1^*(p)2|p|\psi_2(p)dp+\int_{-\infty}^{\infty}\psi_1(x)^*\sigma|x|\psi_2(x)dx\\
&= \langle\psi_1|2|p||\psi_2\rangle+\langle\psi_1|\sigma|x||\psi_2\rangle\\
&= \langle\psi_1|H\psi_2\rangle.
\end{aligned} \tag{4.72}$$

Details on the difference between hermiticity and self-adjointness are to be found in Appendix E. The fact that assuming Fourier symmetry does not lead to self-adjointness has two possible explanations. One is that we do not have Fourier symmetry, i.e. that we need a self-adjoint extension of the operator which breaks Fourier symmetry, or perhaps the method of solving separately for $x > 0$ and $x < 0$ does not lead to a correct solution.

Since it is not clear for instance how $|p|$ acts in position space, we do not attempt to answer the question whether there is a boundary condition for which the Hamiltonian becomes self-adjoint, but we simply demand that the eigenfunctions be orthogonal and see if we can extract a spectrum from that. If we denote with $s(M_1, M_2)$ the scalar product of $\psi_+(x)$ with $M = M_1$ and $\psi_+(x)$ with $M = M_2$, we get the following system of conditions:

$$s(M_1, M_2) = 0, \quad s(M_1, M_3) = 0, \quad s(M_2, M_3) = 0. \tag{4.73}$$

So these three equations should have common solutions if there is an orthogonal set of $\psi_+(x)$ for different values of M. We have made many attempts to find such solutions, first with a routine that tried to solve the above system and later with one that miminizes the following function:

$$\mathrm{S}(M_1, M_2, M_3) = \mathrm{s}(M_1, M_2)^2 + \mathrm{s}(M_1, M_3)^2 + \mathrm{s}(M_2, M_3)^2. \tag{4.74}$$

With the first method we got no solution at all but with the second one we at least got one triple of values that brought S down as far as 10^{-20}. Unfortunately, when looking for the next value which is orthogonal to the three values found (this now is simple one-dimensional root finding), we got discrepancies of M_4 of the order of 10^{-3}. Now this is very small and made us go through all of the calculation again looking for possible errors in the numerics, but unfortunately they are four to five orders of magnitude smaller than the discrepancy.

The method used in this section which lead to the auxiliary Fresnel functions as solutions gives the correct result in the non-relativistic case and thus was definitely worth trying. It is somehow less restrictive than the matrix element calculation in Section 4.4, for it requires (or allows for, depending on the point of view) a boundary condition which one must try to choose in a particular way to make the operator self-adjoint. The fact that it is not possible to find an orthogonal set of these auxiliary Fresnel functions leads to the conclusion that either the operator cannot be made self-adjoint or that the method used is not correct in this case.

Chapter 5

Linear Confining Potential on the Light Cone

5.1 Notation and Conventions

Before we consider in detail light cone coordinates, let's first have a look at the usual Cartesian space-time coordinates. For a more extensive treatment we refer to [1]. As we mentioned earlier, the task of developing relativistic quantum mechanics is reduced to the question of finding expressions for the generators that satisfy the Poincaré algebra, which in general form reads

$$[K_{\mu\nu}, K_{\rho\sigma}] = i(g_{\nu\rho}K_{\mu\sigma} - g_{\mu\rho}K_{\nu\sigma} + g_{\mu\sigma}K_{\nu\rho} - g_{\nu\sigma}K_{\mu\rho}),$$
$$[K_{\mu\nu}, P_\rho] = i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu), \quad [P_\mu, P_\nu] = 0. \tag{5.1}$$

The ten generators are all functions of the coordinates x_μ and momenta p_μ of the particles. In the instant form the initial conditions are specified on the hypersurface $x_0 = 0$. With this $p_0 = \frac{\partial L}{\partial x_0}$ no longer has a meaning and thus needs to be eliminated from the expressions for the ten generators. In the less familiar so called front form the initial conditions are specified on the hypersurface $x_0 - x_1 = 0$. It is now convenient to introduce new, so called light cone coordinates (we concentrate on the case for one space dimension):

$$x_+ = \frac{1}{\sqrt{2}}(x_0 + x_1), \quad x_- = \frac{1}{\sqrt{2}}(x_0 - x_1), \quad p_+ = \frac{1}{\sqrt{2}}(p_0 + p_1), \quad p_- = \frac{1}{\sqrt{2}}(p_0 - p_1). \tag{5.2}$$

In these coordinates the initial conditions are specified on the hypersurface $x_- = 0$ and here now p_+ no longer has a meaning and needs to be eliminated from the expressions for the generators. The metric tensor has the form

$$g_{++} = g_{--} = 0, \quad g_{+-} = g_{-+} = 1. \tag{5.3}$$

The canonical commutation relations of the particle coordinates still have the same form

$$[x_\mu, x_\nu] = [p_\mu, p_\nu] = 0, \quad [p_\mu, x_\nu] = ig_{\mu\nu}. \tag{5.4}$$

Written out we have $[p_-, x_+] = i$. The commutation relations read

$$[P_+, P_-] = 0, \quad [K_{+-}, P_+] = iP_+, \quad [K_{+-}, P_-] = -iP_-. \tag{5.5}$$

The question now is how to implement the confining potential for two particles on the light cone. It is straightforward (though a bit lengthy) to show that the following generators satisfy the commutation relations

$$\begin{aligned} P_+ &= \frac{m_1^2}{2p_{1-}} + \frac{m_2^2}{2p_{2-}} + \sigma\,|x_{1+} - x_{2+}|\,, \\ P_- &= p_{1-} + p_{2-}, \\ K_{+-} &= \frac{1}{2}\left(x_{1+}p_{1-} + p_{1-}x_{1+} + x_{2+}p_{2-} + p_{2-}x_{2+}\right). \end{aligned} \tag{5.6}$$

For details on how one arrives at these expressions we refer to [1], which, however, uses a different convention for the metric tensor. At this point we also want to pay attention to work by Bardeen, Bars, Hanson, and Peccei, who have treated very similar problems, however, from a string point of view [8–10]. We regain the spectrum through

$$2P_+P_- = 2\left(\frac{1}{\sqrt{2}}(H+P)\frac{1}{\sqrt{2}}(H-P)\right) = H^2 - P^2 = M^2. \tag{5.7}$$

In light cone coordinates P_+ plays the role of the Hamiltonian. If one can solve the eigenvalue problem for P_+ the spectrum then is determined by the above equation. So the equation to be solved is

$$\left(\frac{m_1^2}{2p_{1-}} + \frac{m_2^2}{2p_{2-}} + \sigma\,|x_{1+} - x_{2+}|\right)|\psi\rangle = G|\psi\rangle. \tag{5.8}$$

We introduce $P_- = p_{1-} + p_{2-}$, $p_- = \frac{p_{1-} - p_{2-}}{2}$ and $x_+ = x_{1+} - x_{2+}$. The Hamiltonian P_+ now has the form ($m_1 = m_2 = m$)

$$P_+ = \frac{m^2}{2}\left(\frac{1}{\frac{P_-}{2} + p_-} + \frac{1}{\frac{P_-}{2} - p_-}\right) + \sigma|x_+| = 2m^2\left(\frac{P_-}{P_-^2 + 4p_-^2}\right) + \sigma|x_+|. \tag{5.9}$$

For the commutator of p_- and x_+ we get

$$[p_-, x_+] = \frac{1}{2}[p_{1-} - p_{2-}, x_{1+} - x_{2+}] = \frac{1}{2}\left([p_{1-}, x_{1+}] + [p_{2-}, x_{2+}]\right) = i. \tag{5.10}$$

There is an important difference to usual coordinates where one has $[p, x] = -i$. This does not change things much, we just have to be aware of it. Another important difference to the usual coordinates is that the momenta are no longer unrestricted but (only allowing for positive energies) we now have $p_{i0} = \sqrt{m^2 + p_{i1}^2} \geq m$ and thus $p_{i-} = p_{i0} - p_{i1} > 0$ for $m > 0$. From this follows that $P_- = p_{1-} + p_{2-} > 0$. Further since $p_{1-} = \frac{P_-}{2} + p_- > 0$ and $p_{2-} = \frac{P_-}{2} - p_- > 0$ we have $|p_-| < \frac{P_-}{2}$ for $m > 0$. The boundaries for $|p|$ are different for different m, but for all positive values of m the above restrictions hold. In the case of $m = 0$ we have $P_- \geq 0$ and $|p_-| \leq \frac{P_-}{2}$. So the relative momentum p_- is bounded and the boundary depends on the total momentum P_-.

5.2 Semi-Classical Treatment

Let's now solve the problem semi-classically and show that the spectrum obtained is the same as in the semi-classical calculation in the usual coordinates. We do the calculation differently

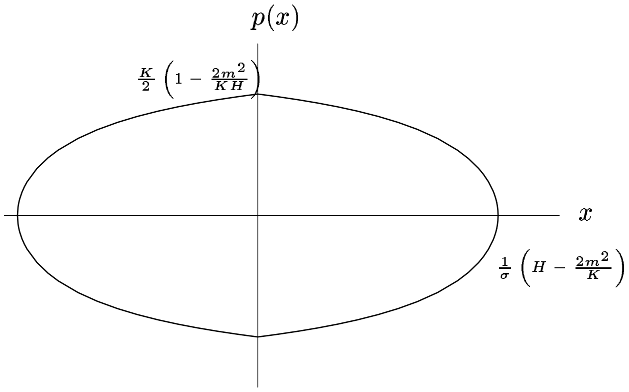

Figure 5.1: **Phase space:** *path in phase space for the linear confining potential.*

than in the usual coordinates. Instead of solving the equations of motion and integrating $p_-\dot{x}_+$ over time, we directly do the integral over the path in phase space, i.e.

$$J = \oint p_- \; dx_+. \tag{5.11}$$

We may do this because — in contrast to the usual coordinates— one can resolve in this case H with respect to p_- and give the integration boundary explicitly. We obtain for p_-^2 as a function of x_+

$$p_-^2(x_+) = \frac{P_-^2}{4}\left(1 - \frac{2m^2}{P_-\left(H - \sigma|x_+|\right)}\right). \tag{5.12}$$

This closed curve crosses the x_+-axis at $\pm|x_{+\max}| = \frac{1}{\sigma}\left(H - \frac{2m^2}{P_-}\right)$. The maximum value of p_- it can reach is $p_{-\max} = \frac{P_-}{2}\left(1 - \frac{2m^2}{P_- H}\right) = \frac{P_-}{2}\left(1 - \frac{4m^2}{M^2}\right)$, which goes to $\frac{P_-}{2}$ in the massless limit. The path is shown in Fig. 5.2. Since the path has parity symmetry with respect to x_+ and p_-, we need to perform the following integral

$$J = 2P_- \int_0^{x_{+\max}} \sqrt{1 - \frac{2m^2}{P_-\left(H - \sigma|x_+|\right)}}\, dx_+ \tag{5.13}$$

Also using $2HP_- = M^2$ we obtain

$$\begin{aligned} J &= \frac{1}{\sigma}\left(M\sqrt{M^2 - 4m^2} - 4m^2 \ln\left(\frac{1}{2m}\sqrt{M^2 - 4m^2} + \frac{M}{2m}\right)\right) \\ &= \frac{1}{\sigma}\left(M\sqrt{M^2 - 4m^2} - 4m^2 \mathrm{Arsh}\left(\frac{\sqrt{M^2 - 4m^2}}{2m}\right)\right). \end{aligned} \tag{5.14}$$

This is exactly the same result as we obtained in the semi-classical calculation (4.28) in the usual coordinates. In the limit $m \to 0$ we thus of course also have the same result, namely

$$M_{n-1} = \sqrt{2\pi\sigma n}, \quad n = 1, 2, \ldots \quad . \tag{5.15}$$

5.3 Quantum Mechanical Approach

We deal with the Hamiltonian

$$P_+ = 2m^2 \frac{P_-}{P_-^2 - 4p_-^2} + \sigma |x_+|, \tag{5.16}$$

however, with the complications that $[p_-, x_+] = i$, $P_- > 0$ and p_- is bounded, where the boundary depends on m and P_-. Now the restriction of p_- would imply, that x_+ is quantized! However, since these restrictions are the result of classical kinematic calculations, we take the point of view that the classically forbidden regions might have non-vanishing probability at the quantum level. Thus we assume that x_+ and p_- are connected through the usual Fourier transform and we may thus apply the already twice used method of solving the problem separately for $x_+ > 0$ and $x_+ < 0$. We concentrate on $x_+ > 0$ where we have

$$\left(2m^2 \frac{P_-}{P_-^2 - 4p_-^2} + \sigma x_+\right) \psi_+(x_+) = G\psi_+(x_+). \tag{5.17}$$

Written in momentum space this reads (since $[p_-, x_+] = i$ we have $x_+ = -i\partial_{p_-}$)

$$\left(2m^2 \frac{P_-}{P_-^2 - 4p_-^2} - i\partial_{p_-}\right) \tilde{\psi}_+(p_-) = G\tilde{\psi}_+(p_-). \tag{5.18}$$

Thus we have the following first order differential equation

$$\partial_{p_-} \tilde{\psi}_+(p_-) = \frac{i}{\sigma} \left(G - 2m^2 \frac{P_-}{P_-^2 - 4p_-^2}\right) \tilde{\psi}_+(p_-), \tag{5.19}$$

with the solution

$$\tilde{\psi}_+(p_-) = \psi_0 \exp\left(\frac{i}{\sigma} \int_0^{p_-} \left(G - 2m^2 \frac{P_-}{P_-^2 - 4q^2}\right) dq\right). \tag{5.20}$$

For the integral we obtain

$$\begin{aligned} \int_0^{p_-} \left(G - 2m^2 \frac{P_-}{P_-^2 - 4q^2}\right) dq &= \begin{cases} Gp_- - \frac{m^2}{2} \ln\left(\frac{1+\frac{2p_-}{P_-}}{1-\frac{2p_-}{P_-}}\right) & : \ |p_-| < \frac{P_-}{2} \\ Gp_- - \frac{m^2}{2} \ln\left(\frac{\frac{2p_-}{P_-}-1}{1+\frac{2p_-}{P_-}}\right) & : \ |p_-| > \frac{P_-}{2} \end{cases} \\ &= \begin{cases} Gp_- - m^2 \mathrm{Arth}\left(\frac{2p_-}{P_-}\right) & : \ |p_-| < \frac{P_-}{2} \\ Gp_- - m^2 \mathrm{Arcth}\left(\frac{2p_-}{P_-}\right) & : \ |p_-| > \frac{P_-}{2} \end{cases} . \end{aligned} \tag{5.21}$$

So we have

$$\tilde{\psi}_+(p_-) = \begin{cases} \psi_0 \exp\left(\frac{i}{\sigma}\left(Gp_- - m^2 \mathrm{Arth}\left(\frac{2p_-}{P_-}\right)\right)\right) & : \ |p_-| < \frac{P_-}{2} \\ \psi_0 \exp\left(\frac{i}{\sigma}\left(Gp_- - m^2 \mathrm{Arcth}\left(\frac{2p_-}{P_-}\right)\right)\right) & : \ |p_-| > \frac{P_-}{2} \end{cases} . \tag{5.22}$$

To obtain $\psi_+(x)$ we need to inverse Fourier transform $\tilde{\psi}_+(p)$. Due to $[p_-, x_+] = i$ we have $\exp(-ix_+p_-)$ instead of $\exp(ix_+p_-)$. We obtain

$$\begin{aligned}
\psi_+(x) &= \frac{\psi_0}{2\pi}\left(\int_{-\infty}^{-\frac{P_-}{2}} \exp\left(\frac{i}{\sigma}\left((G-x_+)p_- - m^2\mathrm{Arcth}\left(\frac{2p_-}{P_-}\right)\right)\right) dp_-\right. \\
&\quad + \int_{-\frac{P_-}{2}}^{\frac{P_-}{2}} \exp\left(\frac{i}{\sigma}\left((G-x_+)p_- - m^2\mathrm{Arth}\left(\frac{2p_-}{P_-}\right)\right)\right) dp_- \\
&\quad \left. + \int_{\frac{P_-}{2}}^{\infty} \exp\left(\frac{i}{\sigma}\left((G-x_+)p_- - m^2\mathrm{Arcth}\left(\frac{2p_-}{P_-}\right)\right)\right) dp_-\right) \\
&= \frac{\psi_0}{\pi}\left(\int_{0}^{\frac{P_-}{2}} \cos\left(\frac{i}{\sigma}\left((G-x_+)p_- - m^2\mathrm{Arth}\left(\frac{2p_-}{P_-}\right)\right)\right) dp_-\right. \\
&\quad \left. + \int_{\frac{P_-}{2}}^{\infty} \cos\left(\frac{i}{\sigma}\left((G-x_+)p_- - m^2\mathrm{Arcth}\left(\frac{2p_-}{P_-}\right)\right)\right) dp_-\right). \qquad (5.23)
\end{aligned}$$

Unfortunately we have not been able to solve this integral and thus we stop here with our attempt to solve the linear confining potential on the light cone fully quantum mechanically.

Summary and Conclusion

We have seen that the only relativistic invariant interaction of point particles in one dimension is given by a linear confining potential. Subsequently we have tried to quantize this theory in order to obtain a relativistically invariant quantum theory for interacting particles. In usual coordinates we have not been able to prove Poincaré invariance at the quantum level explicitly and all attempts to solve the problem and show that the spectrum is relativistically invariant have led to the conclusion that either the Hamiltonian has no self-adjoint extension or that the system is not invariant on the quantum level, i.e. that we have an anomaly. In light cone coordinates, however, the Poincaré algebra closes and we have been able to show that in a semi-classical calculation the results agree with the ones obtained in a semi-classical calculation in usual coordinates. Unfortunately, when trying to solve the problem fully quantum mechanically, we faced serious problems originating from the fact that in light cone coordinates the momentum is restricted. Perhaps one might gain further insight into this subject by going in detail into issues such as self-adjoint extensions. At this point, however, it seems that an easy access to relativistic bound state problems does not exist.

Apparently, there are similar problems in bosonic string theory. The bosonic string is only free of anomalies in 26 space-time dimensions. Even in $d = 26$, i.e. when the anomalies cancel, there is still a tachyon, and thus an inconsistency in the theory. Adding supersymmetry, the string theory becomes anomaly-free in 10 space-time dimensions and there is no tachyon. Since in the massless limit two particles with a linear confining potential are basically the same as a string, it would be interesting to have a look at the calculation leading to the result that the bosonic string is anomaly-free only in one particular dimension and to see, whether this could be adapted to our problem. This would perhaps lead to a deeper understanding of the problems discussed in this thesis.

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Appendix A

Derivation of the Poincaré Algebra

We use the following conventions for the metric tensor $g_{\mu\nu}$:

$$g_{00} = 1, g_{11} = g_{22} = g_{33} = -1. \tag{A.1}$$

The condition on a homogeneous Lorentz transformation is

$$g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}. \tag{A.2}$$

Inserting an infinitesimal transformation

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} \tag{A.3}$$

into the condition yields

$$g_{\mu\nu}(\delta^{\mu}{}_{\rho} + \omega^{\mu}{}_{\rho})(\delta^{\nu}{}_{\sigma} + \omega^{\nu}{}_{\sigma}) = g_{\rho\sigma}. \tag{A.4}$$

Only keeping term of order ω we obtain

$$g_{\rho\sigma} + \omega_{\sigma\rho} + \omega_{\rho\sigma} = g_{\rho\sigma}, \tag{A.5}$$

which implies

$$\omega_{\rho\sigma} = -\omega_{\sigma\rho}. \tag{A.6}$$

A Poincaré transformation (an inhomogeneous Lorentz transformation) has the general form

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} + a^{\mu}. \tag{A.7}$$

Performing two such transformations

$$x'^{\mu} = \Lambda_1{}^{\mu}{}_{\nu}x^{\nu} + a_1{}^{\mu},\;\; x''^{\mu} = \Lambda_2{}^{\mu}{}_{\nu}x'^{\nu} + a_2{}^{\mu} \tag{A.8}$$

in a row yields

$$x''^{\mu} = \Lambda_2{}^{\mu}{}_{\nu}(\Lambda_1{}^{\nu}{}_{\rho}x^{\rho} + a_1{}^{\nu}) + a_2{}^{\mu} \;\; = \Lambda_2{}^{\mu}{}_{\nu}\Lambda_1{}^{\nu}{}_{\rho}x^{\rho} + \Lambda_2{}^{\mu}{}_{\nu}a_1{}^{\nu} + a_2{}^{\mu}. \tag{A.9}$$

Hence, for two successive Poincaré transformations we have the following composition law:

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U(\Lambda_2\Lambda_2, \Lambda_2 a_1 + a_2). \tag{A.10}$$

An infinitesimal transformation may be written as

$$U(1+\omega,\epsilon) = 1 - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} - i\epsilon_\mu P^\mu. \tag{A.11}$$

In order to derive the commutation relations we use the commutator of two infinitesimal transformations. First we use the composition law for two infinitesimal transformations,

$$\begin{aligned}\mathcal{U}_{12} &= U(1+\omega_2,\epsilon_2)U(1+\omega_1,\epsilon_1) - U(1+\omega_1,\epsilon_1)U(1+\omega_2,\epsilon_2)\\ &= U(1+\omega_1+\omega_2+\omega_2\omega_1,\epsilon_1+\epsilon_2+\omega_2\epsilon_1)\\ &\quad -U(1+\omega_1+\omega_2+\omega_1\omega_2,\epsilon_1+\epsilon_2+\omega_1\epsilon_2).\end{aligned} \tag{A.12}$$

Inserting the expansion yields

$$\mathcal{U}_{12} = -\frac{i}{2}(\omega_2\omega_1)_{\mu\nu}J^{\mu\nu} + \frac{i}{2}(\omega_1\omega_2)_{\mu\nu}J^{\mu\nu} + i(\omega_1\epsilon_2)_\mu P^\mu - i(\omega_2\epsilon_1)_\mu P^\mu. \tag{A.13}$$

Next we calculate the same commutator but this time only using the infinitesimal expansion and not the composition law

$$\begin{aligned}\mathcal{U}_{12} &= (1-\frac{i}{2}\omega_{2\mu\nu}J^{\mu\nu} - i\epsilon_{2\mu}P^\mu)(1-\frac{i}{2}\omega_{1\rho\sigma}J^{\rho\sigma} - i\epsilon_{1\rho}P^\rho)\\ &\quad -(1-\frac{i}{2}\omega_{1\rho\sigma}J^{\rho\sigma} - i\epsilon_{1\rho}P^\rho)(1-\frac{i}{2}\omega_{2\mu\nu}J^{\mu\nu} - i\epsilon_{2\mu}P^\mu)\\ &= \frac{1}{4}\omega_{1\rho\sigma}\omega_{2\mu\nu}(J^{\rho\sigma}J^{\mu\nu} - J^{\mu\nu}J^{\rho\sigma}) - \frac{1}{2}\omega_{2\mu\nu}\epsilon_{1\rho}(J^{\mu\nu}P^\rho - P^\rho J^{\mu\nu})\\ &\quad -\frac{1}{2}\omega_{1\rho\sigma}\epsilon_{2\mu}(P^\mu J^{\rho\sigma} - J^{\rho\sigma}P^\mu) + \frac{1}{2}\epsilon_{1\rho}\epsilon_{2\mu}(P^\rho P^\mu - P^\mu P^\rho).\end{aligned} \tag{A.14}$$

Comparing the coefficients in the two expansions of $\mathcal{U}_{12}$ one obtains the commutation relations for the generators. From the terms with $\omega_1\omega_2$ we obtain

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} + g^{\mu\sigma}J^{\nu\rho} - g^{\nu\sigma}J^{\mu\rho}). \tag{A.15}$$

The terms with $\omega_1\epsilon_2$ and $\omega_2\epsilon_1$ yield

$$[P^\rho, J^{\mu\nu}] = i(g^{\rho\mu}P^\nu - g^{\rho\nu}P^\mu). \tag{A.16}$$

Finally, from the $\epsilon_1\epsilon_2$ term we get

$$[P^\mu, P^\nu] = 0. \tag{A.17}$$

Introducing the following notation

$$H = P^0, \vec{P} = (P^1, P^2, P^3), \vec{J} = (J^{23}, J^{31}, J^{12}), \vec{K} = (J^{10}, J^{20}, J^{30}), \tag{A.18}$$

one obtains the usual commutation relations of the Poincaré algebra as written down in Chapter 1.

Appendix B

Some Facts about Lie Groups

In order to state a theorem on the number of Casimir operators we first want to give some definitions (for more details see [6, 11])

- A *Lie Group* is a group that is at the same time a finite dimensional manifold of the differentiability class C^2 in such a way that

$$\mu \quad : \quad (x,y) \to xy : G \times G \to G, \tag{B.1}$$
$$\imath \quad : \quad x \to x^{-1} : G \to G \tag{B.2}$$

  are C^2 mappings.

- A Lie group G is said to be *connected* if for every two elements $a_1, a_2 \in G$ there is a continuous path (in parameter space) which connects them.

- A *Lie algebra* A is a vector space over a field $\mathbf{K}$ with a linear composition $[\ ,\ ]$, such that with $J_i, J_k \in A$ also $[J_i, J_k] \in A$ and the bracket (commutator) satisfies

$$[J_i, J_k] = -[J_k, J_i], \tag{B.3}$$
$$[\alpha J_i + \beta J_k, J_l] = \alpha[J_i, J_l] + \beta[J_k, J_l], \tag{B.4}$$
$$[J_i, [J_k, J_l]] + [J_l, [J_i, J_k]] + [J_k, [J_l, J_i]] = 0 \quad \text{(Jacoby identity)}. \tag{B.5}$$

- A' is a *subalgebra* of A if $A' \subset A$ and A' is itself a Lie algebra.

- $S \subset A$ is an *invariant subalgebra* if $[J_i, J_k] \in S$ for each $J_i \in S$ and $J_k \in A$.

- An algebra A or subalgebra S is said to be *abelian* if $[J_i, J_k] = 0$ for each $J_i, J_k \in A$ or S.

- An Algebra A is said to be *simple* if it has no invariant subalgebras besides A and $\{0\}$.

- An algebra A is said to be *semi-simple* if it does not contain an invariant abelian subalgebra.

- The *Rank* of a Lie algebra is the maximum number of mutually commuting infinitesimal generators.

The commutation relations of a Lie algebra may be written as

$$[X_i, X_k] = \sum_l c_{ik}^l X_l, \tag{B.6}$$

where the c_{ik}^l are the structure constants. One may define the *Killing form* g_{ik} as

$$g_{ik} := \sum_{l,n} c_{il}^n c_{kn}^l = g_{ki}. \tag{B.7}$$

The *theorem of Cartan* then states:
A Lie algebra is semi-simple if and only if the determinant of the Killing form is different from zero: $\det(g_{ik}) \neq 0$.

We are now ready to state a theorem by Beltrametti and Blasi [12] on the number of Casimir operators associated with any Lie Group. Let N_c denote the number of Casimir operators of a Lie Group. We then have

Theorem: *If the Lie algebra is semi-simple, i.e if* $\det(g_{ik}) \neq 0$*, then* N_c *equals the Rank of the Lie algebra. If the Lie algebra is non-semi-simple, i.e if* $\det(g_{ik}) = 0$*, then*

$$N_c = r - \text{Rank}\left(\sum_{l=1}^{r} c_{ik}^l \alpha_l\right), \tag{B.8}$$

where r *is the dimension of the Lie algebra and the* α_l *are to be treated as independent variables.*

The Poincaré algebra is non-semi-simple since (H, P_1, P_2, P_3) form an invariant abelian subalgebra, as one easily sees from the commutation relations. In the one-dimensional case an invariant abelian subalgebra is given by (H, P).

Appendix C

Hermite Functions

In this appendix we want to prove that the Hermite functions are eigenfunctions of the Fourier transformation and form a complete orthonormal set in the space $\mathcal{L}^2$ of square integrable functions. The Hermite functions are given by

$$\phi_n(x) = AH_n(x)\exp(-\frac{x^2}{2}), \tag{C.1}$$

where A is determined by normalization and $H_n(x)$ is the Hermite polynomial of order n which is defined as

$$H_n(x) = (-1)^n \exp(x^2)\left(\frac{d}{dx}\right)^n \exp(-x^2). \tag{C.2}$$

We find

$$\begin{aligned}
\tilde{\phi}_n(p) &= (-1)^n A\int_{-\infty}^{\infty}\exp(-ipx)\exp\left(-\frac{x^2}{2}\right)\exp(x^2)\left(\frac{d}{dx}\right)^n\exp(-x^2)dx \\
&= (-1)^n A\int_{-\infty}^{\infty}\exp\left(-ipx+\frac{x^2}{2}\right)\left(\frac{d}{dx}\right)^n\exp(-x^2)dx \\
&= (-1)^n A\int_{-\infty}^{\infty}\exp\left(\frac{1}{2}(x-ip)^2+\frac{p^2}{2}\right)\left(\frac{d}{dx}\right)^n\exp(-(x-ip+ip)^2)dx
\end{aligned} \tag{C.3}$$

Putting $y = x - ip$ we obtain

$$\tilde{\phi}_n(p) = (-1)^n A\int_{-\infty-ip}^{\infty-ip}\exp\left(\frac{p^2}{2}\right)\exp\left(\frac{y^2}{2}\right)\left(\frac{d}{dy}\right)^n\exp(-(y+ip)^2)dy. \tag{C.4}$$

For $y \to \infty$ the integrand goes to zero and it has no poles. Thus we may move the integration path back to the real axis. We find

$$\begin{aligned}
\tilde{\phi}_n(p) &= (-1)^n A\exp\left(\frac{p^2}{2}\right)\int_{-\infty}^{\infty}\exp\left(\frac{y^2}{2}\right)(-i)^n\left(\frac{d}{dp}\right)^n\exp(-(y+ip)^2)dy \\
&= (i)^n A\exp\left(\frac{p^2}{2}\right)\left(\frac{d}{dp}\right)^n\int_{-\infty}^{\infty}\exp\left(-p^2-\frac{1}{2}(y+2ip)^2\right)dy.
\end{aligned} \tag{C.5}$$

Now we put $z = y+2ip$ and use the same argument as above concerning the path of integration. This yields

$$\begin{aligned}
\tilde{\phi}_n(p) &= (i)^n A \exp\left(\frac{p^2}{2}\right)\left(\frac{d}{dp}\right)^n \exp\left(-p^2\right)\int_{-\infty}^{\infty}\exp\left(-\frac{z^2}{2}\right)dz \\
&= \sqrt{2\pi}(-i)^n A \exp\left(-\frac{p^2}{2}\right)(-1)^n \exp(p^2)\left(\frac{d}{dp}\right)^n \exp(-p^2) \\
&= \sqrt{2\pi}(-i)^n \phi_n(p). \qquad \text{(C.6)}
\end{aligned}$$

So ϕ_n is indeed an eigenfunction of the Fourier transformation with eigenvalue $(-i)^n$.

Instead of just going through the calculation one might instead use a different argument. The Hermite functions are eigenfunctions to the following differential operator (harmonic oscillator)

$$\mathcal{D}\phi(x) = -\partial_x^2\phi(x) + x^2\phi(x). \qquad \text{(C.7)}$$

This operator commutes with the Fourier transformation $\mathcal{F}$

$$\begin{aligned}
\mathcal{F}\mathcal{D}\phi(x) &= \mathcal{F}(-\partial_x^2\phi(x) + x^2\phi(x)) = p^2\tilde{\phi}(p) - \partial_p^2\tilde{\phi}(p) \\
&= \mathcal{D}\tilde{\phi}(p) = \mathcal{D}\mathcal{F}\phi(x), \qquad \text{(C.8)}
\end{aligned}$$

so the two operators must have the same eigensystem and thus the Hermite functions are eigenfunctions of the Fourier transformation.

The harmonic oscillator Hamiltonian is self-adjoint and thus its eigenfunctions form a complete basis in $\mathcal{L}^2$. We may also show this explicitly, i.e. that if $f \in \mathcal{L}^2$ and $\langle f|\phi_n\rangle = 0, \forall n$ then $f = 0$. We may write x^n for arbitrary n as

$$x^n = \alpha_n H_n + \alpha_{n-1}H_{n-1} + \cdots + \alpha_1 H_1 + \alpha_0 H_0. \qquad \text{(C.9)}$$

Since $\langle f|\phi_n\rangle = 0, \forall n$, we also have $\langle f|x^n \exp\left(-\frac{x^2}{2}\right)\rangle = 0, \forall n$, which written out reads

$$\int_{-\infty}^{\infty} f(x)\exp\left(-\frac{x^2}{2}\right)x^n dx = 0. \qquad \text{(C.10)}$$

If $f \in \mathcal{L}^2$ than also $ff_0 = f(x)\exp\left(-\frac{x^2}{2}\right)$ and thus we have

$$\begin{aligned}
\mathcal{F}ff_0(p) &= \int_{-\infty}^{\infty} f(x)\exp\left(-\frac{x^2}{2}\right)\exp\left(-ipx\right)dx \\
&= \int_{-\infty}^{\infty}\sum_{n=0}^{\infty} f(x)\exp\left(-\frac{x^2}{2}\right)\frac{(-ipx)^n}{n!}dx. \qquad \text{(C.11)}
\end{aligned}$$

Since

$$\begin{aligned}
\left|\sum_{n=0}^{\infty} f(x)\exp\left(-\frac{x^2}{2}\right)\frac{(-ipx)^n}{n!}\right| &\leq |f(x)|\exp\left(-\frac{x^2}{2}\right)\sum_{n=0}^{\infty}\frac{|px|^n}{n!} \\
&= |f(x)|\exp\left(-\frac{x^2}{2}\right)\exp\left(|px|\right), \qquad \text{(C.12)}
\end{aligned}$$

we may interchange summation and integration and obtain (using (C.10))

$$\mathcal{F} f f_0(p) = \sum_{n=0}^{\infty} \frac{(-ip)^n}{n!} \int_{-\infty}^{\infty} f(x) \exp\left(-\frac{x^2}{2}\right) x^n dx = 0. \tag{C.13}$$

Thus we have $\mathcal{F} f f_0(p) = 0$, from this follows $f f_0 = 0$ and we therefore have $f = 0$.

Appendix D

Boundary Conditions following from Fourier Symmetry

In this appendix we show that assuming Fourier symmetry and integrability of the wave function leads to the condition $\psi(0) = 0$ for odd functions and $\psi'(0) = 0$ for even functions. Using $\psi_-(-x) = \psi_+(x)$ we obtain

$$\begin{aligned}
\tilde{\psi}_{\text{odd}}(p) &= \int_0^\infty \exp(-ipx)\psi_+(x)dx - \int_{-\infty}^0 \exp(-ipx)\psi_-(x)dx \\
&= -2i \int_0^\infty \left(\frac{\exp(ipx) - \exp(-ipx)}{2i} \right) \psi_+(x)dx \\
&= -2i\psi_0 \sqrt{\frac{\sigma}{2\pi}} \int_0^\infty \sin(px)\, \mathrm{g}\left(-\sqrt{\frac{\sigma}{2\pi}} \left(\frac{E}{\sigma} - x \right) \right) dx. \qquad \text{(D.1)}
\end{aligned}$$

Since we assume that $\psi(x)$ is an eigenfunction of the Fourier transform, $\tilde{\psi}(p) = \psi(p)$ has exactly the same behavior at the origin. So we are interested in the following limit:

$$\lim_{p\to 0} \tilde{\psi}_{\text{odd}}(p) = -2i\psi_0 \sqrt{\frac{\sigma}{2\pi}} \lim_{p\to 0} \int_0^\infty \sin(px)\, \mathrm{g}\left(-\sqrt{\frac{\sigma}{2\pi}} \left(\frac{E}{\sigma} - x \right) \right) dx \qquad \text{(D.2)}$$

If one is allowed to exchange the order of limit and integration on the right hand side we have of course $\tilde{\psi}(0) = 0$. Since we have

$$\left| \sin(px)\mathrm{g}\left(\sqrt{\frac{\sigma}{2\pi}} \left(x - \frac{E}{\sigma} \right) \right) \right| \le \left| \mathrm{g}\left(\sqrt{\frac{\sigma}{2\pi}} \left(x - \frac{E}{\sigma} \right) \right) \right|, \qquad \text{(D.3)}$$

one may interchange limit and integration if

$$\int_0^\infty \left| \mathrm{g}\left(\sqrt{\frac{\sigma}{2\pi}} \left(x - \frac{E}{\sigma} \right) \right) \right| \qquad \text{(D.4)}$$

is finite. This is the case for finite values of E, since $\mathrm{g}(x)$ is bounded and for $x \to \infty$ goes like x^{-3}. Thus we may interchange limit and integration, which leads to $\tilde{\psi}_{\text{odd}}(0) = 0$. Fourier symmetry then tells us that we have indeed

$$\psi_{\text{odd}}(0) = 0. \qquad \text{(D.5)}$$

In the even case we may proceed very similarly. We have

$$\begin{aligned} \tilde{\psi}_{\text{even}}(p) &= \int_0^\infty \exp(-ipx)\psi_+(x)dx + \int_{-\infty}^0 \exp(-ipx)\psi_-(x)dx \\ &= 2\int_0^\infty \cos(px)\psi_+(x). \end{aligned} \tag{D.6}$$

To be even, $\tilde{\psi}_{\text{even}}(p)$ must be continuous at the origin. But what happens with the derivative? We find

$$\frac{d\tilde{\psi}_{\text{even}}(p)}{dp} = 2\frac{d}{dp}\int_0^\infty \cos(px)\psi_+(x)dx. \tag{D.7}$$

As above

$$\int_0^\infty |\cos(px)\psi_+(x)|\, dx \tag{D.8}$$

is finite and thus we may interchange integration and derivative

$$\lim_{p\to 0} \frac{d\tilde{\psi}_{\text{even}}(p)}{dp} = 2\psi_0\sqrt{\frac{\sigma}{2\pi}} \lim_{p\to 0}\int_0^\infty \sin(px)x\, \mathrm{g}\left(\sqrt{\frac{\sigma}{2\pi}}\left(x - \frac{E}{\sigma}\right)\right). \tag{D.9}$$

Using the same argument as in the odd case we may interchange limit and integration and thus obtain due to the Fourier symmetry

$$\left.\frac{d\psi_{\text{even}}(x)}{dx}\right|_{x=0} = 0. \tag{D.10}$$

So indeed Fourier symmetry applied to the auxiliary Fresnel function implies $\psi'(0) = 0$ for even functions and $\psi(0) = 0$ for odd functions.

Appendix E

Hermiticity and Self-Adjointness

This appendix closely follows [13].

It turns out that on an infinite dimensional Hilbert space it is not sufficient for an operator to be hermitian in order to have only real eigenvalues and a complete set of eigenvectors, but that in order to have these properties the operator must be self-adjoint.

The domain of definition $\mathcal{D}(A) \subset \mathcal{H}$ of an operator A is the set of vectors ψ on which $A\psi$ is defined and belongs again to the Hilbert space $\mathcal{H}$.

An operator A defined on $\mathcal{D}(A)$ is called *hermitian* if it satisfies

$$\langle \phi | A\psi \rangle = \langle A\phi | \psi \rangle, \quad \text{for any} \quad \phi, \psi \in \mathcal{D}(A) \subset \mathcal{H}. \tag{E.1}$$

One denotes $A^\dagger$ the *adjoint operator* to A if

$$\langle \phi | A\psi \rangle = \langle A^\dagger \phi | \psi \rangle, \quad \text{for } \psi \in \mathcal{D}(A), \phi \in \mathcal{D}(A^\dagger). \tag{E.2}$$

A symmetric operator A is *self-adjoint* if $A^\dagger = A$, which also means that $\mathcal{D}(A^\dagger) = \mathcal{D}(A)$.

The hermitian operator A_1 is a *self-adjoint extension* of the hermitian operator A if

- $\mathcal{D}(A) \subset \mathcal{D}(A_1)$
- A_1 coincides with A on $\mathcal{D}(A)$
- $\mathcal{D}(A_1^\dagger) = \mathcal{D}(A_1)$

The question whether there exists such a self-adjoint extension for a hermitian operator is highly non-trivial.

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