Effective field theory for charge carriers in an antiferromagnet

Inauguraldissertation
der Philosophisch-naturwissenschaftlichen Fakultät
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Prof. Dr. P. Messerli

Mosquito hits car
Those unenlightened think SPLAT
We know Δp

Andrew W., Physics Haiku

Abstract

The low-energy physics of antiferromagnets is described by an effective field theory in terms of magnons—the corresponding Goldstone bosons. This is in complete analogy to QCD, where the low-energy physics is described by chiral perturbation theory (χPT)—the effective theory for QCD formulated in terms of the Goldstone pions. In this thesis we construct a low-energy effective field theory for charge carriers in an antiferromagnet which describes the interactions of electrons and holes through magnon exchange. This is again in complete analogy to QCD where baryon chiral perturbation theory (BχPT) describes the interactions of baryons through pion exchange. The construction of an effective theory is based on the symmetries of the underlying microscopic system. Whereas for χPT and BχPT the underlying microscopic system is fundamental QCD, we choose the Hubbard model, which is the standard model for antiferromagnets, as the underlying microscopic system for the effective theory to be constructed. We present a detailed symmetry analysis of the Hubbard model and deduce how the various symmetry transformations translate to the effective theory. The resulting effective transformation laws are then used to construct a complete set of linearly independent leading contributions to the effective action. Possible applications of the effective theory include magnon-hole and magnon-electron scattering, the determination of the resulting long-range potential between the charge carriers, as well as the reduction of the staggered magnetization upon doping. Furthermore, since high-temperature superconductors result from antiferromagnets upon doping, one may investigate potential mechanisms for the preformation of electron or hole pairs in the antiferromagnetic phase. The condensation of such pairs would then lead to superconductivity.

Contents

1 Introduction

The goal of this thesis is to construct a low-energy effective field theory for charge carriers in an antiferromagnet. To this end we sketch in the following sections briefly the domain of effective theories. After looking at how effective theories are constructed, we discuss the effective theory for QCD which is an important and successful example for an effective theory. Further, we discuss the impact of lattices before we finally outline our project and point out the relations to the effective theory for QCD.

1.1 Construction of effective theories

Despite the fact that the haiku that inaugurates this thesis describes a 'high-energy' process, namely the inelastic scattering of mosquitos off car windows[1], this thesis is entirely about low-energy processes. The low-energy behavior of a given theory is most economically described by its effective theory. Naively speaking, the low-energy effective theory emerges from the underlying 'fundamental' microscopic theory by integrating out all high-energy degrees of freedom in the underlying theory and thus leaving alone only the low-energy degrees of freedom. In practice, however, this procedure turns out to be a highly non-trivial task, except for the most simplest cases. Consequently, one has to find a different, more practicable way to construct the low-energy effective theory for a given underlying theory.

The approach used in practice is based on symmetry considerations [43]. Clearly, if the effective theory should really represent the underlying theory in the low-energy region, it must share the same symmetry properties as the underlying theory[2]. Hence, if the underlying theory is invariant under the symmetry group G, so must be the effective theory. According to this insight one can now just 'invent' and write down all possible expressions that respect the above symmetries in order to construct the effective theory. However, there is a priori a large number of expressions one can write down following this rule. Hence, in order to use the effective theory for feasible calculations, one has to single out a relevant finite subset of expressions. If we remind ourselves that derivatives correspond to momenta p, we realize that the effective theory is a systematic expansion in powers of p/Λ, where Λ is a high-energy scale of the underlying microscopic theory at which the power expansion of the effective theory breaks down. Consequently, expressions

[1] Of course, this is a high energy process only with respect to the mosquito, but it is a low-energy process for the atoms the mosquito consists of.

[2] Aside from this, it is possible to explicitly break Lorentz invariance in order to construct a non-relativistic low-energy effective theory to an underlying relativistic theory.

with a higher number of derivatives are suppressed such that in practice one uses only all linearly independent expressions up to a given small number of derivatives.

Each of these leading expressions is then multiplied by a low-energy constant, which determines the coupling strength of the corresponding interaction in the effective theory. The numerical values of these low-energy constants originate, at least in principle, from the high-momenta integrals carried out during the transition from the microscopic to the effective theory. This means that the high-energy physics of the underlying microscopic theory is absorbed into the low-energy constants and enters in this way the low-energy effective theory. As a result, the effective theory is insensitive to the microscopic details of the underlying microscopic theory. In this sense, an effective theory makes universal predictions for a whole class of underlying microscopic theories. This also means that the exact details of the high-energy physics are irrelevant for the construction of the effective theory. They only find their way into the effective theory by different values for the low-energy constants.

In practice the numerical values of the low-energy constants are determined by matching calculations. This means that an appropriate process (i.e. one that involves the low-energy constants to be determined) is on the one hand calculated in the framework of the effective theory. On the other hand, the same processes is also studied using the microscopic theory. This can be done either with analytic calculations, Monte Carlo simulations or even experiments. Since both predictions describe the same physics, a comparison of the two can then be used to determine the so far unknown low-energy constants. After the matching is complete, i.e. the low-energy constants are known, the effective theory can be engaged to predict new results. This is especially interesting for results that are not accessible within the underlying microscopic theory using known methods. Indeed, this is the main reason why one considers effective theories at all.

We did not yet talk about how the relevant degrees of freedom are identified in the effective theory. In systems in which all symmetries stay intact, even at low energies, they look in principle the same as in the underlying microscopic theory, despite that their relation to the original degrees of freedom of the underlying microscopic theory is highly non-trivial. On the other hand, there are systems in which at low energies a symmetry group G spontaneously breaks down to a subgroup H of G. This means that the dynamically created ground state of the theory is only invariant under the subgroup H but not under the full symmetry G. For such systems the Goldstone theorem predicts the existence of massless bosonic excitations that live in the coset space G/H. Since these so-called Goldstone bosons are massless, they play a dominant role in the low-energy region and hence they are relevant degrees of freedom in the effective theory for such systems. In this thesis we will deal exclusively with systems in which spontaneous symmetry breaking occurs.

It should be pointed out that an effective theory is not just a model that tries to mimic the underlying microscopic theory for low energies. It is completely equivalent to the underlying theory at low energies, but formulated in the degrees of freedom that best describe the physics in the low-energy region in which the effective theory is applicable.

Above this region, the effective theory breaks down and no longer describes the underlying theory. The limited validity range is the price one has to pay for the excellent adaption to the low-energy physics.

1.2 Pions in Quantum Chromodynamics (QCD)

An important and well-known example of a system in which spontaneous symmetry breaking occurs is Quantum Chromodynamics (QCD), which describes the strong interactions of quarks via gluon exchange. In QCD at low energies the $G = SU(2)_L \otimes SU(2)_R$ chiral symmetry of massless up and down quarks is spontaneously broken down to the isospin symmetry $H = SU(2)_{L=R}$. This results in three massless Goldstone bosons—the pions—that live in the coset space $G/H = SU(2)$. The associated low-energy effective theory was first introduced by Weinberg [41] and later formulated as a systematic expansion by Gasser and Leutwyler [42] resulting in their chiral perturbation theory (χPT). This low-energy effective theory, which involves only pions as the degrees of freedom, correctly describes the purely pionic sector of QCD at low energies in terms of a few low-energy parameters such as the pion decay constant, the chiral condensate, and the Gasser-Leutwyler coefficients. Note that the up and down quark masses in Nature are small but not exactly zero, resulting in a small explicit breaking of chiral symmetry. Although pions are hence only pseudo-Goldstone bosons and pick up a mass as well, chiral perturbation theory remains perfectly well valid.

It is an experimental fact that the QCD spectrum at low energies does not consist solely of pions, but of baryons as well. Hence, it is natural to ask if the baryonic degrees of freedom can be incorporated into the effective theory as well. This is indeed possible and leads to baryon chiral perturbation theory (BχPT) [70–74]. For its construction it is necessary to realize the spontaneously broken chiral symmetry G in a non-linear, i.e. pion field dependent way [75, 76]. The chiral symmetry, under which the baryon fields transform, then appears as a local symmetry in the unbroken isospin subgroup H. This is in contrast to the underlying microscopic QCD, where the quark fields transform under the linearly realized global symmetry G. Note that the introduced baryon fields are effective degrees of freedom and their relation to the underlying microscopic QCD, which deals exclusively with quark and gluon fields, is highly non-trivial. Whereas with ordinary chiral perturbation theory only the baryon number sector $B = 0$ was accessible, it is by means of baryon chiral perturbation theory now also possible to study the baryon number sectors with $B \neq 0$.

Before one can do loop-calculations in the effective theory, one must establish a consistent power-counting scheme enabling a systematic loop-expansion of the effective theory. This has been done originally for ordinary chiral perturbation theory, i.e. for the $B = 0$ sector [42]. Since the baryons are massive they set a new scale in the effective theory. Hence the issue of power-counting must be reconsidered for baryon chiral perturbation theory, i.e. in the $B \neq 0$ sectors. For the baryon number $B = 1$ sector a consistent power-counting

scheme was established by Becher and Leutwyler [74]. On the other hand, in the sectors with $B \geq 2$ systematic power-counting is still a controversial issue. The Weinberg power-counting scheme [53] seems to work in most (but not necessarily in all) cases. Its relation to the alternative Kaplan-Savage-Wise scheme [54] should be clarified further [60, 62].

Solving QCD directly at low energies turns out to be a notoriously hard problem. Analytical perturbative calculations are prevented by the strong coupling strength in this energy range. Furthermore, the QCD spectrum at low energies consists, due to confinement, of mesons and baryons and not of the quark and gluon fields native to QCD. On the other hand, it is possible to study chiral symmetry breaking in the QCD vacuum with numerical simulations of lattice QCD. However, simulating lattice QCD at non-zero baryon chemical potential, i.e. in the $|B| \gg 0$ sectors, is prevented by a severe complex action problem. Instead, it is also possible to study the effective theory on the lattice. In [93, 94] for example, baryon chiral perturbation theory was regularized on a space-time lattice in order to address non-perturbative questions.

At very large baryon densities, such as the ones found in the core of compact neutron or quark stars, one expects that QCD becomes a superconductor for the color charge carried by quarks and gluons [52]. This phenomenon of color-superconductivity is well understood within microscopic QCD in terms of one-gluon exchange. In addition, at more moderate baryon densities, such as the ones found inside large nuclei or neutron stars, also superconductivity or superfluidity of ordinary electric charge exists. However, the mechanism behind this phenomenon, which involves the pairing of protons or neutrons, is very hard to understand within the framework of microscopic QCD. Instead, it is much more adequate to employ the low-energy effective theory which deals with the appropriate degrees of freedom, and whose parameters can be determined from the underlying microscopic QCD. Indeed, in nuclear physics effective field theory has recently led to some progress in describing the forces between nucleons in terms of just a few low-energy parameters [53–62]. Also steps towards describing nuclear matter with effective field theories have already been taken [63–67]. On the other hand, one could also invent phenomenological models to describe the low-energy physics. However, since these are usually not considering the underlying microscopic theory, they involve in general a much larger number of adjustable parameters.

1.3 Phonons in crystals and low-temperature superconductors

Another example of spontaneous symmetry breaking comes from condensed matter physics. While the underlying fundamental forces in this domain are Poincaré-invariant, some of this space-time symmetries spontaneously break down during the formation of a crystal lattice. In particular, the spontaneous breakdown of translation invariance during this process gives rise to phonons—the corresponding massless Goldstone bosons. Naively speaking, the phonons describe the vibrations of the crystal lattice caused e.g. by thermal excitations.

There is general agreement that the phonons play the key role in ordinary low-temperature superconductors. In these materials the electrons that are responsible for the effect are only weakly coupled. Furthermore, the phonons mediate an attractive interaction that can overcome the Coulomb repulsion between these electrons and the thermal vibrations at very low-temperatures. For specific spin and phase-space configurations the phonon-mediated force may lead to the formation of electron pairs—the Cooper pairs—whose condensation induces superconductivity. The phenomenon of low-temperature superconductivity is successfully quantified in BCS (Bardeen, Cooper, Schrieffer) theory.

It should be pointed out that the spontaneous symmetry breaking and the resulting existence of phonons strictly relies on the dynamical formation of the crystal lattice. If one imposes, on the other hand, a rigid lattice right from the beginning by hand, such that the lattice does not have its own degrees of freedom, no spontaneous but explicit symmetry breaking occurs. As a result, no phonons exist in such systems. In particular this is true for models natively defined on the lattice, like e.g. the Hubbard model of the next section. On the other hand, in the real materials which these models attempt to mimic, the lattice is formed dynamically and hence phonons exist. This is no contradiction, it simply means that such models are too simple to contain the necessary degrees of freedom to describe the phonon physics. Hence, if one wants to model a material in which the phonons play an important role for the physics to capture, rigid lattice models are inappropriate and must be augmented with appropriate degrees of freedom. If, however, the phonons are irrelevant for the physics under investigation, models with a rigid lattice are perfectly well suited.

1.4 Magnons in antiferromagnets and high-temperature superconductors

Another example from condensed matter physics in which spontaneous symmetry breaking occurs are antiferromagnets. In such systems the $SU(2)_s$ spin rotational symmetry spontaneously breaks down to the subgroup $U(1)_s$ by the formation of a staggered magnetization vector. Accordingly there are in this case two massless Goldstone bosons—the magnons—that live in the coset space S^2. Note that this is in complete analogy to QCD, i.e. the magnons of antiferromagnets correspond to the pions of the strong interactions. Naively speaking, the single degree of freedom of the unbroken subgroup $U(1)_s$ corresponds to rotations around the staggered magnetization vector, whereas the two magnon degrees of freedom wiggle the staggered magnetization vector in the two possible directions.

Furthermore it is known that high-temperature superconductors originate from antiferromagnets upon doping. The superconductivity in these high-T_c cuprate materials takes place in copper oxide layers (hence the name cuprates) in which electrons are strongly correlated as an antiferromagnet. These cuprate layers are separated by charge reservoir layers in which upon doping atoms are substituted such that either additional electrons or holes (missing electrons) are present in the cuprate layers. Hence doping corresponds to

a chemical potential which controls the density of electrons or holes with respect to half-filling, i.e. the antiferromagnetic background. This is in complete analogy to QCD, where the chemical potential gives access to the baryon number $B \neq 0$ sectors corresponding to baryons or antibaryons in the hadronic vacuum.

Still, 20 years after the discovery of high-temperature superconductivity in the layered cuprate materials [1], identifying the dynamical mechanism behind it remains one of the great challenges in condensed matter physics. In contrast to the ordinary low-temperature superconductors, the high-T_c cuprates are systems of strongly correlated electrons to which the weak coupling BCS theory is not readily applicable. Furthermore, the high transition temperatures of the cuprate superconductors and the smallness of the isotope effect suggest that mechanisms other than phonon exchange may be responsible for Cooper pair formation. Since the high-temperature superconductors are antiferromagnets before doping, it is natural to suspect (but not generally accepted) that the magnons may be important for binding electrons or holes into preformed pairs. In this thesis we assume therefore that phonons do not play an important role in the cuprate materials. Instead we will entirely focus on the magnons.

Even if spin fluctuations were not the key to explain the high-temperature superconductivity, the dynamics of charge carriers (i.e. electrons and holes) in an antiferromagnet is an interesting topic in itself. There is a vast literature on this subject. For example, the dynamics of holes in an antiferromagnet has been investigated in [2–36]. Understanding the dynamics of even just a single hole propagating in an antiferromagnet is a challenging problem. For example, experiments [37–40] as well as theoretical investigations [6, 7, 17, 34] indicate that the minimum of the dispersion relation (i.e. of the energy) of a single hole is located at lattice momenta $(\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$ in the Brillouin zone. The situation becomes even more complicated as one considers a second hole. For example, there is no general agreement if two holes can build a bound state or not. If this is indeed possible, the condensation of such bound pairs would provide a potential mechanism for high-temperature superconductivity. At a non-zero density of holes, i.e. at sufficient doping, experiments show that high-temperature superconductivity may arise. On the other hand, theoretical arguments suggest that even an infinitesimal amount of doping may affect the antiferromagnetic phase and turn it into a spiral phase [6].

One approach to shed light on the possible mechanisms behind the phenomenon of high-temperature superconductivity is to invent phenomenological models that are formulated directly in terms of magnon and electron or hole fields [11, 13, 25, 32, 36]. However, although they may provide qualitative insights, such models do not lead to unambiguous predictions. Another standard approach is the use of microscopic models. The standard microscopic models for antiferromagnets as well as for high-temperature superconductors are the Hubbard and t-J models. In order to capture the relevant physics, these models are equally strongly coupled as the electrons in the actual materials. Unfortunately, this strong coupling prevents systematic analytic calculations, such that again often uncontrolled approximations are applied. In addition, reliable numerical simulations of these microscopic models, away from half-filling, are prevented by a severe fermion sign problem.

Yet another approach to understand antiferromagnets and the high-T_c materials is to construct and engage a systematic low-energy effective theory. At half-filling, i.e. in the purely magnonic sector, this has indeed already been done for both ferro- [44, 45] and antiferromagnetic magnons [46–51]. This is in complete analogy to the chiral perturbation theory of QCD, which involves only pions. For antiferromagnetic magnons the leading order low-energy parameters of the effective theory are the spin stiffness ρ_s and the spin-wave velocity c. Once these two parameters have been determined in terms of microscopic parameters, like e.g. the Hubbard model parameters t and U, numerical simulations or even experiments, the low-energy physics of the underlying half-filled microscopic system is completely and correctly described by the effective theory. Remarkably, some physical phenomena that are practically inaccessible within microscopic models can be tackled analytically in the effective theory.

If one wants to address questions in the doped materials, i.e. away from half-filling, an effective theory corresponding to baryon chiral perturbation theory of QCD must be developed. To our best knowledge, no such effective theory was constructed so far in a fully systematic way. In this thesis, for the first time we introduce a systematic low-energy effective field theory for magnons and charge carriers in an antiferromagnet. The resulting theory will describe the interactions of electrons and holes (with respect to half-filling) in an antiferromagnet through magnon exchange. This is indeed in complete analogy to the baryon chiral perturbation theory of QCD, which describes the interactions of baryons and antibaryons through pion exchange. Furthermore, and most important, in contrast to the strongly correlated electrons and holes of the microscopic systems, the electrons and holes of the effective theory are effective degrees of freedom that are only weakly coupled to the magnons. Consequently, one may expect that the effective theory is easier to solve than the corresponding underlying microscopic systems.

There is one difference with respect to the QCD case, however, in that the doping destroys the antiferromagnetic background, whereas in QCD the hadronic vacuum stays intact when one adds baryons. In particular, it is known experimentally that the antiferromagnetism is destroyed before one enters the superconducting phase. Nevertheless, the destruction of antiferromagnetism just means the absence of infinite-range antiferromagnetic correlations, yet such of only a finite-range still exist. As a result of the finite correlation length, the magnons pick up a finite mass. In addition, in $2+1$ dimensions, according to the Mermin-Wagner theorem, a continuous symmetry can break down spontaneously only at exactly zero temperature. More precisely, the magnons pick up a mass that is exponentially small in the inverse temperature [68, 69]. Even so, the effective theory stays valid, as long as the magnons still exist among the lightest and hence relevant degrees of freedom in the low-energy range. This is similar to the QCD case, where the pions are not exactly massless either. In addition, one knows, since antiferromagnetism at finite inverse temperature is an experimental fact, that there must be at least small interactions between the layers in the actual cuprate materials.

The construction of baryon chiral perturbation theory is based on a symmetry analysis of the underlying fundamental microscopic system—the QCD. This suggests that, in

order to construct the effective theory for the cuprates, one should start with a concrete microscopic system as well. The actual cuprate materials, however, are quite complex and contain many features assumed to be irrelevant for potential mechanisms leading to high-temperature superconductivity. For this reason we choose the much simpler and clearly defined Hubbard model as the underlying microscopic system. Note that in contrast to QCD, which is generally accepted as a fundamental theory of Nature, the Hubbard model is, as its name already indicates, just a model. However, it is believed that the Hubbard model shares the relevant symmetry properties with the real cuprate materials. Thus, the Hubbard model is a good starting point in order to construct the effective theory for the cuprates, without having to deal with irrelevant details.

The symmetries that are captured by the Hubbard model are the $SU(2)_s$ spin rotational symmetry which is spontaneously broken down to the $U(1)_s$ subgroup in the antiferromagnetic phase (at least at zero temperature), as well as the $U(1)_Q$ fermion number symmetry whose breakdown would signal superconductivity. Another important symmetry is the translation by one lattice spacing which toggles the orientation of the staggered magnetization vector due to the antiferromagnetic alignment of the spins. Further, there are the space-time symmetries of 90 degrees rotations and reflections of the square crystal lattice, as well as time-reversal. In addition to these symmetries, that are all present in the actual cuprate materials, the Hubbard model possesses also an $SU(2)_Q$ symmetry which is a non-Abelian extension of the $U(1)_Q$ charge symmetry. This symmetry, which was first discussed by Yang and Zhang [77, 78], is specific to the Hubbard model and hence not expected to be generally present in the actual cuprate materials. However, it is not excluded that it may be a relevant approximate symmetry in specific samples. A summary of correspondences between QCD and antiferromagnets is presented in Table 1.1. Connections between QCD and condensed matter physics have also been discussed in [80].

It is a priori not clear, however, how all these symmetries manifest themselves in the effective theory. In order to figure out how the effective fermionic degrees of freedom transform under the various symmetries, we couple, as an intermediate step, the Hubbard model to a magnon background field. In this way the effective fermion fields inherit their transformation properties from the underlying microscopic degrees of freedom. Still, there is more than one way to establish such an interface. Consequently one has to make a choice. In this thesis as well as in the corresponding paper [95] we assume the simplest such interface. The resulting effective theory then describes holes and electrons that are located at lattice momenta $(0, 0)$ and (π, π) in the Brillouin zone. On the other hand, experiments indicate that the charge carriers are located at lattice momenta $(\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$ in the Brillouin zone. It is in fact possible to also construct an effective theory for this case, if one chooses the interface appropriately. In a further work [96] such an effective theory for holes only was indeed already constructed.

Possible basic applications of the effective theory to be constructed in this thesis include magnon-hole and magnon-electron scattering and the determination of the resulting long-range effective potential between the charge carriers. More ambitious applications could comprise the reduction of the staggered magnetization upon doping, the formation of a

spiral phase, or even a systematic study of potential mechanisms for the preformation of electron or hole pairs in the antiferromagnetic phase. Except for the derivation of the dispersion relation of the charge carriers, however, we do not consider applications yet, but concentrate entirely on the construction of the effective theory itself.

	QCD	Antiferromagnets
Broken phase	Hadronic vacuum	Antiferromagnetic phase
Global symmetry	Chiral symmetry	Spin rotations
Symmetry group G	$SU(2)_L \otimes SU(2)_R$	$SU(2)_s$
Unbroken subgroup H	$SU(2)_{L=R}$	$U(1)_s$
Goldstone boson	Pion	Magnon
Goldstone field in G/H	$U(x) \in SU(2)$	$\vec{e}(x) \in S^2$
Order parameter	Chiral condensate	Staggered magnetization
Coupling strength	Pion decay constant F_π	Spin stiffness ρ_s
Propagation speed	Velocity of light	Spin-wave velocity
Conserved charge	Baryon number $U(1)_B$	Electric charge $U(1)_Q$
Charged particle	Nucleon or antinucleon	Electron or hole
Long-range force	Pion exchange	Magnon exchange
Dense phase	Nuclear or quark matter	High-T_c superconductor
Microscopic description	Lattice QCD	Hubbard-type models
Effective description of Goldstone bosons	Chiral perturbation theory	Magnon effective theory
Effective description of charged fields	Baryon chiral perturbation theory	Effective theory presented here

Table 1.1: *Correspondences between QCD and antiferromagnets.*

1.5 Organization of this thesis

The rest of this thesis is organized as follows. In the first part we briefly review the effective theory for QCD. To this end we first present in Chapter 2 the QCD Lagrangian and its symmetries as the underlying fundamental microscopic theory for the effective theory. In Chapter 3 we then introduce chiral perturbation theory (χPT) which is the effective theory for QCD involving pions as the only degrees of freedom. Finally, in Chapter 4 we present baryon chiral perturbation theory (BχPT) which involves both pions and fermions as the degrees of freedom.

In the second part, which is also the main part of this thesis, we construct, in complete analogy to the QCD case, the effective theory for charge carriers in an antiferromagnet. As a first step of this project we introduce in Chapter 5 the Hubbard model which serves as the underlying microscopic system for the effective theory to be constructed. To this end

we also perform a detailed symmetry analysis of the Hubbard model. In Chapter 6 we then introduce the effective theory for antiferromagnets which involves magnons as the only degrees of freedom and corresponds to the χPT of QCD. As a preparation to the introduction of the effective fermionic degrees of freedom we also construct in this chapter the non-linear realization of the $SU(2)_s$ spin symmetry. To figure out how the effective fermionic degrees of freedom transform under the various symmetries in the effective theory, we couple in Chapter 7, as an intermediate step, the Hubbard model to a magnon background field. The resulting transformation laws are then used to construct in Chapter 8 the effective theory for charge carriers in an antiferromagnet which involves both magnons and fermions as the degrees of freedom and corresponds to the BχPT of QCD. This chapter also contains a first application of the effective theory, namely the calculation of the dispersion relation of the charge carriers. Most high-T_c materials are hole-doped, such that the holes are the relevant charge carriers in these materials, whereas the electrons only play a minor role. Hence, it is natural to ask how the constructed effective theory for holes and electrons simplifies if one only allows holes as charge carriers. This is studied in Chapter 9 by means of the t-J model as the underlying microscopic system. Finally in Chapter 10 we present our conclusions while some technical details and the explicit construction of the leading expressions of the effective theory are discussed in five appendices.

PART I

Effective field theory for QCD

2 The QCD action and its symmetries

In this chapter we briefly review Quantum Chromodynamics (QCD) which is generally accepted as to be the fundamental microscopic theory of Nature which describes quarks and gluons, and their interactions. The QCD action can be regarded as to consist of three parts, i.e.

$$S_{\mathrm{QCD}}[\overline{\psi},\psi,A] = S_{\mathrm{YM}}[A] + S_{\mathrm{F}}[\overline{\psi},\psi,A] + S_{\mathrm{M}}[\overline{\psi},\psi], \tag{2.1}$$

where ψ and $\overline{\psi}$ represent the quark fields and A the gluon fields to be introduced below. The Yang-Mills part $S_{\mathrm{YM}}[A]$ describes the purely gluonic self-interactions, whereas the fermion part $S_{\mathrm{F}}[\overline{\psi},\psi,A]$ describes on the one hand the propagation of massless free quarks and on the other hand their interactions with the gluons via a gauge coupling. Finally, the mass term $S_{\mathrm{M}}[\overline{\psi},\psi]$ renders the quarks massive and at the same time explicitly breaks the chiral symmetry.

The quark fields exist in different colors that are distinct degrees of freedom and are exact copies of each other. In addition, they exist in different flavors that are again copies of each other but with individual masses. QCD can be formulated a priori for any number of colors N_{c} and flavors N_{f}. However, there is overwhelming experimental evidence that Nature has chosen $N_{\mathrm{c}} = 3$ and $N_{\mathrm{f}} = 6$ for reasons yet not fully understood. In the next sections we present each part of the QCD action for arbitrary N_{c} and N_{f}.

In doing so, we work in Euclidean space-time, i.e. x always denotes a point in $(3+1)$-dimensional Euclidean space-time. Consequently, we also have to use the Euclidean Dirac matrices which are all Hermitean and obey the anti-commutation relations

$$\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}, \qquad \{\gamma_\mu,\gamma_5\} = 0, \qquad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4. \tag{2.2}$$

Furthermore we use the path integral formalism in which the fermion fields ψ and $\overline{\psi}$ represent two independent Grassmann-valued degrees of freedom, and the quantum theory is defined by a functional integral over all degrees of freedom, i.e.

$$Z = \int \mathcal{D}\overline{\psi}\mathcal{D}\psi\mathcal{D}A \ \exp(-S_{QCD}[\overline{\psi},\psi,A]). \tag{2.3}$$

2.1 $SU(N_{\mathrm{c}})$ Yang-Mills action for the gluon gauge field

The gluons are described by an anti-Hermitean non-Abelian $SU(N_{\mathrm{c}})$ gauge field given by

$$A_\mu(x) = igA^a_\mu(x)T^a, \tag{2.4}$$

where g is the gauge coupling, $a = 1, 2, \ldots, N_\mathrm{c}^2 - 1$ sums over the Hermitean generators T^a of the $SU(N_\mathrm{c})$ algebra, and $A_\mu^a(x) \in \mathbb{R}$ are the vector potentials of the gluon field. A derived quantity of the algebra-valued gauge field $A_\mu(x)$ is the field strength defined as

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)], \tag{2.5}$$

which is algebra-valued as well. It is used to finally formulate the Euclidean Yang-Mills action which takes the form

$$S_\mathrm{YM}[A] = \int d^4x \; \frac{1}{2g^2} \mathrm{Tr}(F_{\mu\nu} F_{\mu\nu}). \tag{2.6}$$

This action is invariant under $SU(N_\mathrm{c})$ gauge transformations—the local group-valued transformations $\Omega(x) \in SU(N_\mathrm{c})$. They act on the gauge field as

$$A'_\mu(x) = \Omega(x)(A_\mu(x) + \partial_\mu)\Omega(x)^\dagger, \tag{2.7}$$

such that the field strength transforms as

$$F'_{\mu\nu}(x) = \Omega(x) F_{\mu\nu}(x) \Omega(x)^\dagger. \tag{2.8}$$

It is now obvious that the Yang-Mills action is indeed invariant under the $SU(N_\mathrm{c})$ gauge transformations, i.e. that

$$S_\mathrm{YM}[A'] = S_\mathrm{YM}[A]. \tag{2.9}$$

Note that there is an additional potential term in the gluon action—the topological charge—which we do not discuss here, since it is not of interest for our symmetry-centric point of view.

2.2 Fermion action for N_f flavors of massless quarks

Now we augment the pure gluon theory with N_f flavors of massless quarks that come each in N_c colors. As usual for fermions, the quark fields are described by anti-commuting Dirac spinors $\psi(x)$ and $\overline{\psi}(x)$. However, since we are adding $N_\mathrm{f} N_\mathrm{c}$ quark fields, these spinors consist of $N_\mathrm{f} N_\mathrm{c}$ single-particle spinors, one for each flavor and color.

These quark fields transform under $\Omega(x) \in SU(N_\mathrm{c})$ gauge transformations as

$$\psi'(x) = \Omega(x)\psi(x), \qquad \overline{\psi}'(x) = \overline{\psi}(x)\Omega(x)^\dagger. \tag{2.10}$$

Note that this transformation intermixes the different quark colors but leaves the quark flavors and the content of the individual single-particle spinors untouched.

The Euclidean fermion action for massless quarks takes the form

$$S_\mathrm{F}[\overline{\psi}, \psi, A] = \int d^4x \; \overline{\psi} \gamma_\mu (\partial_\mu + A_\mu) \psi, \tag{2.11}$$

which is gauge invariant by construction, since the gluon field $A_\mu(x)$ enters via a gauge coupling. Note that the Dirac matrices γ_μ act on each individual single-particle spinor within the multi-flavor multi-color quark fields $\psi(x)$ and $\overline{\psi}(x)$.

In order to point out the chiral symmetry, we now decompose the quark fields into their left- and right-handed spinor components. To this end we first introduce the chiral projectors

$$P_R = \frac{1+\gamma_5}{2}, \qquad P_L = \frac{1-\gamma_5}{2}, \tag{2.12}$$

which obey

$$P_L^2 = P_L, \quad P_R^2 = P_R, \qquad P_L P_R = P_R P_L = 0, \qquad P_L + P_R = \mathbb{1}, \tag{2.13}$$

and by means of which the components can be expressed as

$$\begin{aligned} \psi_L(x) &= P_L\psi(x), \qquad \psi_R(x) = P_R\psi(x), \qquad \psi(x) = \psi_L(x) + \psi_R(x), \\ \overline{\psi}_L(x) &= \overline{\psi}(x)P_R, \qquad \overline{\psi}_R(x) = \overline{\psi}(x)P_L, \qquad \overline{\psi}(x) = \overline{\psi}_L(x) + \overline{\psi}_R(x). \end{aligned} \tag{2.14}$$

Note the mixing of L and R on the $\overline{\psi}(x)$ fields. Using this notation the Euclidean fermion action of Eq. (2.11) takes the form

$$S_{\mathrm{F}}[\overline{\psi}, \psi, A] = \int d^4x \ \left[\overline{\psi}_L\gamma_\mu(\partial_\mu + A_\mu)\psi_L + \overline{\psi}_R\gamma_\mu(\partial_\mu + A_\mu)\psi_R\right], \tag{2.15}$$

in which the left- and right-handed quark field components completely decouple. This observation gives rise to the chiral symmetry $U(N_{\mathrm{f}})_L \otimes U(N_{\mathrm{f}})_R$ under which the left- and right-handed components transform individually as

$$\begin{aligned} \psi_L(x)' &= L\,\psi_L(x), \qquad \overline{\psi}_L(x)' = \overline{\psi}_L(x)L^+, \qquad L \in U(N_{\mathrm{f}})_L, \\ \psi_R(x)' &= R\,\psi_R(x), \qquad \overline{\psi}_R(x)' = \overline{\psi}_R(x)R^+, \qquad R \in U(N_{\mathrm{f}})_R. \end{aligned} \tag{2.16}$$

Note that these transformations intermix the different quark flavors but leave the quark colors and the content of the individual single-particle spinors untouched. It is obvious that the action of Eq. (2.15) is indeed invariant under the chiral symmetry

$$\begin{aligned} U(N_{\mathrm{f}})_L \otimes U(N_{\mathrm{f}})_R &\cong SU(N_{\mathrm{f}})_L \otimes SU(N_{\mathrm{f}})_R \otimes U(1)_L \otimes U(1)_R \\ &\cong SU(N_{\mathrm{f}})_L \otimes SU(N_{\mathrm{f}})_R \otimes U(1)_B \otimes U(1)_A. \end{aligned} \tag{2.17}$$

Here we have introduced the baryon number conservation symmetry $U(1)_B = U(1)_{L=R}$ and the axial symmetry $U(1)_A = U(1)_{L=-R}$. Due to an anomaly in the axial symmetry $U(1)_A$, the symmetry of the quantum theory is reduced to

$$SU(N_{\mathrm{f}})_L \otimes SU(N_{\mathrm{f}})_R \otimes U(1)_B. \tag{2.18}$$

We do not discuss the origin of the axial anomaly here, however, since this topic is again out of scope for our needs.

2.3 Quark mass term and explicit breaking of chiral symmetry

The introduced quarks are up to now still massless. In Nature, however, the quarks are massive. To account for this experimental fact we now add the quark mass term to the Euclidean action. Using the left- and right-handed component notation of Eq. (2.14) the quark mass term takes the form

$$S_{\mathrm{M}}[\overline{\psi},\psi] = \int d^4x \;\; \left[\overline{\psi}_R \mathcal{M} \psi_L + \overline{\psi}_L \mathcal{M}^\dagger \psi_R\right], \tag{2.19}$$

where the quark mass matrix, which operates in flavor space, is given by

$$\mathcal{M} = \mathrm{diag}(m_u, m_d, m_s, \ldots, m_{N_\mathrm{f}}). \tag{2.20}$$

The mass term of Eq. (2.19) is again invariant under the $SU(N_\mathrm{c})$ gauge transformations of Eq. (2.10). However, in contrast to the fermion action of Eq. (2.15), the mass term mixes the left- and right-handed components of the quark fields such that it is not invariant under the chiral transformations of Eq. (2.16), i.e. it explicitly breaks the chiral symmetry.

If the quark masses were all equal (however, in Nature they are not), i.e. if $\mathcal{M} = m\mathbb{1}$, the mass term would be at least invariant under simultaneous chiral transformations $L = R$. The chiral symmetry would then be explicitly broken down to

$$U(N_\mathrm{f})_{L=R} \cong SU(N_\mathrm{f})_{L=R} \otimes U(1)_{L=R} = SU(N_\mathrm{f})_F \otimes U(1)_B, \tag{2.21}$$

which would correspond to flavor and baryon number symmetry.

In Nature the quark masses are indeed all different, such that the chiral symmetry is in fact explicitly broken down to

$$\prod_{f=1}^{N_\mathrm{f}} U(1)_f = U(1)_u \otimes U(1)_d \otimes U(1)_s \otimes \cdots \otimes U(1)_{N_\mathrm{f}}. \tag{2.22}$$

However, this statement has to be refined. To this end, let us first have a look at the actual quark masses as they appear in Nature, i.e.

$$\begin{aligned}
m_u &= (\;\; 1.5 \text{ to } \;\; 5\,) \text{ MeV}, \qquad & m_c &= (\;\; 1.1 \text{ to } \;\; 1.4\,) \text{ GeV},\\
m_d &= (\;\; 3 \;\; \text{to } \;\; 9\,) \text{ MeV}, \qquad & m_b &= (\;\; 4.1 \text{ to } \;\; 4.4\,) \text{ GeV},\\
m_s &= (\, 60 \;\; \text{to } 170\,) \text{ MeV}, \qquad & m_t &= (\, 168.6 \text{ to } 179.0\,) \text{ GeV}.
\end{aligned} \tag{2.23}$$

If we compare these quark masses to a typical scale of QCD, like e.g. the mass of the proton $m_p = 938$ MeV which is the lightest baryon, we realize that the up and down quark masses are in contrast vanishingly small. Also the strange quark mass is much smaller, whereas the other three quark flavors are all heavier. The explicit breaking of chiral symmetry caused by the three light quark flavors is therefore only of weak nature. As a result, the

part of the total chiral symmetry which corresponds to the up and down quark flavors, i.e.

$$SU(2)_L \otimes SU(2)_R, \tag{2.24}$$

is a very good approximate global symmetry, whereas the part corresponding to all three light quark flavors, i.e.

$$SU(3)_L \otimes SU(3)_R, \tag{2.25}$$

is broken more strongly. One can therefore expect that these symmetries manifest themselves in some way in the QCD spectrum. We will illuminate this issue in the next section when we discuss the spontaneous breakdown of the chiral symmetry.

Note that the knowledge of the actual quark mass values as presented in Eq. (2.23) is based on experimental data in comparison with theory. However, there is at present not at all a theoretical understanding where this specific values originate from. In particular, it is mysterious why there are three light quark flavors. Hence, the approximate symmetries of Eqs. (2.24) and (2.25) seem to be arbitrary, unless one understands the relevant physics beyond the Standard Model behind it.

2.4 Spontaneous breakdown of chiral symmetry

The approximate chiral symmetries of Eqs. (2.24) and (2.25) operate in flavor space and hence they translate the different flavors into each other. As a result, one expects in the hadronic spectrum a corresponding set of almost mass-degenerate particles that reflect these symmetries. In particular, since the left- and right-handed components can be rotated individually, one expects mass-degenerate parity doublets of hadrons. However, no such doublets exist in the QCD spectrum. On the other hand, the hadrons can be indeed classified as multiplets, but only according to simultaneous chiral transformations $L = R$. Hence, the approximate chiral symmetries of Eqs. (2.24) and (2.25) are visible in the spectrum only as approximate $SU(2)_F = SU(2)_{L=R}$ and $SU(3)_F = SU(3)_{L=R}$ flavor symmetries, respectively, thereby being the $SU(3)_F$ symmetry less accurate. Furthermore, the spectrum of QCD contains particles that are significantly lighter than the typical QCD scale. These are the three very light pions and the somewhat heavier four kaons and the eta, i.e.

$$\begin{aligned} m_{\pi^0} &= 135.0 \text{ MeV}, & m_{K^+}, m_{K^-} &= 493.7 \text{ MeV}, \\ m_{\pi^+}, m_{\pi^-} &= 139.6 \text{ MeV}, & m_{K^0}, m_{\bar{K}^0} &= 497.7 \text{ MeV}, \\ & & m_\eta &= 547.3 \text{ MeV}. \end{aligned} \tag{2.26}$$

One concludes from all this experimental evidences that the chiral symmetry in QCD must be spontaneously broken.

According to the Goldstone theorem, the spontaneous breakdown of an exact global continuous symmetry group G to a subgroup H gives rise to massless bosonic excitations—the Goldstone bosons—that live in the coset space G/H. The number of massless Goldstone

bosons is hence given by $\dim(G) - \dim(H)$, i.e. the difference of the number of generators of the full symmetry group G and the subgroup H that remains unbroken.

In QCD with massless quarks, where the chiral symmetry is still intact, the full symmetry group is

$$G = SU(N_\mathrm{f})_L \otimes SU(N_\mathrm{f})_R \otimes U(1)_B, \tag{2.27}$$

while the unbroken subgroup is the flavor symmetry

$$H = SU(N_\mathrm{f})_{L=R} \otimes U(1)_B, \tag{2.28}$$

such that there would be $N_\mathrm{f}^2 - 1$ massless Goldstone bosons that are described by fields in the coset space

$$G/H = SU(N_\mathrm{f}). \tag{2.29}$$

However, in the QCD of Nature most of the chiral symmetry is explicitly broken down due to the non-vanishing quark masses. There are only the residual approximate chiral symmetries of Eqs. (2.24) and (2.25) one can expect to deliver Goldstone bosons on their spontaneous breakdown. Still, the small explicit breaking of chiral symmetry renders these Goldstone bosons massive as well.

Indeed, the spontaneous breakdown of the approximate chiral symmetry of Eq. (2.24) leads to the three pions, i.e.

$$SU(2)_L \otimes SU(2)_R \;\;\rightarrow\;\; SU(2)_{L=R} \;\;\Rightarrow\;\; \pi^0, \pi^+, \pi^-, \tag{2.30}$$

which are according the Eq. (2.26) in fact very light but not massless. Furthermore, the almost unbroken flavor subgroup $SU(2)_{L=R}$ causes the three pions to be almost mass-degenerate.

Likewise, the spontaneous breakdown of the larger but less accurate approximate chiral symmetry of Eq. (2.25) leads to eight massive Goldstone bosons—the four kaons and the eta in addition to the three pions from before, i.e.

$$SU(3)_L \otimes SU(3)_R \;\;\rightarrow\;\; SU(3)_{L=R} \;\;\Rightarrow\;\; \pi^0, \pi^+, \pi^-, K^+, K^-, K^0, \bar{K}^0, \eta. \tag{2.31}$$

According to Eq. (2.26) the kaons and the eta are quite a bit heavier than the pions, and as a consequence also the mass-degeneracy among these eight mesons is much larger. Clearly, this is due to the strange quark mass, which is quite large with respect to the ones of the up and down quarks. This breaks both chiral and flavor symmetry more explicitly.

Numerical simulations in lattice QCD confirm that the chiral symmetry is indeed broken. One detects the spontaneous breakdown of the chiral symmetry by investigating the chiral condensate, i.e.

$$\langle \overline{\psi}\psi \rangle = \langle 0|\overline{\psi}(x)\psi(x)|0\rangle = \langle 0|\overline{\psi}_R(x)\psi_L(x) + \overline{\psi}_L(x)\psi_R(x)|0\rangle, \tag{2.32}$$

which is the corresponding order parameter. It is invariant under simultaneous chiral transformations $L = R$, but not under general independent ones. Hence it vanishes as long as chiral symmetry is intact, but it is nonzero when chiral symmetry is broken.

3 Chiral perturbation theory

As we saw in the previous chapter, the Goldstone bosons are the lightest particles in the QCD spectrum. As a consequence they dominate the low-energy physics of QCD. It is therefore possible to construct a low-energy effective theory which involves exclusively Goldstone bosons as degrees of freedom.

The Goldstone bosons are described by fields in the coset space $G/H = SU(N_\mathrm{f})$, i.e. by special unitary matrices $U(x) \in SU(N_\mathrm{f})$. Under the global chiral transformations of Eq. (2.16) they transform as

$$U(x)' = LU(x)R^\dagger, \qquad L \in SU(N_\mathrm{f})_L, \quad R \in SU(N_\mathrm{f})_R. \tag{3.1}$$

The effective action is organized as a systematic expansion in powers of the momentum, which in configuration space corresponds to an expansion in derivatives. The leading order terms of the effective action for pions is given by

$$S[U] = \int d^4x \, \left[\frac{F_\pi^2}{4} \mathrm{Tr}(\partial_\mu U^\dagger \partial_\mu U) - \frac{\langle \overline{\psi}\psi \rangle}{2N_\mathrm{f}} \mathrm{Tr}(\mathcal{M}U^\dagger + U\mathcal{M}^\dagger) \right]. \tag{3.2}$$

The first term describes the interactions between the Goldstone bosons. It is obvious that this term is invariant under the chiral transformations of Eq. (3.1). The second term, on the other hand, is the mass term which accounts for the quark masses and explicitly breaks the chiral symmetry. It transforms under the chiral symmetry as

$$\mathrm{Tr}(\mathcal{M}U'^\dagger + U'\mathcal{M}^\dagger) = \mathrm{Tr}(\mathcal{M}RU^\dagger L^\dagger + LUR^\dagger \mathcal{M}^\dagger). \tag{3.3}$$

Hence, if all quark masses are equal, i.e. the mass matrix $\mathcal{M} = m\mathbb{1}$, the mass term is at least invariant under the flavor symmetry $SU(N_\mathrm{f})_{L=R}$. However, the flavor symmetry is reduced to $\prod_{f=1}^{N_\mathrm{f}} U(1)_f$, if the quark masses are all different. This is indeed exact the same behavior as in the underlying microscopic QCD.

At leading order, the low-energy dynamics of massless QCD is completely determined by chiral symmetry and only two low-energy parameters, i.e. the pion decay constant F_π and the chiral condensate $\langle \overline{\psi}\psi \rangle$. However, at higher energies, additional terms arise in the action such that the number of low-energy parameters rapidly increases.

Chiral perturbation theory is a systematic low-energy expansion around the classical vacuum given by $U(x) = \mathbb{1}$. One parameterizes the Goldstone boson field $U(x)$, for example, as

$$U(x) = \exp(2i\pi^a(x)T^a/F_\pi), \tag{3.4}$$

where the T^a (with $a = 1, 2, \ldots, N_\mathrm{c}^2 - 1$) are the generators of $SU(N_\mathrm{f})$. This parameterization is then expanded in powers of $\pi^a(x)$.

4 Baryon chiral perturbation theory

The effective theory described so far operates only in the baryon number sector $B = 0$. Now we want to extend its capabilities to the $B \neq 0$ sectors.

Nucleons, for example, are described in the effective theory by Dirac spinor fields $\psi(x)$ and $\overline{\psi}(x)$. They do not transform directly under the global chiral symmetry $L \otimes R \in SU(2)_L \otimes SU(2)_R$, but under the non-linearly realized chiral symmetry which is a local transformation $V(x)$ in the flavor subgroup $SU(2)_{L=R}$, i.e.

$$\psi(x)' = V(x)\psi(x), \qquad \overline{\psi}(x)' = \overline{\psi}(x)V(x)^\dagger. \tag{4.1}$$

The non-linearly realized local chiral transformation $V(x)$ depends on the global chiral transformations L and R as well as on the local Goldstone boson field $U(x)$ and it can be written as

$$\begin{aligned} V(x) &= R(R^\dagger L U(x))^{1/2}(U(x)^{1/2})^\dagger \\ &= L(L^\dagger R U(x)^\dagger)^{1/2} U(x)^{1/2}. \end{aligned} \tag{4.2}$$

Note that despite the fact that $V(x)$ has the form of a local $SU(2)_{L=R}$ transformation, it represents the global $SU(2)_L \otimes SU(2)_R$ symmetry. Also note that for transformations in the unbroken flavor subgroup $SU(2)_{L=R}$, i.e. for $L = R$, also the local transformation $V(x)$ reduces to the global flavor transformation $V(x) = L = R$.

In order to couple the nucleons to the pion field, one needs an $SU(2)$ flavor 'gauge' field. To this end we first introduce the field $u(x) \in SU(2)$ which is defined as

$$u(x) = U(x)^{1/2}, \tag{4.3}$$

and represents the middle of the geodesic connecting $U(x)$ with the unit-matrix $\mathbb{1}$ in the group space of $SU(2)$. The field $u(x)$ transforms under the global chiral symmetry as

$$u(x)' = Lu(x)V(x)^\dagger = V(x)u(x)R^\dagger. \tag{4.4}$$

Finally we introduce the anti-Hermitean composite field

$$v_\mu(x) = \frac{1}{2}[u(x)^\dagger \partial_\mu u(x) + u(x)\partial_\mu u(x)^\dagger], \tag{4.5}$$

which indeed transforms as a 'gauge' field, i.e.

$$\begin{aligned} v_\mu(x)' = \frac{1}{2}\big[&V(x)u(x)^\dagger L^\dagger \partial_\mu (Lu(x)V(x)^\dagger) \\ &+ V(x)u(x)R^\dagger \partial_\mu (Ru(x)^\dagger V(x)^\dagger)\big] = V(x)(v_\mu(x) + \partial_\mu)V(x)^\dagger. \end{aligned} \tag{4.6}$$

Furthermore, there is also a Hermitean composite field, defined as

$$a_\mu(x) = \frac{i}{2}[u(x)^\dagger \partial_\mu u(x) - u(x)\partial_\mu u(x)^\dagger], \tag{4.7}$$

which transforms as

$$a_\mu(x)' = \frac{1}{2}\big[V(x)u(x)^\dagger L^\dagger \partial_\mu (Lu(x)V(x)^\dagger) \\ - V(x)u(x)R^\dagger \partial_\mu (Ru(x)^\dagger V(x)^\dagger)\big] = V(x)a_\mu(x)V(x)^\dagger. \tag{4.8}$$

Using these fields, the leading order terms in the Euclidean action of the low-energy effective theory for pions and nucleons are given by

$$S[\overline{\psi},\psi,U] = \int d^4x \, \Big\{ M\overline{\psi}\psi + \overline{\psi}\gamma_\mu(\partial_\mu + v_\mu)\psi + ig_{\mathrm{A}}\overline{\psi}\gamma_\mu\gamma_5 a_\mu \psi \\ + \frac{F_\pi^2}{4}\mathrm{Tr}[\partial_\mu U^\dagger \partial_\mu U] - \frac{\langle\overline{\psi}\psi\rangle}{4}\mathrm{Tr}[\mathcal{M}U^\dagger + \mathcal{M}^\dagger U]\Big\}. \tag{4.9}$$

The parameter M represents the nucleon mass dynamically generated by the spontaneous breakdown of chiral symmetry. Hence, it is also non-zero for vanishing quark masses. As a result, the nucleon mass term is chirally invariant. Finally the parameter g_{A} is the coupling to the isovector axial current. Note that there are also 4-fermion contact interactions for the nucleons at leading order which are not shown here. This is since one considers usually only the one nucleon sector of the theory to which these terms do not contribute.

PART II

Effective field theory for antiferromagnets

5 The Hubbard model as the underlying microscopic system

The construction of a low-energy effective theory is mainly based on symmetry considerations. For this reason it is helpful to have a concrete appropriate microscopic system whose symmetries can then be adopted in the effective theory. In this thesis we consider the Hubbard model as the underlying microscopic system for the effective theory to be constructed. However, the Hubbard model just serves as one representative of a large class of systems, including the actual high-T_c materials. The essential point thereby is that the Hubbard model shares important symmetries, e.g. an $SU(2)_s$ spin symmetry and a $U(1)_Q$ fermion number symmetry with the actual materials. At half-filling the $U(1)_Q$ symmetry of the Hubbard model even extends to an $SU(2)_Q$ symmetry. Although this extended symmetry is not exact in actual materials, it may still be approximately realized. Consequently we will also investigate the $SU(2)_Q$ symmetry in the framework of the effective theory as a non-mandatory option.

5.1 Definition of the Hubbard model

The Hubbard model is defined by the Hamiltonian

$$\begin{aligned} H = &- t \sum_{x,i} (c^\dagger_{x\uparrow} c_{x+\hat{i}\uparrow} + c^\dagger_{x+\hat{i}\uparrow} c_{x\uparrow} + c^\dagger_{x\downarrow} c_{x+\hat{i}\downarrow} + c^\dagger_{x+\hat{i}\downarrow} c_{x\downarrow}) \\ &+ U \sum_x c^\dagger_{x\uparrow} c_{x\uparrow} c^\dagger_{x\downarrow} c_{x\downarrow} - \mu' \sum_x (c^\dagger_{x\uparrow} c_{x\uparrow} + c^\dagger_{x\downarrow} c_{x\downarrow}). \end{aligned} \tag{5.1}$$

Here x denotes the sites of a 2-dimensional spatial square lattice with lattice spacing a and $\hat{i}$ is a vector of length a pointing in the i-direction. Furthermore, t is the nearest-neighbor hopping parameter, while $U > 0$ is the strength of the screened on-site Coulomb repulsion. Finally μ' is the chemical potential for fermion number. The fermion creation and annihilation operators obey the standard anticommutation relations

$$\{c^\dagger_{xs}, c_{ys'}\} = \delta_{xy}\delta_{ss'}, \qquad \{c_{xs}, c_{ys'}\} = \{c^\dagger_{xs}, c^\dagger_{ys'}\} = 0. \tag{5.2}$$

5.2 The $SU(2)_s$ and $U(1)_Q$ symmetries

As already mentioned, the Hubbard model possesses an $SU(2)_s$ spin symmetry and a $U(1)_Q$ fermion number symmetry. In order to state the Hamiltonian in a manifestly $SU(2)_s$

invariant form, we introduce the $SU(2)_s$ Pauli spinor

$$c_x = \begin{pmatrix} c_{x\uparrow} \\ c_{x\downarrow} \end{pmatrix}. \tag{5.3}$$

Indeed, in terms of this spinor the Hamiltonian of Eq. (5.1) takes up to an irrelevant constant the manifestly $SU(2)_s$ invariant form

$$H = -t \sum_{x,i} (c_x^\dagger c_{x+\hat{i}} + c_{x+\hat{i}}^\dagger c_x) + \frac{U}{2} \sum_x (c_x^\dagger c_x - 1)^2 - \mu \sum_x (c_x^\dagger c_x - 1). \tag{5.4}$$

At the same time we have introduced here $\mu = \mu' - \frac{1}{2}U$ as the chemical potential for fermion number relative to half-filling, i.e. $\mu = 0$ implies an average density of one fermion per lattice site. The corresponding $SU(2)_s$ symmetry is generated by the total spin operator

$$\vec{S} = \sum_x \vec{S}_x = \sum_x c_x^\dagger \, \frac{\vec{\sigma}}{2} \, c_x, \tag{5.5}$$

where $\vec{\sigma}$ are the Pauli matrices, such that the individual components of $\vec{S}$ obey the standard $SU(2)$ commutation relations $[S_a, S_b] = i\varepsilon_{abc} S_c$. On the other hand, the $U(1)_Q$ symmetry is generated by the charge operator

$$Q = \sum_x Q_x = \sum_x (c_x^\dagger c_x - 1), \tag{5.6}$$

where we again count the fermion number relative to half-filling. It is easy to see that the above Hamiltonian conserves both spin and fermion number, i.e. $[H, \vec{S}] = [H, Q] = 0$, and that $[Q, \vec{S}] = 0$.

The infinitesimal generators $\vec{S}$ of $SU(2)_s$ can be used to construct the unitary operator

$$V = \exp(i\vec{\eta} \cdot \vec{S}), \tag{5.7}$$

which implements the corresponding symmetry transformations in the Hilbert space of the theory. As a result, the transformed $SU(2)_s$ spinors take the form

$$c'_x = V^\dagger c_x V = \exp(i\vec{\eta} \cdot \frac{\vec{\sigma}}{2}) c_x = g c_x, \qquad g = \exp(i\vec{\eta} \cdot \frac{\vec{\sigma}}{2}) \in SU(2)_s. \tag{5.8}$$

Similarly, the $U(1)_Q$ transformations are implemented by the unitary operator

$$W = \exp(i\omega Q), \tag{5.9}$$

such that

$${}^Q c_x = W^\dagger c_x W = \exp(i\omega) c_x, \qquad \exp(i\omega) \in U(1)_Q. \tag{5.10}$$

5.3 Antiferromagnetic ground state

For large positive U, the (very repulsive) Hubbard model at half-filling (i.e. $\mu = 0$) reduces to the antiferromagnetic spin $\frac{1}{2}$ quantum Heisenberg model given by the Hamiltonian

$$H = J \sum_{x,i} \vec{S}_x \cdot \vec{S}_{x+\hat{i}}, \tag{5.11}$$

in which the exchange coupling J is related to the original Hubbard model parameters as $J = 2t^2/U$. This result follows from second order of the perturbation theory in t/U. To leading order, i.e. completely neglecting the kinetic term proportional to t, there is an enormous number of degenerate ground states. Irrespective of spin, any state with exactly one fermion occupying each lattice site avoids the large on-site Coulomb repulsion and consequently represents a ground state for $t = 0$. There is no correction at first order in t/U, since the state resulting after a single hop of a spin cannot be again a ground state as just described. However, at second order of degenerate perturbation theory, a spin can virtually hop to a neighboring site occupied by a fermion with opposite spin and then hop back. On the other hand, virtual hops to sites occupied by a fermion with the same spin orientation are forbidden by the Pauli principle. Since the contributions of second order perturbation theory are always negative, this favors antiparallel spins and leads to the antiferromagnetic Heisenberg model of Eq. (5.11).

5.4 Displacement symmetries D and D'

Let us now consider displacements (i.e. translations) by one lattice spacing in the i-direction. In ferromagnetic systems, where all spins are aligned parallel, this operation does not affect the magnetization vector

$$\vec{M} = \sum_x \vec{S}_x, \tag{5.12}$$

which is the corresponding order parameter. Due to its trivial consequences, displacements by one lattice spacing do not play a particular role in ferromagnetic systems.

On the other hand, in antiferromagnetic systems like the Hubbard model at half-filling (i.e. $\mu = 0$), the spins are aligned antiparallel and consequently the corresponding order parameter is the staggered magnetization vector

$$\vec{M}_s = \sum_x (-1)^x \vec{S}_x. \tag{5.13}$$

The factor $(-1)^x = (-1)^{(x_1+x_2)/a}$ distinguishes between the sites of the even and the odd sublattice, i.e. the points on the even sublattice A have $(-1)^x = 1$ while the points on the

odd sublattice B have $(-1)^x = -1$. As a result, the staggered magnetization vector $\vec{M}_s$ flips the sign under a single displacement.

The displacement is generated by the unitary operator D which acts as

$$^{D}c_x = D^\dagger c_x D = c_{x+\hat{i}}, \tag{5.14}$$

and it turns out to be a symmetry of the Hubbard model Hamiltonian, i.e. $[H, D] = 0$. Obviously, both the $U(1)_Q$ and the $SU(2)_s$ symmetries commute with the displacement symmetry D, i.e. $[Q, D] = [\vec{S}, D] = 0$.

In the effective theory it will be useful to also consider the related symmetry D' which combines D with the special $SU(2)_s$ spin rotation $g = i\sigma_2$. This symmetry acts as

$$^{D'}c_x = D'^\dagger c_x D' = (i\sigma_2)\ ^{D}c_x = (i\sigma_2)c_{x+\hat{i}}. \tag{5.15}$$

Note that $[H, D'] = [D, D'] = [Q, D'] = 0$, but $[\vec{S}, D'] \neq 0$ because of the non-Abelian $SU(2)_s$ part of D'.

5.5 Space-time symmetries

In non-relativistic physics orbital angular momentum and spin are separately conserved and thus spin plays the role of an internal quantum number. Indeed, in the Hubbard model the $SU(2)_s$ spin symmetry is completely independent of the 90 degrees rotational invariance of the spatial square lattice. The 90 degrees rotation O acts on a spatial point $x = (x_1, x_2)$ as $Ox = (-x_2, x_1)$ and the fermion operators transform as

$$^{O}c_x = O^\dagger c_x O = c_{Ox}. \tag{5.16}$$

Parity turns x into $(-x_1, -x_2)$ and hence is equivalent to a 180 degrees rotation in two dimensions. Hence, it is more useful to consider the spatial reflection R at the x_1-axis which turns x into $Rx = (x_1, -x_2)$. Under this transformation the fermion operators transform as

$$^{R}c_x = R^\dagger c_x R = c_{Rx}. \tag{5.17}$$

The reflection at the orthogonal x_2-axis is a combination of the reflection R and the rotation O. One can also consider the reflection at an axis half between lattice points. This transformation is a combination of R with the displacement symmetry D. Similarly, a reflection at a lattice diagonal is a combination of R and O. Another important symmetry is time-reversal which is implemented by an antiunitary operator T.

It should be pointed out that, unlike the actual high-T_c materials, the Hubbard model is not Galilean invariant: In the actual materials translation as well as Galilean invariance are spontaneously broken by the formation of the crystal lattice. The corresponding Nambu-Goldstone bosons are the phonons which are known to play a central role in ordinary low-T_c superconductivity. In the Hubbard model, on the other hand, the lattice is imposed by hand, and thus translation and Galilean invariance are explicitly broken. In particular, phonons cannot arise because the lattice does not have its own physical degrees of freedom.

5.6 $SU(2)_Q$ symmetry

As first noted by Yang and Zhang [77, 78], the Abelian fermion number symmetry $U(1)_Q$ of the Hubbard model at half-filling (i.e. for $\mu = 0$) can be extended to a non-Abelian $SU(2)_Q$ symmetry generated by

$$Q^+ = \sum_x (-1)^x c^\dagger_{x\uparrow} c^\dagger_{x\downarrow}, \qquad Q^- = \sum_x (-1)^x c_{x\downarrow} c_{x\uparrow},$$
$$Q^3 = \sum_x \frac{1}{2}(c^\dagger_{x\uparrow} c_{x\uparrow} + c^\dagger_{x\downarrow} c_{x\downarrow} - 1) = \frac{1}{2} Q. \tag{5.18}$$

Writing $Q^\pm = Q^1 \pm iQ^2$, it is straightforward to show that indeed $[H, \vec{Q}] = 0$ for $\mu = 0$, and that in general $[H, Q^\pm] \neq 0$ for $\mu \neq 0$. Note that the factors $(-1)^x$ in $Q^\pm$ are required in order that the kinetic part of the Hamiltonian commutes with $Q^\pm$. Furthermore the $SU(2)_Q$ symmetry commutes with the $SU(2)_s$ symmetry, i.e. $[S^a, Q^b] = 0$, but it does not commute with the displacement symmetries D and D', i.e. $[D, Q^\pm] \neq 0$ and $[D', Q^\pm] \neq 0$, because $D^\dagger Q^\pm D = -Q^\pm$.

In complete analogy to the $SU(2)_s$ symmetry, one can state the Hamiltonian for $\mu = 0$ in a manifestly $SU(2)_Q$ invariant form. To this end we introduce the $SU(2)_Q$ spinor

$$d_x = \begin{pmatrix} c_{x\uparrow} \\ (-1)^x c^\dagger_{x\downarrow} \end{pmatrix}, \tag{5.19}$$

which also obeys the standard anticommutation relations

$$\{d^\dagger_{xa}, d_{yb}\} = \delta_{xy}\delta_{ab}, \qquad \{d_{xa}, d_{yb}\} = \{d^\dagger_{xa}, d^\dagger_{yb}\} = 0. \tag{5.20}$$

By means of this spinor one can express the generators of Eq. (5.18) as

$$\vec{Q} = \sum_x \vec{Q}_x = \sum_x d^\dagger_x \frac{\vec{\sigma}}{2} d_x, \tag{5.21}$$

where $\vec{\sigma}$ are again the Pauli matrices.

As before, the infinitesimal generators $\vec{Q}$ of $SU(2)_Q$ can be used to construct the unitary operator

$$W = \exp(i\vec{\omega} \cdot \vec{Q}), \tag{5.22}$$

which implements the corresponding symmetry transformations in the Hilbert space of the theory. The transformed $SU(2)_Q$ spinors are then given by

$${}^{\vec{Q}}d_x = W^\dagger d_x W = \exp(i\vec{\omega} \cdot \frac{\vec{\sigma}}{2}) d_x = \Omega d_x, \qquad \Omega = \exp(i\vec{\omega} \cdot \frac{\vec{\sigma}}{2}) \in SU(2)_Q. \tag{5.23}$$

As promised, in terms of the $SU(2)_Q$ spinor the Hamiltonian takes the form

$$H = -t \sum_{x,i} (d^\dagger_x d_{x+\hat{i}} + d^\dagger_{x+\hat{i}} d_x) - \frac{U}{2} \sum_x (d^\dagger_x d_x - 1)^2 - \mu \sum_x d^\dagger_x \sigma_3 d_x. \tag{5.24}$$

Indeed the first two terms on the right-hand side are manifestly $SU(2)_Q$ invariant, while away from half-filling (i.e. for $\mu \neq 0$) the chemical potential term explicitly breaks the $SU(2)_Q$ symmetry down to $U(1)_Q$ due to the Pauli matrix σ_3.

5.7 Manifestly $SU(2)_s$ and $SU(2)_Q$ invariant matrix notation

So far we have introduced the $SU(2)_s$ and $SU(2)_Q$ spinors c_x and d_x respectively, in order to show the corresponding symmetry of the Hamiltonian in a manifest form. However, to display both the $SU(2)_s$ and the $SU(2)_Q$ symmetries at the same time, we have to combine the c_x and d_x spinors in an appropriate way. To this end we introduce the matrix-valued fermion operator

$$C_x = \begin{pmatrix} c_{x\uparrow} & (-1)^x c^\dagger_{x\downarrow} \\ c_{x\downarrow} & -(-1)^x c^\dagger_{x\uparrow} \end{pmatrix}. \tag{5.25}$$

Here the first column is the $SU(2)_s$ spinor c_x, while the second column is another $SU(2)_s$ spinor which transforms exactly like c_x. On the other hand, the first row of C_x is the $SU(2)_Q$ spinor d_x^T, while the second row is another $SU(2)_Q$ spinor which transforms exactly like d_x^T. Hence, under combined $SU(2)_s$ and $SU(2)_Q$ transformations C_x transforms as

$${}^{\vec{Q}}C'_x = gC_x\Omega^T. \tag{5.26}$$

Since the $SU(2)_s$ symmetry acts on the left while the $SU(2)_Q$ symmetry acts on the right, it is now manifest that the two symmetry operations commute.

Under the displacement symmetries D and D' one obtains

$$\begin{aligned} {}^{D}C_x &= C_{x+\hat{i}}\sigma_3, \\ {}^{D'}C_x &= (i\sigma_2)C_{x+\hat{i}}\sigma_3. \end{aligned} \tag{5.27}$$

The appearance of σ_3 on the right is due to the factor $(-1)^x$ in the second column of C_x and confirms that the displacement symmetries commute with all $SU(2)_s$ transformations, but only with the Abelian $U(1)_Q$ (and not with all $SU(2)_Q$) transformations.

The Hamiltonian can now be expressed in a manifestly $SU(2)_s$, $U(1)_Q$ (or even $SU(2)_Q$), D, and D' invariant form

$$H = -\frac{t}{2}\sum_{x,i} \mathrm{Tr}[C_x^\dagger C_{x+\hat{i}} + C^\dagger_{x+\hat{i}} C_x] + \frac{U}{12}\sum_x \mathrm{Tr}[C_x^\dagger C_x C_x^\dagger C_x] - \frac{\mu}{2}\sum_x \mathrm{Tr}[C_x^\dagger C_x \sigma_3]. \tag{5.28}$$

Again, the chemical potential term is only $U(1)_Q$ invariant, while the other two terms are manifestly $SU(2)_Q$ invariant.

6 Effective theory for magnons

Prior to doping, all high-T_c layered cuprate materials are quantum antiferromagnets in which the $SU(2)_s$ spin symmetry is spontaneously broken down to $U(1)_s$. Consequently, the low-energy physics of antiferromagnets is governed by the corresponding Nambu-Goldstone bosons—the magnons. The resulting chiral perturbation theory, which was originally developed for the Nambu-Goldstone pions of QCD, is a systematic low-energy expansion that has already been applied to magnons as well [44–51].

In this chapter we review the basic features of magnon chiral perturbation theory. In addition, as a necessary prerequisite for the coupling of magnons to fermionic charge carriers (i.e. electrons and holes), we also construct the non-linear realization of the spontaneously broken $SU(2)_s$ symmetry, which then appears as a local symmetry in the unbroken subgroup $U(1)_s$ of $SU(2)_s$. This is in complete analogy to the baryon chiral perturbation theory of QCD in which the spontaneously broken $SU(2)_L \otimes SU(2)_R$ chiral symmetry is implemented on the nucleon fields as a local transformation in the unbroken isospin subgroup $SU(2)_{L=R}$ of $SU(2)_L \otimes SU(2)_R$.

6.1 Spontaneous breakdown of $SU(2)_s \to U(1)_s$ and the resulting effective action for the magnon field $\vec{e}(x) \in S^2$

As already mentioned, the undoped precursors of high-temperature superconductors are quantum antiferromagnets. At half-filling, also the repulsive Hubbard model displays antiferromagnetism. In all these systems, at least at zero temperature[1], the spin rotational symmetry $G = SU(2)_s$ is spontaneously broken down to the subgroup $H = U(1)_s$ by the formation of a staggered magnetization $\vec{M}_s$. On the other hand, the $U(1)_Q$ symmetry remains unbroken until one reaches the superconducting phase. In the Hubbard model even the $SU(2)_Q$ symmetry remains unbroken at half-filling but is explicitly broken down to $U(1)_Q$ for $\mu \neq 0$.

As a consequence of Goldstone's theorem, there are two massless bosons—the antiferromagnetic spin-waves or magnons, which are described by the unit-vector field

$$\vec{e}(x) = (e_1(x), e_2(x), e_3(x)), \qquad \vec{e}(x)^2 = 1 \tag{6.1}$$

in the coset space $G/H = SU(2)_s/U(1)_s = S^2$. Here $x = (x_1, x_2, t)$ denotes a point in Euclidean space-time. The vector $\vec{e}(x)$ describes the direction of the local staggered magnetization.

[1] According to the Mermin-Wagner theorem, in $2+1$ dimensions, a continuous symmetry can break down spontaneously only at exactly zero temperature. More precisely, this means that the magnons pick up a mass that is exponentially small in the inverse temperature [68, 69]. This issue is further discussed in Section 1.4 of the introduction.

The leading order terms in the Euclidean action of the low-energy effective theory for the antiferromagnetic magnons take the form [68, 69]

$$S[\vec{e}] = \int d^2x\ dt\ \frac{\rho_s}{2}\left(\partial_i\vec{e}\cdot\partial_i\vec{e} + \frac{1}{c^2}\partial_t\vec{e}\cdot\partial_t\vec{e}\right). \tag{6.2}$$

Here the index $i \in \{1,2\}$ labels the two spatial directions, while the index t refers to the Euclidean time-direction. The parameter ρ_s is the spin stiffness and c is the spin-wave velocity. For the antiferromagnetic Heisenberg model of Eq. (5.11) these low-energy parameters have been determined very precisely in Monte Carlo calculations [81, 82] resulting in $\rho_s = 0.186(4)J$ and $c = 1.68(1)Ja$, where J is the exchange coupling of the Heisenberg model and a is the lattice spacing.

Note that the leading order terms in the magnon effective action of Eq. (6.2) are 'Poincaré' invariant with the spin-wave velocity c playing the role of the velocity of light. Consequently, antiferromagnetic magnons have a 'relativistic' spectrum. However, the 'Poincaré' symmetry emerges only at low energies as a consequence of the discrete 90 degrees rotational invariance of the lattice. Higher-derivative terms relevant at higher energies are in general not invariant. Note that in contrast ferromagnetic magnons have a non-relativistic spectrum due to the WZW term as discussed in Appendix E.

6.2 $P(x) \in CP(1)$ representation of the magnon field

In the following we prefer to work with an alternative representation of the magnon field using 2×2 Hermitean projection matrices $P(x)$ that obey

$$P(x)^\dagger = P(x), \qquad \mathrm{Tr}P(x) = 1, \qquad P(x)^2 = P(x), \tag{6.3}$$

and are given by

$$P(x) = \frac{1}{2}(\mathbb{1} + \vec{e}(x)\cdot\vec{\sigma}) = \frac{1}{2}\begin{pmatrix} 1+e_3(x) & e_1(x)-ie_2(x) \\ e_1(x)+ie_2(x) & 1-e_3(x)\end{pmatrix}. \tag{6.4}$$

In this $CP(1)$ language, the lowest-order effective action of Eq. (6.2) takes the form

$$S[P] = \int d^2x\ dt\ \rho_s \mathrm{Tr}\left[\partial_i P\partial_i P + \frac{1}{c^2}\partial_t P\partial_t P\right]. \tag{6.5}$$

6.3 Symmetry transformations of the magnon field $P(x)$

Under the global $SU(2)_s$ spin transformations $g \in SU(2)_s$ of Eq. (5.8), the magnon field $P(x)$ transforms as

$$P(x)' = gP(x)g^\dagger. \tag{6.6}$$

On the other hand, the magnon field $P(x)$ is invariant under both the $U(1)_Q$ and $SU(2)_Q$ charge transformations $\exp(i\omega) \in U(1)_Q$ and $\Omega \in SU(2)_Q$ of Eqs. (5.10) and (5.23) respectively, i.e.

$$^{\vec{Q}}P(x) = P(x). \tag{6.7}$$

Under the displacement D by one lattice spacing the staggered magnetization $\vec{M}_s$ changes sign. Consequently, also the sign of the local staggered magnetization changes, i.e. $^{D}\vec{e}(x) = -\vec{e}(x)$. This manifests itself on the magnon field $P(x)$ as

$$^{D}P(x) = \mathbb{1} - P(x). \tag{6.8}$$

Let us again consider the transformation D', which is the combination of the displacement D with the special $SU(2)_s$ spin rotation $g = i\sigma_2$. The transformation D' acts on the magnon field $P(x)$ as

$$^{D'}P(x) = (i\sigma_2)\ ^{D}P(x)(i\sigma_2)^\dagger = (i\sigma_2)[\mathbb{1} - P(x)](i\sigma_2)^\dagger = P(x)^*, \tag{6.9}$$

reminiscent of charge conjugation in particle physics. It is easy to see that all these transformations leave the action of Eq. (6.5) invariant and hence indeed represent symmetries of the effective theory.

The Hubbard model is invariant under translations by an integer multiple of the lattice spacing. As we have seen, due to the antiferromagnetic order, the displacement D by one lattice spacing (which connects the two sublattices A and B) plays a special role. In particular, in the effective theory it manifests itself as an internal symmetry that changes the sign of $\vec{e}(x)$. Translations by an even number of lattice spacings (which do not mix the sublattices), on the other hand, manifest themselves as ordinary translations in the effective theory. It should be noted that in the effective theory one need not distinguish between the displacement symmetries D for the two spatial directions, since they are related by an ordinary translation by two lattice spacings (one in the 1- and one in the 2-direction).

When we decompose a space-time vector $x = (x_1, x_2, t)$ into its spatial and temporal components, the 90 degrees rotation O acts on x as $Ox = (-x_2, x_1, t)$. Under the symmetry O the magnon field transforms as

$$^{O}P(x) = P(Ox). \tag{6.10}$$

Similarly, under the spatial reflection R at the x_1-axis, which turns x into $Rx = (x_1, -x_2, t)$, the magnon field transforms as

$$^{R}P(x) = P(Rx). \tag{6.11}$$

Had we not treated spin as an internal quantum number, it would also be directly affected by the spatial reflection. Since spin is a form of angular momentum, it transforms like the orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$ of a particle, which is a pseudo-vector and thus changes into $^{R}\vec{L} = (-L_1, L_2, -L_3)$ under the reflection R. This is equivalent to a 180 degrees $SU(2)_s$ rotation around the 2-direction. Since we treat $SU(2)_s$ as an exact internal

symmetry, the pure spatial inversion R (without 180 degrees rotation of the spin) is also a symmetry.

Another important symmetry is the time-reversal T which turns $x = (x_1, x_2, t)$ into $Tx = (x_1, x_2, -t)$. In a Hamiltonian description time-reversal is represented by an anti-unitary operator. Here we discuss time-reversal in the framework of the Euclidean path integral. Again, the spin transforms like the orbital angular momentum $\vec{L}$ of a particle. The momentum $\vec{p}$ changes sign under time-reversal and so does $\vec{L}$, i.e. ${}^T\vec{L} = -\vec{L}$.[2] Consequently, under the time-reversal T the local staggered magnetization vector $\vec{e}(x)$ (which is built from microscopic spins) transforms as ${}^T\vec{e}(x) = -\vec{e}(Tx)$. Accordingly, the magnon field $P(x)$ transforms as

$$ {}^TP(x) = \mathbb{1} - P(Tx) = {}^DP(Tx). \tag{6.12}$$

Hence, time-reversal is closely related to the displacement symmetry of Eq. (6.8). Just like the displacement symmetry D, time-reversal is spontaneously broken in an antiferromagnet. However, in contrast to a ferromagnet, the combination TD of time-reversal and the displacement symmetry remains unbroken. Previously we have combined the displacement symmetry D with the $SU(2)_s$ spin rotation $i\sigma_2$ in order to obtain the unbroken symmetry D'. In order to obtain an unbroken variant T' of time-reversal we now combine T with the spin rotation $i\sigma_2$ which yields

$$ {}^{T'}P(x) = (i\sigma_2)\ {}^TP(x)(i\sigma_2)^\dagger = (i\sigma_2)\ {}^DP(Tx)(i\sigma_2)^\dagger = {}^{D'}P(Tx). \tag{6.13}$$

6.4 Diagonalizing field $u(x) \in SU(2)$ and the non-linear realization $h(x) \in U(1)_s$ of the symmetry transformation $g \in SU(2)_s$

In order to couple electron or hole fields to the magnons one must construct a non-linear realization of the spontaneously broken $SU(2)_s$ symmetry which then manifests itself as a local symmetry in the unbroken $U(1)_s$ subgroup of $SU(2)_s$. This local transformation is constructed from the global transformation $g \in SU(2)_s$ as well as from the local magnon field $P(x)$ as follows: One first diagonalizes the magnon field by a unitary transformation $u(x) \in SU(2)$, i.e.

$$ u(x)P(x)u(x)^\dagger = \frac{1}{2}(\mathbb{1} + \sigma_3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad u_{11}(x) \geq 0. \tag{6.14}$$

Note that, due to its projector properties, $P(x)$ has eigenvalues 0 and 1. In order to make $u(x)$ uniquely defined, we demand that the element $u_{11}(x)$ is real and non-negative. Otherwise the diagonalizing matrix $u(x)$ would be defined only up to a $U(1)$ phase. Since the magnon field $P(x)$ can be reconstructed from the diagonalizing field $u(x)$, i.e. $P(x) =$

[2] Note that $-\vec{L}$ does not obey the angular momentum commutation relations. This is a consequence of the antiunitary nature of T which does not represent an ordinary symmetry (implemented by a unitary transformation) in Hilbert space.

$u(x)^\dagger(\mathbb{1}+\sigma_3)u(x)/2$, one can think of $u(x)$ as a further alternative representation of the magnon field. Using Eq. (6.4) and spherical coordinates for $\vec{e}(x)$, i.e.

$$\vec{e}(x) = (\sin\theta(x)\cos\varphi(x), \sin\theta(x)\sin\varphi(x), \cos\theta(x)), \tag{6.15}$$

one obtains

$$\begin{aligned} u(x) &= \frac{1}{\sqrt{2(1+e_3(x))}} \begin{pmatrix} 1+e_3(x) & e_1(x)-ie_2(x) \\ -e_1(x)-ie_2(x) & 1+e_3(x) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\frac{1}{2}\theta(x)) & \sin(\frac{1}{2}\theta(x))\exp(-i\varphi(x)) \\ -\sin(\frac{1}{2}\theta(x))\exp(i\varphi(x)) & \cos(\frac{1}{2}\theta(x)) \end{pmatrix}. \end{aligned} \tag{6.16}$$

Under a global $SU(2)_s$ transformation g the diagonalizing field $u(x)$ transforms as

$$u(x)' = h(x)u(x)g^\dagger, \qquad u_{11}(x)' \geq 0, \tag{6.17}$$

which implicitly defines the non-linear symmetry transformation

$$h(x) = \exp(i\alpha(x)\sigma_3) = \begin{pmatrix} \exp(i\alpha(x)) & 0 \\ 0 & \exp(-i\alpha(x)) \end{pmatrix} \in U(1)_s. \tag{6.18}$$

The transformation $h(x)$ is uniquely defined since we demand that $u_{11}(x)'$ is again real and non-negative. Note that with this definition of $h(x)$ indeed

$$u(x)'P(x)'u(x)'^\dagger = \frac{1}{2}(\mathbb{1}+\sigma_3). \tag{6.19}$$

Interestingly, the global $SU(2)_s$ transformation g manifests itself in the form of a local transformation $h(x) \in U(1)_s$ which inherits its x-dependence from the magnon field $P(x)$.

We still need to show that the $SU(2)_s$ group structure $g = g_2g_1$ is inherited by the non-linear $U(1)_s$ realization, i.e. $h(x) = h_2(x)h_1(x)$. First, we perform the global $SU(2)_s$ transformation g_1, i.e.

$$P(x)' = g_1P(x)g_1^\dagger, \qquad u(x)' = h_1(x)u(x)g_1^\dagger, \tag{6.20}$$

which defines the non-linear realization $h_1(x)$. Then we perform the subsequent global transformation g_2 which defines the non-linear realization $h_2(x)$, i.e.

$$\begin{aligned} P(x)'' &= g_2P(x)'g_2^\dagger \quad &&= g_2g_1P(x)(g_2g_1)^\dagger &&= gP(x)g^\dagger, \\ u(x)'' &= h_2(x)u(x)'g_2^\dagger &&= h_2(x)h_1(x)u(x)(g_2g_1)^\dagger &&= h(x)u(x)g^\dagger. \end{aligned} \tag{6.21}$$

This indeed implies the correct group structure $h(x) = h_2(x)h_1(x)$.

Under the displacement symmetry D the sign-change of the staggered magnetization $\vec{e}(x)$ implies

$$\begin{aligned} {}^{D}u(x) &= \frac{1}{\sqrt{2(1-e_3(x))}} \begin{pmatrix} 1-e_3(x) & -e_1(x)+ie_2(x) \\ e_1(x)+ie_2(x) & 1-e_3(x) \end{pmatrix} \\ &= \begin{pmatrix} \sin(\frac{1}{2}\theta(x)) & -\cos(\frac{1}{2}\theta(x))\exp(-i\varphi(x)) \\ \cos(\frac{1}{2}\theta(x))\exp(i\varphi(x)) & \sin(\frac{1}{2}\theta(x)) \end{pmatrix} \\ &= \tau(x)u(x), \end{aligned} \tag{6.22}$$

where

$$\begin{aligned} \tau(x) &= \frac{1}{\sqrt{e_1(x)^2+e_2(x)^2}} \begin{pmatrix} 0 & -e_1(x)+ie_2(x) \\ e_1(x)+ie_2(x) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\exp(-i\varphi(x)) \\ \exp(i\varphi(x)) & 0 \end{pmatrix}. \end{aligned} \tag{6.23}$$

Note that ${}^{D}\tau(x) = -\tau(x) = \tau(x)^\dagger$, such that

$${}^{DD}u(x) = {}^{D}\tau(x)\,{}^{D}u(x) = \tau(x)^\dagger\tau(x)u(x) = u(x), \tag{6.24}$$

as one would expect for the displacement symmetry. It should also be noted that—like the $SU(2)_s$ symmetry—the displacement symmetry D is also spontaneously broken and hence realized in a non-linear (i.e. magnon field dependent) manner. Similarly, under the displacement symmetry D' one finds ${}^{D'}u(x) = h(x)\,{}^{D}u(x)g^\dagger$ with $g = i\sigma_2$. For this particular g the local transformation takes the form $h(x) = (i\sigma_2)\tau(x)^\dagger$, such that

$${}^{D'}u(x) = u(x)^*. \tag{6.25}$$

In contrast to the displacement symmetry D, the symmetry D' is not spontaneously broken and is thus realized in a linear (i.e. magnon field independent) manner.

6.5 Composite magnon field $v_\mu(x)$

Now we use the diagonalizing field $u(x)$ of the last section to introduce the anti-Hermitean composite field

$$v_\mu(x) = u(x)\partial_\mu u(x)^\dagger, \tag{6.26}$$

which transforms under $SU(2)_s$ as

$$v_\mu(x)' = h(x)u(x)g^\dagger\partial_\mu[gu(x)^\dagger h(x)^\dagger] = h(x)[v_\mu(x)+\partial_\mu]h(x)^\dagger. \tag{6.27}$$

Writing

$$v_\mu(x) = iv_\mu^a(x)\sigma_a = i\begin{pmatrix} v_\mu^3(x) & v_\mu^+(x) \\ v_\mu^-(x) & -v_\mu^3(x) \end{pmatrix}, \qquad v_\mu^\pm(x) = v_\mu^1(x) \mp iv_\mu^2(x) \tag{6.28}$$

and using Eq. (6.18) this implies

$$v_\mu^3(x)' = v_\mu^3(x) - \partial_\mu \alpha(x), \qquad v_\mu^\pm(x)' = \exp(\pm 2i\alpha(x)) v_\mu^\pm(x). \tag{6.29}$$

Hence, $v_\mu^3(x)$ transforms like an Abelian gauge field for $U(1)_s$, while $v_\mu^\pm(x)$ represent vector fields 'charged' under $U(1)_s$. For later convenience we also introduce the Hermitean charged vector field

$$V_\mu(x) = v_\mu^1(x)\sigma_1 + v_\mu^2(x)\sigma_2 = v_\mu^+(x)\sigma_+ + v_\mu^-(x)\sigma_- = \begin{pmatrix} 0 & v_\mu^+(x) \\ v_\mu^-(x) & 0 \end{pmatrix}, \tag{6.30}$$

where $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$ are raising and lowering operators of spin. Under the $SU(2)_s$ symmetry the charged vector field transforms as

$$V_\mu(x)' = h(x) V_\mu(x) h(x)^\dagger. \tag{6.31}$$

The magnon action can also be written as

$$S[v_\mu] = \int d^2x\ dt\ 2\rho_s \left(v_i^+ v_i^- + \frac{1}{c^2} v_t^+ v_t^- \right) = \int d^2x\ dt\ \rho_s \mathrm{Tr} \left[V_i^\dagger V_i + \frac{1}{c^2} V_t^\dagger V_t \right]. \tag{6.32}$$

It should be pointed out that the fields $v_\mu^a(x)$ do not represent independent degrees of freedom, but are composed of magnon fields. In particular, what looks like a mass term for a charged vector field is indeed just the kinetic term of a massless Nambu-Goldstone boson.

Under the displacement symmetry D the composite vector field transforms as

$$\begin{aligned} {}^D v_\mu(x) = \tau(x)[v_\mu(x) + \partial_\mu]\tau(x)^\dagger \quad \Rightarrow \quad & {}^D v_\mu^3(x) = -v_\mu^3(x) + \partial_\mu \varphi(x), \\ & {}^D v_\mu^\pm(x) = -\exp(\mp 2i\varphi(x)) v_\mu^\mp(x), \\ & {}^D V_\mu(x) = \tau(x) V_\mu(x) \tau(x)^\dagger. \end{aligned} \tag{6.33}$$

Similarly, under the symmetry D' one finds

$$\begin{aligned} {}^{D'} v_\mu(x) = v_\mu(x)^* \quad \Rightarrow \quad & {}^{D'} v_\mu^3(x) = -v_\mu^3(x), \\ & {}^{D'} v_\mu^\pm(x) = -v_\mu^\mp(x), \\ & {}^{D'} V_\mu(x) = -V_\mu(x)^*. \end{aligned} \tag{6.34}$$

This is exactly how an ordinary non-Abelian gauge field in particle physics behaves under charge conjugation.

Under the 90 degrees spatial rotation O the composite field $v_\mu(x)$ transforms as

$${}^O v_i(x) = \varepsilon_{ij} v_j(Ox), \qquad {}^O v_t(x) = v_t(Ox), \tag{6.35}$$

while under the reflection R one obtains

$${}^R v_1(x) = v_1(Rx), \qquad {}^R v_2(x) = -v_2(Rx), \qquad {}^R v_t(x) = v_t(Rx). \tag{6.36}$$

Finally, under the time-reversal symmetry T the field v_μ transforms as

$$\begin{aligned}
{}^T v_i(x) &= {}^D v_i(Tx) \quad \Rightarrow & {}^T v_t(x) &= -{}^D v_t(Tx) \quad \Rightarrow \\
{}^T v_i^3(x) &= -v_i^3(Tx) + \partial_i \varphi(Tx), & {}^T v_t^3(x) &= v_t^3(Tx) - \partial_t \varphi(Tx), \\
{}^T v_i^\pm(x) &= -\exp(\mp 2i\varphi(Tx)) v_i^\mp(Tx), & {}^T v_t^\pm(x) &= \exp(\mp 2i\varphi(Tx)) v_t^\mp(Tx), \\
{}^T V_i(x) &= \tau(Tx) V_i(Tx) \tau(Tx)^\dagger, & {}^T V_t(x) &= -\tau(Tx) V_t(Tx) \tau(Tx)^\dagger,
\end{aligned} \tag{6.37}$$

and under its unbroken variant T' one finds

$$\begin{aligned}
{}^{T'} v_i(x) &= {}^{D'} v_i(Tx) \quad \Rightarrow & {}^{T'} v_t(x) &= -{}^{D'} v_t(Tx) \quad \Rightarrow \\
{}^{T'} v_i^3(x) &= -v_i^3(Tx), & {}^{T'} v_t^3(x) &= v_t^3(Tx), \\
{}^{T'} v_i^\pm(x) &= -v_i^\mp(Tx), & {}^{T'} v_t^\pm(x) &= v_t^\mp(Tx), \\
{}^{T'} V_i(x) &= -V_i(Tx)^T, & {}^{T'} V_t(x) &= V_t(Tx)^T.
\end{aligned} \tag{6.38}$$

Note that the upper index T on the right denotes transpose, while on the left it denotes time-reversal. The above relations are equivalent to time-reversal of an ordinary non-Abelian gauge field.

6.6 Complex doublet $z(x)$ representation of the magnon field

We have already used the unit-vector $\vec{e}(x)$ and the projection matrix $P(x)$ to represent the magnon field. There is another equivalent representation in terms of a complex doublet

$$\begin{aligned}
z(x) &= \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}, \qquad z(x)^\dagger = (z_1(x)^*, z_2(x)^*), \\
z(x)^\dagger z(x) &= |z_1(x)|^2 + |z_2(x)|^2 = 1,
\end{aligned} \tag{6.39}$$

which is related to the other two representations by

$$\begin{aligned}
\vec{e}(x) = z(x)^\dagger \vec{\sigma} z(x) \quad \Rightarrow \quad & e_1(x) = z_1(x)^* z_2(x) + z_2(x)^* z_1(x), \\
& e_2(x) = i[z_2(x)^* z_1(x) - z_1(x)^* z_2(x)], \\
& e_3(x) = |z_1(x)|^2 - |z_2(x)|^2, \\
P(x) = z(x) z(x)^\dagger = & \begin{pmatrix} |z_1(x)|^2 & z_1(x) z_2(x)^* \\ z_2(x) z_1(x)^* & |z_2(x)|^2 \end{pmatrix}.
\end{aligned} \tag{6.40}$$

The field $z(x)$ is defined in terms of $\vec{e}(x)$ or $P(x)$ only up to a $U(1)$ gauge transformation

$$z(x)' = \exp(i\beta(x)) z(x). \tag{6.41}$$

It is therefore necessary to also introduce the auxiliary real-valued $U(1)$ gauge field

$$a_\mu(x) = \frac{1}{2i}[z(x)^\dagger \partial_\mu z(x) - \partial_\mu z(x)^\dagger z(x)], \tag{6.42}$$

which under the symmetry of Eq. (6.41) transforms as

$$a_\mu(x)' = a_\mu(x) + \partial_\mu \beta(x). \tag{6.43}$$

The complex doublet $z(x)$ is closely related to the field $u(x)$. Fixing the gauge freedom of Eq. (6.41) such that $z_1(x)$ is real and non-negative, it is easy to show that

$$u(x) = \begin{pmatrix} z_1(x) & z_2(x)^* \\ -z_2(x) & z_1(x) \end{pmatrix}, \qquad v_\mu^3(x) = a_\mu(x). \tag{6.44}$$

Hence, the description in terms of the complex doublet $z(x)$ and the additional auxiliary gauge field $a_\mu(x)$ is physically equivalent to what we described before. It should again be pointed out that $a_\mu(x)$ (or equivalently $v_\mu^3(x)$) does not represent a dynamical Abelian gauge field, but is simply a composite field constructed from the underlying magnon field $P(x)$.

6.7 Baby-Skyrmions

It is interesting to note that magnon fields support topological solitons known as baby-Skyrmions—a lower-dimensional variant of the Skyrme soliton which represents a baryon in the low-energy pion effective theory for QCD [83]. Baby-Skyrmions are solitons whose topological charge

$$B = \frac{1}{8\pi} \int d^2x \; \varepsilon_{ij} \vec{e} \cdot (\partial_i \vec{e} \times \partial_j \vec{e}), \tag{6.45}$$

defined at every instant in time, is an element of the homotopy group $\Pi_2[S^2] = \mathbb{Z}$. The corresponding topological current

$$j_\mu(x) = \frac{1}{8\pi} \varepsilon_{\mu\nu\rho} \vec{e}(x) \cdot [\partial_\nu \vec{e}(x) \times \partial_\rho \vec{e}(x)] \tag{6.46}$$

is conserved, i.e. $\partial_\mu j_\mu = 0$, independent of the equations of motion. Baby-Skyrmions are massive excitations inaccessible to the systematic low-energy expansion of chiral perturbation theory. Still, the existence of the conserved current $j_\mu(x)$ may have physical consequences even for the pure magnon dynamics.

Under the various symmetries the topological current transforms as

$$\begin{aligned}
SU(2)_s: &\quad j_\mu(x)' = j_\mu(x), \\
SU(2)_Q: &\quad {}^{\vec{Q}}j_\mu(x) = j_\mu(x), \\
D: &\quad {}^{D}j_\mu(x) = -j_\mu(x), \\
D': &\quad {}^{D'}j_\mu(x) = -j_\mu(x), \\
O: &\quad {}^{O}j_t(x) = j_t(Ox), \qquad {}^{O}j_i(x) = \varepsilon_{ij} j_j(Ox), \\
R: &\quad {}^{R}j_t(x) = -j_t(Rx), \qquad {}^{R}j_1(x) = -j_1(Rx), \qquad {}^{R}j_2(x) = j_2(Rx), \\
T: &\quad {}^{T}j_t(x) = -j_t(Tx), \qquad {}^{T}j_1(x) = j_1(Tx), \qquad {}^{T}j_2(x) = j_2(Rx), \\
T': &\quad {}^{T'}j_t(x) = -j_t(Tx), \qquad {}^{T'}j_1(x) = j_1(Tx), \qquad {}^{T'}j_2(x) = j_2(Tx).
\end{aligned} \tag{6.47}$$

One might be tempted to add a term $j_\mu(x)v_\mu^3(x)$ to the magnon Lagrangian because this is how an Abelian gauge field couples to a conserved current. Indeed, this term is invariant under $SU(2)_s$, $SU(2)_Q$, D, D', and O. However, it violates the reflection and time-reversal symmetries R, T, and T' and is hence forbidden.

There is another non-trivial homotopy group, $\Pi_3[S^2] = \mathbb{Z}$, which is relevant for baby-Skyrmions. It implies that space-time-dependent magnon fields fall into distinct topological classes characterized by the Hopf number $H[\vec{e}] \in \Pi_3[S^2] = \mathbb{Z}$. In $2+1$ dimensions baby-Skyrmions can be quantized as anyons characterized by a statistics angle θ [84]. The cases $\theta = 0$ and $\theta = \pi$ correspond to bosons and fermions, respectively. Including the Hopf term, the magnon path integral takes the form

$$Z = \int \mathcal{D}\vec{e}\ \exp(-S[\vec{e}])\exp(i\theta H[\vec{e}]). \tag{6.48}$$

The Hopf term also changes sign under R, T, and T'. Hence, $\exp(i\theta H[\vec{e}])$ is invariant only if θ is 0 or π. Consequently, in an antiferromagnet with exact R, T, or T' symmetries baby-Skyrmions can only be quantized as bosons or fermions. For the antiferromagnetic quantum Heisenberg model it has been argued that no Hopf term is generated [85–89]. Hence, in that case the baby-Skyrmions should be bosons.

7 The Hubbard model in a magnon background field

The half-filled ground state of the Hubbard model plays a similar role as the Dirac sea in a relativistic quantum field theory. In particular, any fermion added to a half-filled state will be denoted as an electron, while any fermion removed from such a state represents a hole. In this chapter we couple a background magnon field to the microscopic degrees of freedom of the Hubbard model. In this way composite operators are constructed which transform exactly like the fields of the effective theory. Hence, the effective fields inherit their transformation properties under symmetry operations from the Hubbard model degrees of freedom.

7.1 Fermion operators in a magnon background field

In order to analyze the transformation properties of the electron and hole fields, as an intermediate step between the microscopic and effective descriptions, we first add a continuum magnon background field $P(x)$ to the Hubbard model by hand. The corresponding diagonalizing unitary matrix field $u(x)$ is used to turn the matrix-valued Hubbard model operator C_x of Eq. (5.25) into new operators Ψ_x^A and Ψ_x^B defined on the even and odd

sublattices, respectively

$$\Psi_x^A = u(x)C_x = u(x)\begin{pmatrix} c_{x\uparrow} & c_{x\downarrow}^\dagger \\ c_{x\downarrow} & -c_{x\uparrow}^\dagger \end{pmatrix} = \begin{pmatrix} \psi_{x+}^A & \psi_{x-}^{A\dagger} \\ \psi_{x-}^A & -\psi_{x+}^{A\dagger} \end{pmatrix}, \qquad x \in A,$$
$$\Psi_x^B = u(x)C_x = u(x)\begin{pmatrix} c_{x\uparrow} & -c_{x\downarrow}^\dagger \\ c_{x\downarrow} & c_{x\uparrow}^\dagger \end{pmatrix} = \begin{pmatrix} \psi_{x+}^B & -\psi_{x-}^{B\dagger} \\ \psi_{x-}^B & \psi_{x+}^{B\dagger} \end{pmatrix}, \qquad x \in B. \tag{7.1}$$

In order to achieve a consistent representation of the underlying antiferromagnetic structure, it is unavoidable to explicitly split the degrees of freedom according to their location on sublattice A or B. In this context it may be interesting to consider the electron-hole representation of the Hubbard model operators discussed in Appendix A. The operators $\psi_{x\pm}^{A,B}$ obey standard anticommutation relations. It should be noted that here the continuum field $u(x)$ is evaluated only at discrete lattice points x.

The new lattice operators inherit their transformation properties from the operators of the Hubbard model. According to Eqs. (6.17) and (5.8), under the $SU(2)_s$ symmetry one obtains

$$\Psi_x^{A,B\prime} = u(x)'C_x' = h(x)u(x)g^\dagger g C_x = h(x)\Psi_x^{A,B}. \tag{7.2}$$

In components this relation takes the form

$$\psi_{x\pm}^{A,B\prime} = \exp(\pm i\alpha(x))\psi_{x\pm}^{A,B}. \tag{7.3}$$

The components $\psi_{x\pm}^{A,B}$ do not simply correspond to spin up and spin down with respect to an arbitrarily chosen global quantization axis. Instead they correspond to spin parallel $(+)$ or antiparallel $(-)$ to the local staggered magnetization. This follows from considering global symmetry transformations $g \in U(1)_s$ in the unbroken subgroup of $SU(2)_s$ which describe rotations around the spontaneously selected direction of the staggered magnetization vector. In that case, according to Eq. (6.17), $h(x) = g$ becomes a global transformation as well and Eq. (7.3) shows that $\psi_{x\pm}^{A,B}$ indeed has spin parallel or antiparallel to the direction of the staggered magnetization.

Similarly, under the $SU(2)_Q$ symmetry one obtains

$$^{\vec{Q}}\Psi_x^{A,B} = {}^{\vec{Q}}u(x)\,{}^{\vec{Q}}C_x = u(x)C_x\Omega^T = \Psi_x^{A,B}\Omega^T. \tag{7.4}$$

In particular, under the $U(1)_Q$ subgroup of $SU(2)_Q$ the components transform as

$$^{Q}\psi_{x\pm}^{A,B} = \exp(i\omega)\psi_{x\pm}^{A,B}. \tag{7.5}$$

Under the displacement symmetry the new operators transform as

$$^{D}\Psi_x^{A,B} = {}^{D}u(x+\hat{\imath})C_{x+\hat{\imath}}\sigma_3 = \tau(x+\hat{\imath})u(x+\hat{\imath})C_{x+\hat{\imath}}\sigma_3 = \tau(x+\hat{\imath})\Psi_{x+\hat{\imath}}^{B,A}\sigma_3, \tag{7.6}$$

where $\tau(x)$ is the field introduced in Eq. (6.22). Expressed in components this implies

$$^{D}\psi_{x\pm}^{A,B} = \mp\exp(\mp i\varphi(x+\hat{\imath}))\psi_{x+\hat{\imath},\mp}^{B,A}. \tag{7.7}$$

Similarly, under the symmetry D' one finds

$$^{D'}\Psi_x^{A,B} = {}^{D'}u(x+\hat{i})(i\sigma_2)C_{x+\hat{i}}\sigma_3 = u(x+\hat{i})^*(i\sigma_2)C_{x+\hat{i}}\sigma_3 = (i\sigma_2)\Psi_{x+\hat{i}}^{B,A}\sigma_3, \tag{7.8}$$

where we have used $u(x+\hat{i})^*(i\sigma_2) = (i\sigma_2)u(x+\hat{i})$. Again, expressed in components this relation takes the form

$$^{D'}\psi_{x\pm}^{A,B} = \pm\psi_{x+\hat{i},\mp}^{B,A}. \tag{7.9}$$

We have seen before that the symmetry D' acts on the composite field $v_\mu(x)$ exactly like charge conjugation in particle physics. However, it should be noted that D' acts on the electron and hole fields in a different way than the usual charge conjugation of a relativistic Dirac fermion which interchanges electrons and positrons. In particular, D' does not interchange electrons and holes. Instead, it flips the spin of both electrons and holes from + to − and vice versa. Indeed, the spin is the 'charge' that couples to the composite gauge field of Eq. (6.26) constructed from the magnon field.

In the condensed matter literature on high-temperature superconductivity the concept of spin-charge separation (whose existence is established for some systems in one spatial dimension) has often been invoked. The idea is that there may be quasi-particles—so-called holons—which carry charge but no spin, as well as so-called spinons which are neutral and carry spin $\frac{1}{2}$. In order to avoid confusion between holons and the holes of our effective theory, we like to make a few comments: One might think that the fermion operator $\Psi_x^{A,B}$ does not carry spin since it does not transform with the global spin transformation $g \in SU(2)_s$. However, the spin symmetry is non-linearly realized and hence the fermion operator transforms with the local $h(x) \in U(1)_s$. Consequently, $\Psi_x^{A,B}$ still carries spin and hence does not represent a holon. It should also be pointed out that in the weakly coupled effective theory of magnons and holes there are no linearly confining forces that could form a spinless holon out of $\Psi_x^{A,B}$ and the magnon field $z(x)$ of Eq. (6.39).

7.2 Formal continuum limit of the Hubbard model in a magnon background field

In terms of the new operators the Hubbard model Hamiltonian takes the form

$$\begin{aligned} H = &-\frac{t}{2}\sum_{x\in A,i}\mathrm{Tr}[\Psi_x^{A\dagger}\mathcal{V}_{x,i}\Psi_{x+\hat{i}}^{B} + \Psi_{x+\hat{i}}^{B\dagger}\mathcal{V}_{x,i}^{\dagger}\Psi_x^{A}] \\ &-\frac{t}{2}\sum_{x\in B,i}\mathrm{Tr}[\Psi_x^{B\dagger}\mathcal{V}_{x,i}\Psi_{x+\hat{i}}^{A} + \Psi_{x+\hat{i}}^{A\dagger}\mathcal{V}_{x,i}^{\dagger}\Psi_x^{B}] \\ &+\frac{U}{12}\sum_{x\in A}\mathrm{Tr}[\Psi_x^{A\dagger}\Psi_x^{A}\Psi_x^{A\dagger}\Psi_x^{A}] + \frac{U}{12}\sum_{x\in B}\mathrm{Tr}[\Psi_x^{B\dagger}\Psi_x^{B}\Psi_x^{B\dagger}\Psi_x^{B}] \\ &-\frac{\mu}{2}\sum_{x\in A}\mathrm{Tr}[\Psi_x^{A\dagger}\Psi_x^{A}\sigma_3] - \frac{\mu}{2}\sum_{x\in B}\mathrm{Tr}[\Psi_x^{B\dagger}\Psi_x^{B}\sigma_3], \end{aligned} \tag{7.10}$$

where we have introduced the parallel transporter

$$\mathcal{V}_{x,i} = u(x)u(x+\hat{i})^\dagger \in SU(2)_s, \tag{7.11}$$

which transforms under $SU(2)_s$ as

$$\mathcal{V}'_{x,i} = h(x)\mathcal{V}_{x,i}h(x+\hat{i})^\dagger. \tag{7.12}$$

For smooth magnon fields we can put

$$\begin{aligned} u(x) &= u(x+\frac{\hat{i}}{2}) - \frac{a}{2}\partial_i u(x+\frac{\hat{i}}{2}) + \frac{a^2}{8}\partial_i^2 u(x+\frac{\hat{i}}{2}) + \mathcal{O}(a^3), \\ u(x+\hat{i}) &= u(x+\frac{\hat{i}}{2}) + \frac{a}{2}\partial_i u(x+\frac{\hat{i}}{2}) + \frac{a^2}{8}\partial_i^2 u(x+\frac{\hat{i}}{2}) + \mathcal{O}(a^3), \end{aligned} \tag{7.13}$$

where a is the lattice spacing. Similar expressions hold for the other fields. Using the unitarity of $u(x+\frac{\hat{i}}{2})$ one can show that the lattice parallel transporter reduces to

$$\mathcal{V}_{x,i} = \mathbb{1} + av_i(x+\frac{\hat{i}}{2}) + \frac{a^2}{2}v_i(x+\frac{\hat{i}}{2})^2 + \mathcal{O}(a^3), \tag{7.14}$$

with $v_i(x)$ given by Eq. (6.26). Note that both the continuum field $v_i(x)$ and the lattice parallel transporter field $\mathcal{V}_{x,i}$ transform locally only with the unbroken $U(1)_s$ subgroup and not with the full $SU(2)_s$ symmetry.

In the continuum limit we make the replacements

$$\sum_{x\in A}, \sum_{x\in B} \;\to\; \frac{1}{2a^2}\int d^2x, \qquad \Psi_x^{A,B} \;\to\; \sqrt{2}a\Psi^{A,B}(x). \tag{7.15}$$

The factor $\frac{1}{2}$ in front of the integral accounts for the fact that each sublattice covers only half of the space. Similarly the factor $\sqrt{2}a$ in the definition of the continuum field $\Psi^{A,B}(x)$ arises because there is only one degree of freedom of a given type A or B per area $2a^2$. The components $\psi_\pm^{A,B}(x)$ of $\Psi^{A,B}(x)$ again obey standard anticommutation relations, however, with the Dirac δ-function of the continuum theory instead of the Kronecker δ-function of the lattice. It should be noted that, due to the antiferromagnetic order, the number of degrees of freedom per continuum point is twice as large as the number per lattice point.

Taking the formal continuum limit $a \to 0$ (and ignoring an irrelevant constant) the Hamiltonian of Eq. (7.10) takes the form

$$\begin{aligned} H = \int d^2x \,\Big\{ & M\mathrm{Tr}[\Psi^{A\dagger}\Psi^B] + \frac{1}{2M'}\mathrm{Tr}[D_i\Psi^{A\dagger}D_i\Psi^B] \\ & + iK\mathrm{Tr}[D_i\Psi^{A\dagger}V_i\Psi^B + D_i\Psi^{B\dagger}V_i\Psi^A] + N\mathrm{Tr}[\Psi^{A\dagger}V_iV_i\Psi^B] \\ & + \frac{G}{12}\mathrm{Tr}[\Psi^{A\dagger}\Psi^A\Psi^{A\dagger}\Psi^A + \Psi^{B\dagger}\Psi^B\Psi^{B\dagger}\Psi^B] - \frac{\mu}{2}\mathrm{Tr}[\Psi^{A\dagger}\Psi^A\sigma_3 + \Psi^{B\dagger}\Psi^B\sigma_3]\Big\}. \end{aligned} \tag{7.16}$$

It should be noted that, due to the structure of $\Psi^{A,B}(x)$, the individual terms are Hermitean. In the above expression $V_\mu(x)$ is the field defined in Eq. (6.30) and the covariant derivatives are given by

$$\begin{aligned} D_\mu \Psi^{A,B}(x) &= \left(\partial_\mu + i v_\mu^3(x)\sigma_3\right) \Psi^{A,B}(x), \\ D_\mu \Psi^{A,B\dagger}(x) &= [D_\mu \Psi^{A,B}(x)]^\dagger = \partial_\mu \Psi^{A,B\dagger}(x) - \Psi^{A,B\dagger}(x) i v_\mu^3(x)\sigma_3. \end{aligned} \tag{7.17}$$

In terms of the fundamental parameters t and U and the lattice spacing a of the Hubbard model one obtains

$$M = -4t, \qquad M' = \frac{1}{2ta^2}, \qquad K = ta^2, \qquad N = ta^2, \qquad G = 2Ua^2. \tag{7.18}$$

It should be noted that (in contrast to a relativistic theory) the kinetic mass M' is in general different from the rest mass M. The Hamiltonian from above resembles some (but not all) terms in the action of the effective theory to be constructed below. However, the coupling constants resulting from the formal continuum limit get renormalized and will hence be replaced by a priori unknown low-energy parameters in the effective action. The values of the low-energy parameters can be determined in experiments with cuprate materials or through numerical simulations of a microscopic Hubbard-type model.

8 Effective theory for magnons and charge carriers

The low-energy effective theory for magnons is analogous to chiral perturbation theory for pions in QCD. In QCD the baryon number B is a conserved quantity. Thus one can investigate the low-energy QCD dynamics separately in each baryon number sector. Ordinary chiral perturbation theory operates in the $B = 0$ sector. The low-energy physics in the $B = 1$ sector involves a single nucleon interacting with soft pions. The low-energy effective theory describing these dynamics is known as baryon chiral perturbation theory [70–74]. Similar effective theories have been constructed for the $B = 2$ [53, 54] and $B = 3$ sectors [56, 61] in the context of nuclear physics. Even nuclear matter (i.e. a system with non-zero baryon density) has been studied with effective theories [63–67]. The condensed matter analog of baryon number is electron (or hole) number (or equivalently electric charge) which is obviously also conserved. In analogy to QCD it is hence possible to construct a low-energy effective theory describing the interactions of soft magnons with charge carriers doped into an antiferromagnet. Most high-T_c materials result by hole-doping of quantum antiferromagnets, but the effective theory also applies to electron-doping. The key observation is that the spontaneously broken $SU(2)_s$ spin symmetry is non-linearly realized on the electron or hole fields and appears as a local symmetry in the unbroken subgroup $U(1)_s$ of $SU(2)_s$. This is analogous to baryon chiral perturbation theory in which the spontaneously broken $SU(2)_L \otimes SU(2)_R$ chiral symmetry of QCD is implemented on the nucleon fields as a local transformation in the unbroken isospin subgroup $SU(2)_{L=R}$.

8.1 Effective fields for charge carriers

In the low-energy effective theory we will use the Euclidean path integral description instead of the Hamiltonian description used in the Hubbard model. Consequently, the Hermitean conjugate lattice operators $\psi_{x\pm}^{A,B\dagger}$ are then replaced by Grassmann numbers $\psi_{\pm}^{A,B\dagger}(x)$ which are completely independent of $\psi_{\pm}^{A,B}(x)$. Therefore, in the effective theory the electron and hole fields are represented by eight independent Grassmann numbers $\psi_{\pm}^{A,B}(x)$ and $\psi_{\pm}^{A,B\dagger}(x)$ which can be combined to

$$\Psi^A(x) = \begin{pmatrix} \psi_+^A(x) & \psi_-^{A\dagger}(x) \\ \psi_-^A(x) & -\psi_+^{A\dagger}(x) \end{pmatrix}, \qquad \Psi^B(x) = \begin{pmatrix} \psi_+^B(x) & -\psi_-^{B\dagger}(x) \\ \psi_-^B(x) & \psi_+^{B\dagger}(x) \end{pmatrix}. \tag{8.1}$$

In order to avoid confusion with relativistic theories, we do not denote the conjugate fields by $\overline{\psi}_{\pm}^{A,B}(x)$. For notational convenience we also introduce the fields

$$\Psi^{A\dagger}(x) = \begin{pmatrix} \psi_+^{A\dagger}(x) & \psi_-^{A\dagger}(x) \\ \psi_-^A(x) & -\psi_+^A(x) \end{pmatrix}, \qquad \Psi^{B\dagger}(x) = \begin{pmatrix} \psi_+^{B\dagger}(x) & \psi_-^{B\dagger}(x) \\ -\psi_-^B(x) & \psi_+^B(x) \end{pmatrix}. \tag{8.2}$$

Note that $\Psi^{A,B\dagger}(x)$ is not an independent field, but consists of the same Grassmann fields $\psi_{\pm}^{A,B}(x)$ and $\psi_{\pm}^{A,B\dagger}(x)$ as $\Psi^{A,B}(x)$.

It should be pointed out that, since they emerge dynamically, the continuum fields of the low-energy effective theory can not be derived explicitly from the lattice operators of the microscopic Hubbard model. In the following we will assume that the Grassmann fields $\Psi^{A,B}(x)$ describing electrons and holes in the low-energy effective theory transform just like the lattice operators $\Psi_x^{A,B}$ discussed before. However, in contrast to the lattice operators, the fields $\Psi^{A,B}(x)$ are now defined in the continuum. Hence, under the displacement symmetries D and D' one no longer distinguishes between the points x and $x + \hat{i}$. As a result, the transformation rules of the various symmetries take the form

$$\begin{aligned} SU(2)_s: &\quad \Psi^{A,B}(x)' = h(x)\Psi^{A,B}(x), \\ SU(2)_Q: &\quad {}^{\vec{Q}}\Psi^{A,B}(x) = \Psi^{A,B}(x)\Omega^T, \\ D: &\quad {}^{D}\Psi^{A,B}(x) = \tau(x)\Psi^{B,A}(x)\sigma_3, \\ D': &\quad {}^{D'}\Psi^{A,B}(x) = (i\sigma_2)\Psi^{B,A}(x)\sigma_3. \end{aligned} \tag{8.3}$$

In components these symmetry transformations read

$$\begin{aligned} SU(2)_s: &\quad \psi_{\pm}^{A,B}(x)' = \exp(\pm i\alpha(x))\psi_{\pm}^{A,B}(x), \\ U(1)_Q: &\quad {}^{Q}\psi_{\pm}^{A,B}(x) = \exp(i\omega)\psi_{\pm}^{A,B}(x), \\ D: &\quad {}^{D}\psi_{\pm}^{A,B}(x) = \mp\exp(\mp i\varphi(x))\psi_{\mp}^{B,A}(x), \\ D': &\quad {}^{D'}\psi_{\pm}^{A,B}(x) = \pm\psi_{\mp}^{B,A}(x). \end{aligned} \tag{8.4}$$

Under the space-time symmetries, i.e. under the 90 degrees rotation O, the reflection R, time-reversal T, and its unbroken variant T' the fermion fields transform as

$$\begin{aligned}
O: \quad & {}^{O}\Psi^{A,B}(x) = \Psi^{A,B}(Ox), \\
R: \quad & {}^{R}\Psi^{A,B}(x) = \Psi^{A,B}(Rx), \\
T: \quad & {}^{T}\Psi^{A,B}(x) = \tau(Tx)(i\sigma_2)\Psi^{A,B\dagger}(Tx)^T\sigma_3, \\
& {}^{T}\Psi^{A,B\dagger}(x) = -\sigma_3\Psi^{A,B}(Tx)^T(i\sigma_2)^\dagger\tau(Tx)^\dagger, \\
T': \quad & {}^{T'}\Psi^{A,B}(x) = -\Psi^{A,B\dagger}(Tx)^T\sigma_3, \\
& {}^{T'}\Psi^{A,B\dagger}(x) = \sigma_3\Psi^{A,B}(Tx)^T.
\end{aligned} \tag{8.5}$$

Again an upper index T on the right denotes transpose, while on the left it denotes time-reversal. The form of the time-reversal symmetry T in the effective theory with non-linearly realized $SU(2)_s$ symmetry follows from the usual form of time-reversal in the Euclidean path integral of a non-relativistic theory in which the spin symmetry is linearly realized. The fermion fields in the two formulations just differ by a factor $u(x)$. In components the previous relations take the form

$$\begin{aligned}
O: \quad & {}^{O}\psi_\pm^{A,B}(x) = \psi_\pm^{A,B}(Ox), \\
R: \quad & {}^{R}\psi_\pm^{A,B}(x) = \psi_\pm^{A,B}(Rx), \\
T: \quad & {}^{T}\psi_\pm^{A,B}(x) = \exp(\mp i\varphi(Tx))\psi_\pm^{A,B\dagger}(Tx), \\
& {}^{T}\psi_\pm^{A,B\dagger}(x) = -\exp(\pm i\varphi(Tx))\psi_\pm^{A,B}(Tx), \\
T': \quad & {}^{T'}\psi_\pm^{A,B}(x) = -\psi_\pm^{A,B\dagger}(Tx), \\
& {}^{T'}\psi_\pm^{A,B\dagger}(x) = \psi_\pm^{A,B}(Tx).
\end{aligned} \tag{8.6}$$

It should be noted that the components + and − (denoting spin parallel and antiparallel to the direction of the staggered magnetization) are not interchanged under time-reversal. While both the spin of the fermion and the staggered magnetization change sign under time-reversal, the projection of one onto the other does not.

The action to be constructed in the next sections must be invariant under the internal symmetries $SU(2)_s$, $U(1)_Q$ (or even $SU(2)_Q$), D and D', as well as under space-time translations and the other space-time symmetries O, R, and T (or equivalently T').

The fundamental forces underlying condensed matter physics are Poincaré-invariant. However, some of the space-time symmetries may be spontaneously broken by the formation of a crystal lattice. The resulting Nambu-Goldstone bosons are the phonons, which play a central role in ordinary low-temperature superconductors by providing the force that binds Cooper pairs. In high-T_c superconductors, on the other hand, it is expected that phonons alone cannot provide the mechanism for Cooper pair formation. In the Hubbard model (and also in our effective theory) phonons are explicitly excluded because one imposes a rigid lattice by hand. This does not only break continuous translations and rotations down to their discrete counterparts; it also breaks space-time rotations. In a

relativistic context these would be the boosts of the Poincaré group. In a non-relativistic theory the lattice explicitly breaks Galilean boost invariance, thus providing a preferred rest frame (a condensed matter 'ether'). As a consequence, the magnon-mediated forces between a pair of electrons or holes may depend on the center of mass momentum of the pair. In the actual high-T_c materials Galilean (or more precisely Poincaré symmetry) is spontaneously (and not explicitly) broken. If phonons play an important role in the understanding of high-temperature superconductivity, one should construct an effective theory of spontaneously broken (and thus non-linearly realized) $SU(2)_s$ and Galilean symmetry which would automatically include both magnons and phonons. However, in this thesis we assume that phonons play no major role in the cuprates. In that case, it is legitimate to break Galilean invariance explicitly instead of spontaneously.

8.2 Classification of the expressions to be constructed

Since now all symmetry transformations are established we could immediately start to invent symmetry invariant expressions. The challenge, however, is that we must construct all the expressions up to a certain order. To this end it makes sense to first categorize all the expressions that may arise.

First of all, the number of fermion fields n_ψ in a given term must be even. More precisely the $U(1)_Q$ symmetry dictates that every non-daggered field $\psi_\pm^{A,B}(x)$ must be paired with one of the daggered fields $\psi_\pm^{A,B\dagger}(x)$. Otherwise remaining factors of $\exp(\pm i\omega)$ will indicate the non-compliance with the $U(1)_Q$ symmetry of the given term. Furthermore there are eight different fermion fields, i.e. $\psi_+^A(x), \psi_-^A(x), \psi_+^B(x), \psi_-^B(x)$ and their daggered counterparts. As a result every term consisting of more than eight fermion fields contains a field twice and therefore vanishes due to the anticommutation relations (Pauli principle), unless one includes derivatives.

The effective theory provides a systematic low-energy expansion organized according to the number of derivatives in the terms of the effective action. The most important terms are therefore the ones with a small number of derivatives. Since we are only interested in these leading terms, we restrict ourselves to terms with at most two derivatives.

In the purely magnonic sector where $n_\psi = 0$ all terms without derivatives vanish due to the properties of the matrix-valued magnon field P. A single spatial derivative violates space-time symmetries (e.g. spatial rotations) and is hence forbidden. Two spatial derivatives are well compatible with the space-time symmetries, however, and are thus allowed. Since antiferromagnetic magnons have a 'relativistic' dispersion relation we have to count temporal and spatial derivatives on equal footing. Hence, terms with two temporal derivatives are allowed but none with only one temporal derivative. Note that this is in contrast to ferromagnetic magnons which have a non-relativistic dispersion relation. As discussed in Appendix E, this is since the WZW term, which involves a single temporal derivative, does not vanish in this case. Hence, for ferromagnetic magnons terms with only one temporal derivative would be allowed.

In the fermionic sectors with $n_\psi = 2, 4, 6, 8$ things are a little bit different. Here it is possible to write down non-vanishing terms without derivatives. Again, a single spatial derivative is forbidden because of space-time symmetries, but two spatial derivatives are admitted. Since the fermions are massive they have a non-relativistic dispersion relation (at least in the low-energy limit of the effective theory) we have to count one temporal derivative on equal footing as two spatial derivatives. As a result only terms with one temporal derivative are allowed in addition at the given order. Note that there were indeed terms with two temporal derivatives if we would consider terms with four spatial derivatives.

The terms with more than two fermion fields are contact terms and thus describe short-range interactions between the fermions. Since we are mostly interested in the long-range interactions, we content ourselves in the sectors $n_\psi = 4, 6, 8$ with the leading terms without derivatives and hence omit the construction of terms with derivatives. Moreover we omit the construction of terms with more than eight fermion fields with derivatives at all, because these are contact terms involving many fermions and are as such for our needs completely out of interest. When one wants to address questions for which the short-distance forces between the charge carriers are essential, it will be necessary to consider such terms. Their construction can be performed using the same techniques as presented in the next sections.

Let us summarize what we have sketched so far and introduce the notation (n_t, n_i, n_ψ) to classify the terms we want to focus on according to the number of temporal derivatives n_t, the number of spatial derivatives n_i, and the number of fermion fields n_ψ they contain. In the purely magnonic sector we then have the classes $(2, 0, 0)$ and $(0, 2, 0)$ describing the propagation of magnons. Next we have the class $(0, 0, 2)$, which provides the mass terms of the fermions. The propagation of the fermions as well as the interactions between fermions and magnons are governed by the classes $(1, 0, 2)$ and $(0, 2, 2)$. Finally, there are various contact terms describing the short-range interactions between the fermions originating from the classes $(0, 0, 4)$, $(0, 0, 6)$ and $(0, 0, 8)$. All other classes are either empty, are incompatible with the symmetries, or are of no interest for our needs as described above.

Note that we do not mention the number of magnon fields to classify the terms. It will simply turn out that this is dictated by the number of derivatives involved.

8.3 Strategy to systematically construct all expressions

Now we are ready to set up a strategy to construct all terms in the classes we consider. First we notice that the classes as defined above are disjoint, i.e. they are closed under all symmetry transformations. This is simply because the number of fields and derivatives does not change under any of the symmetry transformations. This observation allows us to construct the terms in each class completely independent of the terms in the other classes. Fortunately this divides the complexity of the whole problem into more manageable parts.

So how do we construct all the terms in a given class? Basically there are two directions one can start with: We can use a top-down approach, that is we 'invent' and write down

expressions using the (high-level) trace notation and afterwards decompose the found expressions to the (low-level) component notation. The advantage of this approach is that it is easy to find expressions that are invariant under the $SU(2)_s$ and $U(1)_Q$ (or even $SU(2)_Q$) symmetries, since in trace notation these symmetries are manifest. However, it is not evident if an expression is also invariant under the displacement symmetry D and the time reversal symmetry T. To check this we first have to decompose the traces into components. Another drawback with this approach is that the trace notation is not unique, i.e. there exist numerous relations between the various traces, like e.g.

$$\mathrm{Tr}[\Psi^{A\dagger}\Psi^{B}\Psi^{B\dagger}\Psi^{A}] + \mathrm{Tr}[\Psi^{A\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{B}] = -2\,\mathrm{Tr}[\Psi^{A\dagger}\Psi^{B}\Psi^{A\dagger}\Psi^{B}]. \tag{8.7}$$

Some traces even vanish like e.g.

$$\mathrm{Tr}[\Psi^{A\dagger}D_t\Psi^{A}\sigma_3] = \mathrm{Tr}[\Psi^{B\dagger}D_t\Psi^{B}\sigma_3] = 0. \tag{8.8}$$

This fact may lead to unwanted linear dependencies in the set of the constructed expressions, if one is not cautious. Note that the above examples are by far not exhaustive. See Appendix C for a more comprehensive list of such relations.

The more severe drawback, however, is that one is never sure if one has really found all linearly independent expressions in a given class, since there is no obvious systematic way to enumerate all possible expressions in the trace notation. In other words, it is delicate to determine the dimension of the vector space that corresponds to the given class. For example, is it sufficient to consider single traces or do we need products of traces as well (or even other constructs) to cover the entire vector space?[1]

Even if we succeed to find all possible expressions, there will be a vast amount of linear dependencies in the resulting set, as indicated by Eqs. (8.7) and (8.8) and Appendix C. To identify and eliminate these linear dependencies we have again to decompose all traces into component notation. Then we have to solve a system of linear equations of the order equal to the number of expressions found, which can be quite large. Since we have to work with components anyway, one may ask if it would not be wiser to start in component notation right from the beginning. All these insights lead us to the second approach.

In the bottom-up approach we systematically enumerate and write down all possible terms in a given class using the component notation. Doing so we forget about the symmetries at first, since they are not manifest in component notation anyway. This strategy ensures that we do not forget any terms. At a later stage we will then impose the symmetries one after the other. To achieve this we have to drop some of the terms again and then linear combine the remaining ones appropriately to build expressions that are compatible with all the symmetries. Note that in contrast to the top-down approach the procedure of enumeration is well practicable here, since the component notation is much simpler and has

[1] At least we can do without determinants, since $\mathrm{Det}[A] = \frac{1}{2}\left(\mathrm{Tr}[A]^2 - \mathrm{Tr}[A^2]\right)$. Note that A has to be bosonic, otherwise the determinant is not well defined. This is no problem, since we always have an even number of fermion fields Ψ.

by far less ambiguities. Moreover the residual ambiguities can be eliminated by imposing some simple additional constraints.

Upon completion of the above procedure we end up with a linearly independent set of expressions (each consisting of a linear combination of terms in component notation) which span the given class. However, it is not obvious how to restate these expressions in trace notation. Unfortunately we have to go basically back to the top-down approach again in order to achieve this, i.e. we have to 'invent' symmetry invariant expressions in the trace notation and hence we will face almost all the problems associated with this as described before. The significant difference however is, that now after the construction of all invariant expressions in component notation we know the dimensions of all the individual vector spaces. As a result we know exactly when we have found all trace expressions in demand and hence can safely stop the annoying 'invention' process. During this whole process we have of course to ensure that all the found trace expressions are linearly independent.

To systematize the above 'invention' process a little bit more, let us introduce the 'number of equal sublattice indices' in a given term defined as

$$n_{AB} = \max(n_A, n_B), \tag{8.9}$$

where n_A and n_B are the number of fermion fields with sublattice index A or B, respectively. It is easy to see that this quantity is invariant under all symmetry operations. As such it is suitable to further divide the vector space of a given class into disjoint subspaces that do not mix under all the symmetry transformations. Consequently only terms with equal n_{AB} potentially have to be combined together in order to build expressions that are invariant under all the symmetries. In other words: Every symmetry invariant expression has a certain n_{AB}. This is of great help in the above 'invention' process, since it allows us to selectively search traces for each individual subspace. Due to its subspace indicating property we will state n_{AB} after each expression in parentheses.

It is obvious that this whole construction process involves a lot of work. As a result it is quite unavoidable to delegate most of the calculations to computers with suitable software. To enumerate all possible terms in component notation the scripting language Perl [90] was used. On the other hand for the more complex, algebraic manipulations the package FORM [91] was engaged. One part of the developed FORM program is responsible to decompose expressions in trace notation to component notation. Consequently, the other parts of the program have to deal with the component notation only. These parts are then responsible to do the various symmetry transformations. In addition some simplification routines were implemented.

In the now following construction of all invariant expressions in component notation, we first have to systematically enumerate all possible terms in a given class. To this end it is helpful to introduce a more enumerative notation for the fermion fields, i.e.

$$\{\psi_k\} = \left\{\psi_+^{A\dagger}, \psi_-^{A\dagger}, \psi_+^{B\dagger}, \psi_-^{B\dagger}, \psi_+^{A}, \psi_-^{A}, \psi_+^{B}, \psi_-^{B}\right\}, \quad k = 1, \ldots, 8, \tag{8.10}$$

such that e.g. $\psi_3 = \psi_+^{B\dagger}$.

8.4 Construction of all $U(1)_Q$ and $SU(2)_s$ invariant terms in component notation for the classes $(0,0,n_\psi)$ with $n_\psi = 2,4,6,8$

As the first step in the construction process in component notation we now have to find all $U(1)_Q$ and $SU(2)_s$ symmetry invariant terms. However, in this section we only do this for the classes with fermion fields and no derivatives. All the other classes are then handled in Appendix B.

To begin with let us enumerate all possible terms in each of these classes, irrespective of any symmetries or other constraints. We then find

$$\begin{aligned}
002: &\quad \psi_{k_1}\psi_{k_2}, && 1 \le k_1, k_2 \le 8, && n = 8^2 = 64,\\
004: &\quad \psi_{k_1}\psi_{k_2}\psi_{k_3}\psi_{k_4}, && 1 \le k_1, k_2, k_3, k_4 \le 8, && n = 8^4 = 4096,\\
006: &\quad \psi_{k_1}\psi_{k_2}\dots\psi_{k_5}\psi_{k_6}, && 1 \le k_1, k_2, \dots, k_5, k_6 \le 8, && n = 8^6 = 262144,\\
008: &\quad \psi_{k_1}\psi_{k_2}\dots\psi_{k_7}\psi_{k_8}, && 1 \le k_1, k_2, \dots, k_7, k_8 \le 8, && n = 8^8 = 16777216,
\end{aligned} \tag{8.11}$$

where n is the number of terms in each class so far. Note that we omit the commas and parentheses of the class specifications in the formulas in order to tighten the notation. Of course, here all terms with multiple identical fields vanish due to the Pauli principle. We can get rid of these if we demand that $k_i \neq k_j$ for $i \neq j$. After imposing this constraint the above list shrinks to

$$\begin{aligned}
002: &\quad \psi_{k_1}\psi_{k_2}, && 1 \le k_1, k_2 \le 8, && k_1 \neq k_2, && n = 8!/6! = 56,\\
004: &\quad \psi_{k_1}\psi_{k_2}\psi_{k_3}\psi_{k_4}, && 1 \le k_1, k_2, k_3, k_4 \le 8, && k_i \neq k_j,\ i \neq j, && n = 8!/4! = 1680,\\
006: &\quad \psi_{k_1}\psi_{k_2}\dots\psi_{k_5}\psi_{k_6}, && 1 \le k_1, k_2, \dots, k_5, k_6 \le 8, && k_i \neq k_j,\ i \neq j, && n = 8!/2! = 20160,\\
008: &\quad \psi_{k_1}\psi_{k_2}\dots\psi_{k_7}\psi_{k_8}, && 1 \le k_1, k_2, \dots, k_7, k_8 \le 8, && k_i \neq k_j,\ i \neq j, && n = 8!/0! = 40320.
\end{aligned} \tag{8.12}$$

Still this list is not yet unique, since e.g. $\psi_{k_1}\psi_{k_2} = -\psi_{k_2}\psi_{k_1}$ due to the anticommutative behaviour of the fermion fields. To get rid of this ambiguity we demand that $k_i \le k_j$ for $i < j$. Together with the above constraint we then find the short list

$$\begin{aligned}
002: &\quad \psi_{k_1}\psi_{k_2}, && 1 \le k_1 < k_2 \le 8, && n = 8!/(6!\,2!) = 28,\\
004: &\quad \psi_{k_1}\psi_{k_2}\psi_{k_3}\psi_{k_4}, && 1 \le k_1 < k_2 < k_3 < k_4 \le 8, && n = 8!/(4!\,4!) = 70,\\
006: &\quad \psi_{k_1}\psi_{k_2}\dots\psi_{k_5}\psi_{k_6}, && 1 \le k_1 < k_2 < \dots < k_5 < k_6 \le 8, && n = 8!/(2!\,6!) = 28,\\
008: &\quad \psi_{k_1}\psi_{k_2}\dots\psi_{k_7}\psi_{k_8}, && 1 \le k_1 < k_2 < \dots < k_7 < k_8 \le 8, && n = 8!/(0!\,8!) = 1,
\end{aligned} \tag{8.13}$$

where n now denotes the number of unique terms in each class.

So far our enumeration of all terms in the above classes is unique, but none of the symmetries are guaranteed yet. To cure this we want to impose as a first step the $U(1)_Q$ symmetry. We can achieve this if we transform each term in the list in turn according

to the $U(1)_Q$ transformation law. If the term under consideration is invariant we keep it, otherwise we drop it from the list. For example

$$\begin{aligned} 002: \quad & {}^{Q}[\psi_+^{A\dagger}\psi_-^{A}] = e^{-i\omega}\psi_+^{A\dagger}e^{+i\omega}\psi_-^{A}, \\ 004: \quad & {}^{Q}[\psi_+^{A\dagger}\psi_-^{A\dagger}\psi_-^{A}\psi_+^{B}] = e^{-i\omega}\psi_+^{A\dagger}e^{-i\omega}\psi_-^{A\dagger}e^{+i\omega}\psi_-^{A}e^{+i\omega}\psi_+^{B}, \end{aligned} \tag{8.14}$$

are invariant under the $U(1)_Q$ symmetry and must hence be kept, whereas

$$\begin{aligned} 002: \quad & {}^{Q}[\psi_+^{A\dagger}\psi_-^{A\dagger}] = e^{-i\omega}\psi_+^{A\dagger}e^{-i\omega}\psi_-^{A\dagger}, \\ 002: \quad & {}^{Q}[\psi_+^{A}\psi_+^{B}] = e^{+i\omega}\psi_+^{A}e^{+i\omega}\psi_+^{B}, \\ 004: \quad & {}^{Q}[\psi_-^{B\dagger}\psi_+^{A}\psi_-^{A}\psi_-^{B}] = e^{-i\omega}\psi_-^{B\dagger}e^{+i\omega}\psi_+^{A}e^{+i\omega}\psi_-^{A}e^{+i\omega}\psi_-^{B}, \\ 004: \quad & {}^{Q}[\psi_+^{A\dagger}\psi_-^{A\dagger}\psi_+^{B\dagger}\psi_-^{B\dagger}] = e^{-i\omega}\psi_+^{A\dagger}e^{-i\omega}\psi_-^{A\dagger}e^{-i\omega}\psi_+^{B\dagger}e^{-i\omega}\psi_-^{B\dagger}, \end{aligned} \tag{8.15}$$

violate the $U(1)_Q$ symmetry and must hence be dropped. Note that the remaining prefactors of $\exp(\pm i\omega)$ cannot be cancelled in any way (e.g. by linear combinations of different terms). It is therefore justified to drop these terms. Also note that, metaphorically speaking, the $U(1)_Q$ symmetry 'counts' the number of non-daggered versus daggered fermion fields. All terms that are not balanced in this sense will not survive the $U(1)_Q$ symmetry requirement. After finalizing this step we end up with a shortened list of terms for each class, i.e.

$$\begin{aligned} 002: \quad & \psi_{k_1}\psi_{k_2}, && \text{unique, } U(1)_Q \text{ invariant}, && n = 16, \\ 004: \quad & \psi_{k_1}\psi_{k_2}\psi_{k_3}\psi_{k_4}, && \text{unique, } U(1)_Q \text{ invariant}, && n = 36, \\ 006: \quad & \psi_{k_1}\psi_{k_2}\ldots\psi_{k_5}\psi_{k_6}, && \text{unique, } U(1)_Q \text{ invariant}, && n = 16, \\ 008: \quad & \psi_{k_1}\psi_{k_2}\ldots\psi_{k_7}\psi_{k_8}, && \text{unique, } U(1)_Q \text{ invariant}, && n = 1, \end{aligned} \tag{8.16}$$

where n denotes the number of unique terms remaining in each class after imposing the $U(1)_Q$ symmetry.

The next step is to impose the $SU(2)_s$ symmetry. To achieve this we play the same game again, but now acting with the $SU(2)_s$ transformation law on each of the remaining terms. Again, the terms that are invariant are kept, while the ones that break the symmetry are dropped from the list. For example

$$\begin{aligned} 002: \quad & [\psi_-^{A\dagger}\psi_-^{B}]' = e^{+i\alpha(x)}\psi_-^{A\dagger}e^{-i\alpha(x)}\psi_-^{B}, \\ 004: \quad & [\psi_+^{A\dagger}\psi_-^{A\dagger}\psi_-^{A}\psi_+^{B}]' = e^{-i\alpha(x)}\psi_+^{A\dagger}e^{+i\alpha(x)}\psi_-^{A\dagger}e^{-i\alpha(x)}\psi_-^{A}e^{+i\alpha(x)}\psi_+^{B}, \end{aligned} \tag{8.17}$$

are invariant under the $SU(2)_s$ symmetry and must hence be kept, whereas

$$\begin{aligned} 002: \quad & [\psi_+^{A\dagger}\psi_-^{A}]' = e^{-i\alpha(x)}\psi_+^{A\dagger}e^{-i\alpha(x)}\psi_-^{A}, \\ 004: \quad & [\psi_-^{A\dagger}\psi_-^{B\dagger}\psi_+^{A}\psi_+^{B}]' = e^{+i\alpha(x)}\psi_-^{A\dagger}e^{+i\alpha(x)}\psi_-^{B\dagger}e^{+i\alpha(x)}\psi_+^{A}e^{+i\alpha(x)}\psi_+^{B}, \end{aligned} \tag{8.18}$$

violate the $SU(2)_s$ symmetry and must hence be dropped. Again, the remaining prefactors of $\exp(\pm i\alpha(x))$ cannot be cancelled in any way and the deletion of these terms is thus

legitimate. Note that, metaphorically speaking, the $SU(2)_s$ symmetry 'counts' the number of fermion fields with + versus − spin orientation, whereas the daggered fields contribute with opposite sign. All terms that are not balanced in this sense will not survive the $SU(2)_s$ symmetry requirement. Upon completion of this second step we end up with an even shorter list of terms for each class, i.e.

$$\begin{aligned}
A_r^{002} &= \psi_{k_1}\psi_{k_2}, && \text{unique, } U(1)_Q \text{ and } SU(2)_s \text{ invariant}, && n = 8,\\
A_r^{004} &= \psi_{k_1}\psi_{k_2}\psi_{k_3}\psi_{k_4}, && \text{unique, } U(1)_Q \text{ and } SU(2)_s \text{ invariant}, && n = 18,\\
A_r^{006} &= \psi_{k_1}\psi_{k_2}\ldots\psi_{k_5}\psi_{k_6}, && \text{unique, } U(1)_Q \text{ and } SU(2)_s \text{ invariant}, && n = 8,\\
A_r^{008} &= \psi_{k_1}\psi_{k_2}\ldots\psi_{k_7}\psi_{k_8}, && \text{unique, } U(1)_Q \text{ and } SU(2)_s \text{ invariant}, && n = 1,
\end{aligned} \tag{8.19}$$

where n now denotes the number of unique terms remaining in each class after imposing the $U(1)_Q$ and $SU(2)_s$ symmetry, i.e. $r = 1, \ldots, n$. Note that the explicit representation of these terms is given in the next section for class $(0, 0, 2)$ and for all other classes in Appendix B.

Still we have to impose the displacement symmetry D and the time reversal symmetry T. It will turn out that, in contrast to the previous steps, this will not force us to drop any further terms. Instead we have to combine the remaining terms appropriately to build invariant and linearly independent expressions. What then is still missing, is the treatment of the $SU(2)_Q$ symmetry. Since we handle the $SU(2)_Q$ invariance only as a non-mandatory option we want to separate the $SU(2)_Q$ breaking from the $SU(2)_Q$ invariant expressions. More precisely this means that we have to determine the dimension of the $SU(2)_Q$ invariant subspace. This involves further linear combinations to be made out of the above expressions. In the next section we will show this for the class $(0, 0, 2)$ at full length. For all other classes this is presented in a somewhat shorter form in Appendix B.

8.5 Construction of all symmetry invariant expressions in component notation for the class $(0, 0, 2)$

According to Eq. (8.19), there are eight remaining terms A_r^{002} after imposing the $U(1)_Q$ and $SU(2)_s$ symmetries. Explicitly these are

$$\begin{aligned}
A_1^{002} &= \psi_+^{A\dagger}\psi_+^A, \quad (2), & A_5^{002} &= \psi_+^{B\dagger}\psi_+^A, \quad (1),\\
A_2^{002} &= \psi_+^{A\dagger}\psi_+^B, \quad (1), & A_6^{002} &= \psi_+^{B\dagger}\psi_+^B, \quad (2),\\
A_3^{002} &= \psi_-^{A\dagger}\psi_-^A, \quad (2), & A_7^{002} &= \psi_-^{B\dagger}\psi_-^A, \quad (1),\\
A_4^{002} &= \psi_-^{A\dagger}\psi_-^B, \quad (1), & A_8^{002} &= \psi_-^{B\dagger}\psi_-^B, \quad (2),
\end{aligned} \tag{8.20}$$

where the number in parentheses after each term represents the quantity n_{AB} as defined in Eq. (8.9). Let us first examine the behavior of these terms under Hermitean conjugation. This information will be useful later to determine if the invariant expressions that have been built are Hermitean or Anti-Hermitean. Since in our path integral description

the fermion fields ψ_k are Grassmann numbers, the operation of Hermitean conjugation is a priori not defined. This is because e.g. ψ_+^A and $\psi_+^{A\dagger}$ are completely independent Grassmann numbers and therefore not related in any way. However, we can resolve this problem, if we imagine to make the transition to the Hamiltonian description, where the fermion fields ψ_k turn into operators and, e.g. , ψ_+^A and $\psi_+^{A\dagger}$ would be indeed related by Hermitean conjugation. Actually the properties of the action (to be constructed below) under Hermitean conjugation is inherited from the Hermiticity of the corresponding Hamiltonian in exactly this way. In the following we adopt therefore this point of view to give the operation of Hermitean conjugation a proper meaning and we get

$$\begin{aligned} A_1^{002\dagger} &= A_1^{002}, \quad (2), & A_4^{002\dagger} &= A_7^{002}, \quad (1), \\ A_2^{002\dagger} &= A_5^{002}, \quad (1), & A_6^{002\dagger} &= A_6^{002}, \quad (2), \\ A_3^{002\dagger} &= A_3^{002}, \quad (2), & A_8^{002\dagger} &= A_8^{002}, \quad (2). \end{aligned} \tag{8.21}$$

Already here we can see that an expression containing e.g. A_2^{002} must also contain A_5^{002} in order to have a chance to be Hermitean or anti-Hermitean. It will turn out that all such combinations that are necessary to make an expression either Hermitean or anti-Hermitean will take place automatically as we impose the D and T symmetries.

Now we are ready to impose the displacement symmetry D. To achieve this, we transform each A_r^{002} according to the D transformation law, e.g.

$$\begin{aligned} {}^D A_2^{002} &= {}^D[\psi_+^{A\dagger}\psi_+^B] = (-1)e^{+i\varphi(x)}\psi_-^{B\dagger}(-1)e^{-i\varphi(x)}\psi_-^A = A_7^{002}, \quad (1), \\ {}^D A_7^{002} &= {}^D[\psi_-^{B\dagger}\psi_-^A] = e^{-i\varphi(x)}\psi_+^{A\dagger}e^{+i\varphi(x)}\psi_+^B \qquad\qquad = A_2^{002}, \quad (1). \end{aligned} \tag{8.22}$$

This result suggests that we have to combine A_2^{002} and A_7^{002} together in order to build a displacement symmetry D invariant expression. Indeed we find

$$\begin{aligned} B_1^{002} &= A_1^{002} + A_8^{002}, \quad (2), & B_3^{002} &= A_3^{002} + A_6^{002}, \quad (2), \\ B_2^{002} &= A_2^{002} + A_7^{002}, \quad (1), & B_4^{002} &= A_4^{002} + A_5^{002}, \quad (1), \end{aligned} \tag{8.23}$$

to be invariant under the displacement symmetry D. Note that as already promised no further terms have to be dropped. Note also that indeed only terms with identical n_{AB} have to be combined together.

Last but not least we have to impose the time reversal symmetry T. In order to accomplish this we now act on each B_s^{002} with the T transformation law, which reveals e.g.

$$\begin{aligned} {}^T B_1^{002} &= B_1^{002}, \quad (2), \\ {}^T B_2^{002} &= B_4^{002}, \quad (1), \\ {}^T B_4^{002} &= B_2^{002}, \quad (1). \end{aligned} \tag{8.24}$$

Again this result suggests that B_2^{002} and B_4^{002} have to be combined together in order to build a time reversal T invariant expression, whereas B_1^{002} can be left alone, since it is

already invariant. Indeed we find

$$\begin{aligned} C_1^{002} &= B_1^{002} &&= A_1^{002} + A_8^{002}, && (2), \\ C_2^{002} &= B_2^{002} + B_4^{002} &&= A_2^{002} + A_4^{002} + A_5^{002} + A_7^{002}, && (1), \\ C_3^{002} &= B_3^{002} &&= A_3^{002} + A_6^{002}, && (2), \end{aligned} \tag{8.25}$$

to be invariant under the time reversal symmetry T. Again no further terms have to be dropped and again only terms with identical n_{AB} have to be combined together. Note that the C_q^{002}, $q = 1, 2, 3$ are apparently linearly independent, since each A_r^{002} appears in only one C_q^{002}. Consequently, the $U(1)_Q$, $SU(2)_s$, D and T invariant vector space is 3-dimensional. Note that there is no need to separately consider the D' and T' symmetries, since these are made out of the already imposed $SU(2)_s$, D and T symmetries.

The only symmetry not addressed so far is the full $SU(2)_Q$ symmetry. Let us act therefore with an arbitrary $SU(2)_Q$ symmetry transformation Ω^T on each C_q^{002}. We find that C_2^{002} is already $SU(2)_Q$ invariant, whereas C_1^{002} and C_3^{002} are not. It would be now overhasty (and in fact wrong) to conclude from this that the $SU(2)_Q$ invariant subspace is only 1-dimensional. Indeed we must be more cautious, since an appropriate linear combination of $SU(2)_Q$ breaking expressions can reveal an $SU(2)_Q$ invariant contribution. To examine this, let us form a general linear combination out of the $SU(2)_Q$ breaking terms, i.e.

$$\beta_1 C_1^{002} + \beta_3 C_3^{002}, \quad (2). \tag{8.26}$$

Now we act again with an arbitrary $SU(2)_Q$ symmetry transformation Ω^T on this expression, which yields

$$-\beta_3 C_1^{002} - \beta_1 C_3^{002} + (\beta_1 + \beta_3) \sum_{1 \le k < l \le 8} f_1^{002}(\Omega^T)_{kl} \psi_k \psi_l, \quad (2). \tag{8.27}$$

Here the $f_1^{002}(\Omega^T)_{kl}$ are polynomials in the matrix elements of Ω^T, which represent only the part of the linear combination, that is not $SU(2)_Q$ invariant. If we subtract this result from the original linear combination (8.26) we get

$$(\beta_1 + \beta_3) C_1^{002} + (\beta_1 + \beta_3) C_3^{002} - (\beta_1 + \beta_3) \sum_{1 \le k < l \le 8} f_1^{002}(\Omega^T)_{kl} \psi_k \psi_l, \quad (2). \tag{8.28}$$

Of course the general linear combination (8.26) is $SU(2)_Q$ invariant only if this expression vanishes. Since this happens for $\beta_3 = -\beta_1$, we have actually found an additional $SU(2)_Q$ invariant contribution, such that the $SU(2)_Q$ invariant subspace is now indeed 2-dimensional instead of only 1-dimensional. Since the total, $U(1)_Q$ invariant vector space is 3-dimensional, there must still be one $SU(2)_Q$ breaking expression. For this one we can take an arbitrary linearly independent combination like e.g. $\beta_3 = \beta_1$. Summarizing all these findings we can state our final result while choosing $\beta_1 = 1$ as

$$\begin{aligned} D_1^{002} &= C_2^{002} &&= A_2^{002} + A_4^{002} + A_5^{002} + A_7^{002}, && (1), \\ D_2^{002} &= C_1^{002} - C_3^{002} &&= A_1^{002} - A_3^{002} - A_6^{002} + A_8^{002}, && (2), \\ \widetilde{D}_1^{002} &= C_1^{002} + C_3^{002} &&= A_1^{002} + A_3^{002} + A_6^{002} + A_8^{002}, && (2). \end{aligned} \tag{8.29}$$

Note that we decorate the $SU(2)_Q$ breaking expressions with a tilde, like e.g. $\widetilde{D}_1^{002}$. A consultation of Eq. (8.21) reveals that all these expressions as well as the C_q^{002} are Hermitean. Also note that the A_r^{002} no longer appear in only one of the D_u^{002} and $\widetilde{D}_{u'}^{002}$ as this was the case for C_q^{002}. Nevertheless the set $\{D_u^{002}, \widetilde{D}_{u'}^{002}\}$ is linearly independent.

Finally, we want to restate this result in the trace notation. The question of finding an appropriate representation in trace notation is addressed at full length in Appendix C. Here we just state the result from there and give the relations to the findings in component notation of this section, i.e.

$$\begin{aligned} E_1^{002} &= \mathrm{Tr}[\Psi^{A\dagger}\Psi^B] &&= D_1^{002}, \quad (1), \\ E_2^{002} &= \frac{1}{2}\mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^A - \Psi^{B\dagger}\sigma_3\Psi^B] &&= D_2^{002}, \quad (2), \\ \widetilde{E}_1^{002} &= \frac{1}{2}\mathrm{Tr}[\Psi^{A\dagger}\Psi^A\sigma_3 + \Psi^{B\dagger}\Psi^B\sigma_3] &&= \widetilde{D}_1^{002}, \quad (2). \end{aligned} \tag{8.30}$$

So far everything is said and we can step on to the next class to play the same game again. As already mentioned this material can be found in the Appendices B and C. In the next section we will use all these results to finally construct the effective action.

8.6 Effective action for magnons and charge carriers

We now present the leading expressions in the effective action for magnons and electrons or holes. Since we handle the $SU(2)_Q$ symmetry as a non-mandatory option we decompose the effective Lagrangian into an $SU(2)_Q$ invariant part $\mathcal{L}$ and an $SU(2)_Q$ breaking (but still $U(1)_Q$ invariant) part $\widetilde{\mathcal{L}}$. As before, we classify the individual contributions $\mathcal{L}^{n_t,n_i,n_\psi}$ and $\widetilde{\mathcal{L}}^{n_t,n_i,n_\psi}$ to the effective Lagrangian according to the number of temporal derivatives n_t, the number of spatial derivatives n_i, and the number of fermion fields n_ψ they contain. The total action is then given by

$$S[\psi_\pm^{A,B\dagger}, \psi_\pm^{A,B}, P] = \int d^2x\, dt \sum_{n_t,n_i,n_\psi} (\mathcal{L}^{n_t,n_i,n_\psi} + \widetilde{\mathcal{L}}^{n_t,n_i,n_\psi}) \tag{8.31}$$

and the partition function takes the form

$$Z = \int \mathcal{D}\psi_\pm^{A,B\dagger}\, \mathcal{D}\psi_\pm^{A,B}\, \mathcal{D}P \exp(-S[\psi_\pm^{A,B\dagger}, \psi_\pm^{A,B}, P]). \tag{8.32}$$

Note that all the individual contributions $\mathcal{L}^{n_t,n_i,n_\psi}$ and $\widetilde{\mathcal{L}}^{n_t,n_i,n_\psi}$ to the effective Lagrangian must be either Hermitean with a real prefactor, or anti-Hermitean with a purely imaginary prefactor, such that the corresponding Hamiltonian is Hermitean. Furthermore all contributions must, of course, be invariant under the $U(1)_Q$, $SU(2)_s$, D, D', O, R, T,

and T' symmetries, whereas the contributions $\mathcal{L}^{n_t,n_i,n_\psi}$ must even be invariant under the full $SU(2)_Q$ symmetry in addition.

Let us start with the purely magnonic sector which is implemented by the classes $(2,0,0)$ and $(0,2,0)$. These classes correspond to the leading contributions of Eq. (6.5) and they take the form

$$\mathcal{L}^{200} = \frac{\rho_s}{c^2}\mathrm{Tr}[\partial_t P \partial_t P], \qquad \mathcal{L}^{020} = \rho_s \mathrm{Tr}[\partial_i P \partial_i P]. \tag{8.33}$$

Since antiferromagnetic magnons have a 'relativistic' dispersion relation (with the spin-wave velocity c playing the role of the velocity of light), in pure magnon chiral perturbation theory one counts temporal and spatial derivatives as being of the same order.

Until now we have constructed the effective action in the $Q = 0$ sector, i.e. for a half-filled system which is described entirely in terms of magnons. Next we will consider the terms quadratic in the fermion fields. These contribute to the scattering of magnons off electrons or holes and they generally describe the propagation of the charge carriers in the antiferromagnetic background in the sectors with $Q \neq 0$.

In contrast to magnons, electrons and holes are massive and have a non-relativistic dispersion relation. Hence, it is natural to count one temporal and two spatial derivatives as being of the same order. In order to count derivatives consistently, in the $Q \neq 0$ sectors it may thus be necessary to also consider the pure magnon term $\mathcal{L}^{200}$ with two temporal derivatives as being of higher order.

The $SU(2)_Q$ invariant leading order expressions without any derivatives which emerge from class $(0,0,2)$ take the form

$$\begin{aligned}\mathcal{L}^{002} &= M_1 \mathrm{Tr}[\Psi^{A\dagger}\Psi^B] + \frac{M_2}{2}\mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^A - \Psi^{B\dagger}\sigma_3\Psi^B] \\ &= \quad M_1(\psi_+^{A\dagger}\psi_+^B + \psi_-^{A\dagger}\psi_-^B + \psi_+^{B\dagger}\psi_+^A + \psi_-^{B\dagger}\psi_-^A) \\ &\quad + M_2(\psi_+^{A\dagger}\psi_+^A - \psi_-^{A\dagger}\psi_-^A - \psi_+^{B\dagger}\psi_+^B + \psi_-^{B\dagger}\psi_-^B).\end{aligned} \tag{8.34}$$

The mass parameters M_1 and M_2 (as well as all other low-energy parameters to be introduced below) take real values, in order to ensure Hermiticity of the corresponding Hamiltonian. When we impose only the generic $U(1)_Q$ but not the full $SU(2)_Q$ symmetry, one more fermion mass term can be added

$$\begin{aligned}\widetilde{\mathcal{L}}^{002} &= \frac{m}{2}\mathrm{Tr}[\Psi^{A\dagger}\Psi^A\sigma_3 + \Psi^{B\dagger}\Psi^B\sigma_3] \\ &= m(\psi_+^{A\dagger}\psi_+^A + \psi_-^{A\dagger}\psi_-^A + \psi_+^{B\dagger}\psi_+^B + \psi_-^{B\dagger}\psi_-^B).\end{aligned} \tag{8.35}$$

This expression can be absorbed into a redefinition of the chemical potential. Remarkably, no other symmetry invariant fermion mass terms exist.

The $SU(2)_Q$ invariant expressions with one temporal derivative correspond to class $(1,0,2)$ and are given by

$$\begin{aligned}\mathcal{L}^{102} &= \frac{1}{2}\mathrm{Tr}[\Psi^{A\dagger}D_t\Psi^A + \Psi^{B\dagger}D_t\Psi^B] \\ &\quad + \frac{i\Lambda_1}{2}\mathrm{Tr}[\Psi^{A\dagger}V_t\Psi^A + \Psi^{B\dagger}V_t\Psi^B] + i\Lambda_2\mathrm{Tr}[\Psi^{A\dagger}\sigma_3 V_t\Psi^B] \\ &= \psi_+^{A\dagger}D_t\psi_+^A + \psi_-^{A\dagger}D_t\psi_-^A + \psi_+^{B\dagger}D_t\psi_+^B + \psi_-^{B\dagger}D_t\psi_-^B \\ &\quad + i\Lambda_1(\psi_+^{A\dagger}v_t^+\psi_-^A + \psi_-^{A\dagger}v_t^-\psi_+^A + \psi_+^{B\dagger}v_t^+\psi_-^B + \psi_-^{B\dagger}v_t^-\psi_+^B) \\ &\quad + i\Lambda_2(\psi_+^{A\dagger}v_t^+\psi_-^B + \psi_-^{B\dagger}v_t^-\psi_+^A - \psi_+^{B\dagger}v_t^+\psi_-^A - \psi_-^{A\dagger}v_t^-\psi_+^B). \end{aligned} \tag{8.36}$$

Here V_t is the field defined in Eq. (6.30) and the covariant derivatives are those of Eq. (7.17). In components they take the form

$$\begin{aligned} D_\mu\psi_\pm^{A,B}(x) &= (\partial_\mu \pm iv_\mu^3(x))\psi_\pm^{A,B}(x), \\ D_\mu\psi_\pm^{A,B\dagger}(x) &= (\partial_\mu \mp iv_\mu^3(x))\psi_\pm^{A,B\dagger}(x). \end{aligned} \tag{8.37}$$

Note that v_t^3 as well as $v_t^\pm$ (and hence V_t) count like one temporal derivative because these composite fields indeed contain one temporal derivative of the magnon field.

When one derives the Euclidean path integral from the Hamiltonian formulation of the effective theory, the expression $\psi_+^{A\dagger}\partial_t\psi_+^A + \psi_-^{A\dagger}\partial_t\psi_-^A + \psi_+^{B\dagger}\partial_t\psi_+^B + \psi_-^{B\dagger}\partial_t\psi_-^B$ arises from the pairs of anticommuting fermion operators. However, according to Appendix B there are two additional $SU(2)_Q$ breaking but $U(1)_Q$ invariant expressions with a single temporal derivative, i.e.

$$\begin{aligned}\widetilde{E}_1^{1_D02} &= \mathrm{Tr}[\Psi^{A\dagger}D_t\Psi^B\sigma_3] \\ &= \psi_+^{A\dagger}D_t\psi_+^B + \psi_-^{A\dagger}D_t\psi_-^B + \psi_+^{B\dagger}D_t\psi_+^A + \psi_-^{B\dagger}D_t\psi_-^A, \\ \widetilde{E}_2^{1_D02} &= \frac{1}{2}\mathrm{Tr}[\Psi^{A\dagger}\sigma_3 D_t\Psi^A\sigma_3 - \Psi^{B\dagger}\sigma_3 D_t\Psi^B\sigma_3] \\ &= \psi_+^{A\dagger}D_t\psi_+^A - \psi_-^{A\dagger}D_t\psi_-^A - \psi_+^{B\dagger}D_t\psi_+^B + \psi_-^{B\dagger}D_t\psi_-^B. \end{aligned} \tag{8.38}$$

These expressions need not to be included in the effective Lagrangian, since they would not imply canonical anticommutation relations in the Hamiltonian formulation. In any case, as discussed in Appendix D, if one does include these expressions they can again be removed by an appropriate field redefinition.

Interestingly, there is only one more expression that violates the $SU(2)_Q$ symmetry but still respects the $U(1)_Q$ symmetry, i.e.

$$\begin{aligned}\widetilde{\mathcal{L}}^{102} &= i\lambda\mathrm{Tr}[\Psi^{A\dagger}V_t\Psi^B\sigma_3] \\ &= i\lambda(\psi_+^{A\dagger}v_t^+\psi_-^B + \psi_-^{B\dagger}v_t^-\psi_+^A + \psi_+^{B\dagger}v_t^+\psi_-^A + \psi_-^{A\dagger}v_t^-\psi_+^B). \end{aligned} \tag{8.39}$$

Next we consider the $SU(2)_Q$ invariant expressions with two spatial derivatives. These are covered in the class $(0,2,2)$ and are given by

$$\begin{aligned}
\mathcal{L}^{022} &= \frac{1}{2M_1'}\mathrm{Tr}[D_i\Psi^{A\dagger}D_i\Psi^B] + \frac{1}{4M_2'}\mathrm{Tr}[D_i\Psi^{A\dagger}\sigma_3 D_i\Psi^A - D_i\Psi^{B\dagger}\sigma_3 D_i\Psi^B] \\
&\quad + iK_1\mathrm{Tr}[D_i\Psi^{A\dagger}V_i\Psi^B + D_i\Psi^{B\dagger}V_i\Psi^A] \\
&\quad + iK_2\mathrm{Tr}[D_i\Psi^{A\dagger}\sigma_3 V_i\Psi^A - D_i\Psi^{B\dagger}\sigma_3 V_i\Psi^B] \\
&\quad + N_1\mathrm{Tr}[\Psi^{A\dagger}V_iV_i\Psi^B] + \frac{N_2}{2}\mathrm{Tr}[\Psi^{A\dagger}V_i\sigma_3 V_i\Psi^A - \Psi^{B\dagger}V_i\sigma_3 V_i\Psi^B] \\
&= \quad \frac{1}{2M_1'}(D_i\psi_+^{A\dagger}D_i\psi_+^B + D_i\psi_+^{B\dagger}D_i\psi_+^A + D_i\psi_-^{A\dagger}D_i\psi_-^B + D_i\psi_-^{B\dagger}D_i\psi_-^A) \\
&\quad + \frac{1}{2M_2'}(D_i\psi_+^{A\dagger}D_i\psi_+^A - D_i\psi_-^{A\dagger}D_i\psi_-^A - D_i\psi_+^{B\dagger}D_i\psi_+^B + D_i\psi_-^{B\dagger}D_i\psi_-^B) \\
&\quad + iK_1(D_i\psi_+^{A\dagger}v_i^+\psi_-^B - \psi_-^{B\dagger}v_i^-D_i\psi_+^A + D_i\psi_-^{A\dagger}v_i^-\psi_+^B - \psi_+^{B\dagger}v_i^+D_i\psi_-^A \\
&\qquad + D_i\psi_+^{B\dagger}v_i^+\psi_-^A - \psi_-^{A\dagger}v_i^-D_i\psi_+^B + D_i\psi_-^{B\dagger}v_i^-\psi_+^A - \psi_+^{A\dagger}v_i^+D_i\psi_-^B) \\
&\quad + iK_2(D_i\psi_+^{A\dagger}v_i^+\psi_-^A - \psi_-^{A\dagger}v_i^-D_i\psi_+^A - D_i\psi_-^{A\dagger}v_i^-\psi_+^A + \psi_+^{A\dagger}v_i^+D_i\psi_-^A \\
&\qquad - D_i\psi_+^{B\dagger}v_i^+\psi_-^B + \psi_-^{B\dagger}v_i^-D_i\psi_+^B + D_i\psi_-^{B\dagger}v_i^-\psi_+^B - \psi_+^{B\dagger}v_i^+D_i\psi_-^B) \\
&\quad + N_1(\psi_+^{A\dagger}v_i^+v_i^-\psi_+^B + \psi_-^{A\dagger}v_i^-v_i^+\psi_-^B + \psi_+^{B\dagger}v_i^+v_i^-\psi_+^A + \psi_-^{B\dagger}v_i^-v_i^+\psi_-^A) \\
&\quad - N_2(\psi_+^{A\dagger}v_i^+v_i^-\psi_+^A - \psi_-^{A\dagger}v_i^-v_i^+\psi_-^A - \psi_+^{B\dagger}v_i^+v_i^-\psi_+^B + \psi_-^{B\dagger}v_i^-v_i^+\psi_-^B). \qquad (8.40)
\end{aligned}$$

Note that the imaginary unit i in front of the expressions proportional to K_1 and K_2 is necessary to ensure that the corresponding Hamiltonian is Hermitean. Since the doped electrons or holes are non-relativistic, there is no reason why the kinetic mass parameters M_1' and M_2' should agree with the rest mass parameters M_1 and M_2. In addition, there are again expressions that break the $SU(2)_Q$ symmetry but leave the $U(1)_Q$ symmetry intact

$$\begin{aligned}
\widetilde{\mathcal{L}}^{022} &= \frac{1}{4m'}\mathrm{Tr}[D_i\Psi^{A\dagger}D_i\Psi^A\sigma_3 + D_i\Psi^{B\dagger}D_i\Psi^B\sigma_3] \\
&\quad + i\kappa_1\mathrm{Tr}[D_i\Psi^{A\dagger}\sigma_3 V_i\Psi^B\sigma_3 + D_i\Psi^{B\dagger}V_i\sigma_3\Psi^A\sigma_3] \\
&\quad + i\kappa_2\mathrm{Tr}[D_i\Psi^{A\dagger}V_i\Psi^A\sigma_3 + D_i\Psi^{B\dagger}V_i\Psi^B\sigma_3] \\
&\quad + \frac{\nu}{2}\mathrm{Tr}[\Psi^{A\dagger}V_iV_i\Psi^A\sigma_3 + \Psi^{B\dagger}V_iV_i\Psi^B\sigma_3] \\
&= \frac{1}{2m'}(D_i\psi_+^{A\dagger}D_i\psi_+^A + D_i\psi_-^{A\dagger}D_i\psi_-^A + D_i\psi_+^{B\dagger}D_i\psi_+^B + D_i\psi_-^{B\dagger}D_i\psi_-^B) \\
&\quad + i\kappa_1(D_i\psi_+^{A\dagger}v_i^+\psi_-^B - \psi_-^{B\dagger}v_i^-D_i\psi_+^A - D_i\psi_-^{A\dagger}v_i^-\psi_+^B + \psi_+^{B\dagger}v_i^+D_i\psi_-^A \\
&\qquad - D_i\psi_+^{B\dagger}v_i^+\psi_-^A + \psi_-^{A\dagger}v_i^-D_i\psi_+^B + D_i\psi_-^{B\dagger}v_i^-\psi_+^A - \psi_+^{A\dagger}v_i^+D_i\psi_-^B) \\
&\quad + i\kappa_2(D_i\psi_+^{A\dagger}v_i^+\psi_-^A - \psi_-^{A\dagger}v_i^-D_i\psi_+^A + D_i\psi_-^{A\dagger}v_i^-\psi_+^A - \psi_+^{A\dagger}v_i^+D_i\psi_-^A \\
&\qquad + D_i\psi_+^{B\dagger}v_i^+\psi_-^B - \psi_-^{B\dagger}v_i^-D_i\psi_+^B + D_i\psi_-^{B\dagger}v_i^-\psi_+^B - \psi_+^{B\dagger}v_i^+D_i\psi_-^B) \\
&\quad + \nu(\psi_+^{A\dagger}v_i^+v_i^-\psi_+^A + \psi_-^{A\dagger}v_i^-v_i^+\psi_-^A + \psi_+^{B\dagger}v_i^+v_i^-\psi_+^B + \psi_-^{B\dagger}v_i^-v_i^+\psi_-^B). \qquad (8.41)
\end{aligned}$$

Finally we switch over to the contact terms, which describe the short-range interactions between the charge carriers. Here we first consider expressions quartic in the fermion fields. To lowest order there are five linearly independent $SU(2)_Q$ invariant 4-fermion expressions arising from class $(0,0,4)$, i.e.

$$\begin{aligned}
\mathcal{L}^{004} = \quad & \frac{G_1}{12}\mathrm{Tr}[\Psi^{A\dagger}\Psi^A\Psi^{A\dagger}\Psi^A + \Psi^{B\dagger}\Psi^B\Psi^{B\dagger}\Psi^B] + \frac{G_2}{2}\mathrm{Tr}[\Psi^{A\dagger}\Psi^A\Psi^{B\dagger}\Psi^B] \\
& + \frac{G_3}{2}\mathrm{Tr}[\Psi^{A\dagger}\Psi^B\Psi^{B\dagger}\Psi^A] + \frac{G_4}{2}\mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^A\Psi^{B\dagger}\sigma_3\Psi^B] \\
& + G_5\mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^A\Psi^{A\dagger}\Psi^B - \Psi^{B\dagger}\sigma_3\Psi^B\Psi^{B\dagger}\Psi^A] \\
= \quad & G_1(\psi_+^{A\dagger}\psi_+^A\psi_-^{A\dagger}\psi_-^A + \psi_+^{B\dagger}\psi_+^B\psi_-^{B\dagger}\psi_-^B) \\
& + G_2(\psi_+^{A\dagger}\psi_+^A\psi_+^{B\dagger}\psi_+^B + \psi_+^{A\dagger}\psi_+^A\psi_-^{B\dagger}\psi_-^B + \psi_-^{A\dagger}\psi_-^A\psi_+^{B\dagger}\psi_+^B + \psi_-^{A\dagger}\psi_-^A\psi_-^{B\dagger}\psi_-^B \\
& \quad - 2\psi_+^{A\dagger}\psi_+^B\psi_-^{A\dagger}\psi_-^B - 2\psi_+^{B\dagger}\psi_+^A\psi_-^{B\dagger}\psi_-^A) \\
& + G_3(\psi_+^{A\dagger}\psi_+^A\psi_-^{B\dagger}\psi_-^B + \psi_-^{A\dagger}\psi_-^A\psi_+^{B\dagger}\psi_+^B - \psi_+^{A\dagger}\psi_+^A\psi_+^{B\dagger}\psi_+^B - \psi_-^{A\dagger}\psi_-^A\psi_-^{B\dagger}\psi_-^B \\
& \quad - 2\psi_+^{A\dagger}\psi_-^A\psi_-^{B\dagger}\psi_+^B - 2\psi_-^{A\dagger}\psi_+^A\psi_+^{B\dagger}\psi_-^B) \\
& + G_4(\psi_+^{A\dagger}\psi_+^A\psi_+^{B\dagger}\psi_+^B - \psi_+^{A\dagger}\psi_+^A\psi_-^{B\dagger}\psi_-^B - \psi_-^{A\dagger}\psi_-^A\psi_+^{B\dagger}\psi_+^B + \psi_-^{A\dagger}\psi_-^A\psi_-^{B\dagger}\psi_-^B) \\
& + G_5(\psi_+^{A\dagger}\psi_+^A\psi_-^{A\dagger}\psi_-^B + \psi_+^{A\dagger}\psi_+^A\psi_-^{B\dagger}\psi_-^A - \psi_+^{B\dagger}\psi_+^B\psi_-^{A\dagger}\psi_-^B - \psi_+^{B\dagger}\psi_+^B\psi_-^{B\dagger}\psi_-^A \\
& \quad - \psi_+^{A\dagger}\psi_+^B\psi_-^{A\dagger}\psi_-^A - \psi_+^{B\dagger}\psi_+^A\psi_-^{A\dagger}\psi_-^A + \psi_+^{A\dagger}\psi_+^B\psi_-^{B\dagger}\psi_-^B + \psi_+^{B\dagger}\psi_+^A\psi_-^{B\dagger}\psi_-^B). \qquad (8.42)
\end{aligned}$$

Again, there are three additional expressions that are invariant under the $U(1)_Q$ symmetry but not under the full $SU(2)_Q$ symmetry, i.e.

$$\begin{aligned}
\widetilde{\mathcal{L}}^{004} = \quad & \frac{g_1}{4}\mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^A\Psi^{B\dagger}\Psi^B\sigma_3 - \Psi^{B\dagger}\sigma_3\Psi^B\Psi^{A\dagger}\Psi^A\sigma_3] + \frac{g_2}{2}\mathrm{Tr}[\Psi^{A\dagger}\Psi^A\sigma_3\Psi^{B\dagger}\Psi^B\sigma_3] \\
& + g_3\mathrm{Tr}[\Psi^{A\dagger}\Psi^A\sigma_3\Psi^{A\dagger}\Psi^B + \Psi^{B\dagger}\Psi^B\sigma_3\Psi^{B\dagger}\Psi^A] \\
= \quad & g_1(\psi_+^{A\dagger}\psi_+^A\psi_-^{B\dagger}\psi_-^B - \psi_-^{A\dagger}\psi_-^A\psi_+^{B\dagger}\psi_+^B) \\
& + g_2(\psi_+^{A\dagger}\psi_+^A\psi_+^{B\dagger}\psi_+^B + \psi_+^{A\dagger}\psi_+^A\psi_-^{B\dagger}\psi_-^B + \psi_-^{A\dagger}\psi_-^A\psi_+^{B\dagger}\psi_+^B + \psi_-^{A\dagger}\psi_-^A\psi_-^{B\dagger}\psi_-^B \\
& \quad + 2\psi_+^{A\dagger}\psi_+^B\psi_-^{A\dagger}\psi_-^B + 2\psi_+^{B\dagger}\psi_+^A\psi_-^{B\dagger}\psi_-^A) \\
& - g_3(\psi_+^{A\dagger}\psi_+^B\psi_-^{A\dagger}\psi_-^A + \psi_+^{A\dagger}\psi_+^B\psi_-^{B\dagger}\psi_-^B + \psi_-^{A\dagger}\psi_-^B\psi_+^{A\dagger}\psi_+^A + \psi_-^{A\dagger}\psi_-^B\psi_+^{B\dagger}\psi_+^B \\
& \quad + \psi_+^{B\dagger}\psi_+^A\psi_-^{A\dagger}\psi_-^A + \psi_+^{B\dagger}\psi_+^A\psi_-^{B\dagger}\psi_-^B + \psi_-^{B\dagger}\psi_-^A\psi_+^{A\dagger}\psi_+^A + \psi_-^{B\dagger}\psi_-^A\psi_+^{B\dagger}\psi_+^B). \qquad (8.43)
\end{aligned}$$

Next we consider the expressions containing six fermion fields and no derivatives arising

from class $(0,0,6)$. The $SU(2)_Q$ invariant 6-fermion expressions can be written as

$$\begin{aligned}\mathcal{L}^{006} = &\quad \frac{H_1}{4}\mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^A\Psi^{A\dagger}\sigma_3\Psi^A\Psi^{B\dagger}\sigma_3\Psi^B - \Psi^{B\dagger}\sigma_3\Psi^B\Psi^{B\dagger}\sigma_3\Psi^B\Psi^{A\dagger}\sigma_3\Psi^A] \\ &+ \frac{H_2}{3}\mathrm{Tr}[\Psi^{A\dagger}\Psi^B\Psi^{A\dagger}\Psi^B\Psi^{A\dagger}\Psi^B] \\ = &\quad H_1(\psi_+^{A\dagger}\psi_+^A\psi_+^{B\dagger}\psi_+^B\psi_-^{B\dagger}\psi_-^B + \psi_+^{A\dagger}\psi_+^A\psi_-^{A\dagger}\psi_-^A\psi_-^{B\dagger}\psi_-^B \\ &\quad - \psi_-^{A\dagger}\psi_-^A\psi_+^{B\dagger}\psi_+^B\psi_-^{B\dagger}\psi_-^B - \psi_+^{A\dagger}\psi_+^A\psi_-^{A\dagger}\psi_-^A\psi_+^{B\dagger}\psi_+^B) \\ &+ H_2(\psi_+^{A\dagger}\psi_+^A\psi_+^{B\dagger}\psi_+^B\psi_-^{B\dagger}\psi_-^A + \psi_-^{A\dagger}\psi_-^A\psi_-^{B\dagger}\psi_-^B\psi_+^{B\dagger}\psi_+^A \\ &\quad + \psi_-^{A\dagger}\psi_-^A\psi_-^{B\dagger}\psi_-^B\psi_+^{A\dagger}\psi_+^B + \psi_+^{A\dagger}\psi_+^A\psi_+^{B\dagger}\psi_+^B\psi_-^{A\dagger}\psi_-^B). \end{aligned} \tag{8.44}$$

In addition, there is one $SU(2)_Q$ breaking (but $U(1)_Q$ invariant) 6-fermion expression

$$\begin{aligned}\widetilde{\mathcal{L}}^{006} &= \frac{h}{4}\mathrm{Tr}[\Psi^{A\dagger}\Psi^A\sigma_3\Psi^{A\dagger}\Psi^A\sigma_3\Psi^{B\dagger}\Psi^B\sigma_3 + \Psi^{B\dagger}\Psi^B\sigma_3\Psi^{B\dagger}\Psi^B\sigma_3\Psi^{A\dagger}\Psi^A\sigma_3] \\ &= -h(\psi_+^{A\dagger}\psi_+^A\psi_-^{A\dagger}\psi_-^A\psi_+^{B\dagger}\psi_+^B + \psi_+^{A\dagger}\psi_+^A\psi_-^{A\dagger}\psi_-^A\psi_-^{B\dagger}\psi_-^B \\ &\quad + \psi_+^{A\dagger}\psi_+^A\psi_+^{B\dagger}\psi_+^B\psi_-^{B\dagger}\psi_-^B + \psi_-^{A\dagger}\psi_-^A\psi_+^{B\dagger}\psi_+^B\psi_-^{B\dagger}\psi_-^B). \end{aligned} \tag{8.45}$$

Finally, the only $SU(2)_Q$ invariant 8-fermion expression with no derivatives that arises from class $(0,0,8)$ takes the form

$$\begin{aligned}\mathcal{L}^{008} &= \frac{I}{24}\ \mathrm{Tr}[\Psi^{A\dagger}\Psi^A\Psi^{B\dagger}\Psi^B\Psi^{A\dagger}\Psi^A\Psi^{B\dagger}\Psi^B] \\ &= -I\ \psi_+^{A\dagger}\psi_+^A\psi_+^{B\dagger}\psi_+^B\psi_-^{A\dagger}\psi_-^A\psi_-^{B\dagger}\psi_-^B. \end{aligned} \tag{8.46}$$

No $SU(2)_Q$ breaking 8-fermion expressions without derivatives exist.

The 4-, 6-, and 8-fermion contact terms parameterize short-distance interactions with a large number of undetermined low-energy constants. Since this limits the predictive power of the effective theory at short-distances, it is natural to concentrate on long-distance forces between the charge carriers. For example, one-magnon exchange mediates a long-range force that is unambiguously predicted by the effective theory in terms of just a few low-energy parameters.

According to Appendix C there are many equivalent ways of rewriting the various contributions to the action in terms of traces. Hence, the above choices of expressions are to some extent arbitrary. For example, all determinants or products of traces of fermion fields can be written as linear combinations of the traces listed above. However, it is important that the selected expressions form a maximal linearly independent set.

It is straightforward to include the fermion chemical potential μ in the effective theory. It appears as the temporal component of a purely imaginary $U(1)_Q$ gauge field and thus manifests itself in an additional contribution to the covariant derivative

$$D_t\Psi^{A,B}(x) = \partial_t\Psi^{A,B}(x) + iv_t^3(x)\sigma_3\Psi^{A,B}(x) - \mu\Psi^{A,B}(x)\sigma_3. \tag{8.47}$$

Before one can do consistent loop-calculations in the low-energy effective theory at non-zero Q or at non-zero μ, one must develop a power-counting scheme, e.g. along the lines of [74].

8.7 Dispersion relations of electrons and holes

In this section, as a first application of the effective theory, we consider the dispersion relations of the charge carriers. For this purpose we switch off the magnon field (i.e. $P(x) = \frac{1}{2}(\mathbb{1} + \sigma_3) \;\Rightarrow\; u(x) = \mathbb{1}$, $v_\mu(x) = 0$) and consider the propagation of free charge carriers in the antiferromagnetic medium. In the absence of $SU(2)_Q$ breaking terms, the part of the Lagrangian which is quadratic in the fermion fields (and hence responsible for the propagation of the charge carriers) then reduces to

$$\begin{aligned}\mathcal{L} = {} & \psi_+^{A\dagger}\partial_t\psi_+^A + \psi_-^{A\dagger}\partial_t\psi_-^A + \psi_+^{B\dagger}\partial_t\psi_+^B + \psi_-^{B\dagger}\partial_t\psi_-^B \\ & + (\psi_+^{A\dagger}, \psi_+^{B\dagger}) \begin{pmatrix} M_2 & M_1 \\ M_1 & -M_2 \end{pmatrix} \begin{pmatrix} \psi_+^A \\ \psi_+^B \end{pmatrix} + (\psi_-^{A\dagger}, \psi_-^{B\dagger}) \begin{pmatrix} -M_2 & M_1 \\ M_1 & M_2 \end{pmatrix} \begin{pmatrix} \psi_-^A \\ \psi_-^B \end{pmatrix} \\ & + (\partial_i\psi_+^{A\dagger}, \partial_i\psi_+^{B\dagger}) \begin{pmatrix} \frac{1}{2M_2'} & \frac{1}{2M_1'} \\ \frac{1}{2M_1'} & -\frac{1}{2M_2'} \end{pmatrix} \begin{pmatrix} \partial_i\psi_+^A \\ \partial_i\psi_+^B \end{pmatrix} \\ & + (\partial_i\psi_-^{A\dagger}, \partial_i\psi_-^{B\dagger}) \begin{pmatrix} -\frac{1}{2M_2'} & \frac{1}{2M_1'} \\ \frac{1}{2M_1'} & \frac{1}{2M_2'} \end{pmatrix} \begin{pmatrix} \partial_i\psi_-^A \\ \partial_i\psi_-^B \end{pmatrix}. \end{aligned} \tag{8.48}$$

The eigenstates of free particles propagating with a 2-d momentum vector $\vec{p}$ arise as the eigenvectors of the matrices

$$\begin{aligned} H_+(p^2) &= \begin{pmatrix} M_2 + \frac{p^2}{2M_2'} & M_1 + \frac{p^2}{2M_1'} \\ M_1 + \frac{p^2}{2M_1'} & -M_2 - \frac{p^2}{2M_2'} \end{pmatrix}, \\ H_-(p^2) &= \begin{pmatrix} -M_2 - \frac{p^2}{2M_2'} & M_1 + \frac{p^2}{2M_1'} \\ M_1 + \frac{p^2}{2M_1'} & M_2 + \frac{p^2}{2M_2'} \end{pmatrix}. \end{aligned} \tag{8.49}$$

Due to the lack of Galilean invariance the eigenvectors depend on p^2, i.e. the probability for an electron or hole to be found on the A or B sublattice depends on the momentum. As a consequence of the displacement symmetries D and D' the eigenvalues of $H_+(p^2)$ and $H_-(p^2)$ are the same. Both matrices have two eigenvalues

$$\begin{aligned} E_{1,2}(p^2) &= \pm\sqrt{\left(M_1 + \frac{p^2}{2M_1'}\right)^2 + \left(M_2 + \frac{p^2}{2M_2'}\right)^2} \\ &= \pm\left(M + \frac{p^2}{2M'} + \mathcal{O}(p^4)\right). \end{aligned} \tag{8.50}$$

The positive energy states correspond to electrons, while the negative energy states correspond to holes. Not surprisingly, due to the $SU(2)_Q$ symmetry electrons and holes have the same dispersion relation. The rest mass M and the kinetic mass M' are given by

$$M = \sqrt{M_1^2 + M_2^2}, \qquad \frac{M}{M'} = \frac{M_1}{M_1'} + \frac{M_2}{M_2'}. \tag{8.51}$$

Next we take into account the additional terms that reduce the $SU(2)_Q$ symmetry to the $U(1)_Q$ symmetry. Then there are additional contributions to the energy

$$\widetilde{H}_+(p^2) = \widetilde{H}_-(p^2) = \begin{pmatrix} m + \frac{p^2}{2m'} & 0 \\ 0 & m + \frac{p^2}{2m'} \end{pmatrix}, \tag{8.52}$$

and the corresponding eigenvalues now take the form

$$E_{1,2}(p^2) = m + \frac{p^2}{2m'} \pm \left(M + \frac{p^2}{2M'} \right) + \mathcal{O}(p^4). \tag{8.53}$$

Still, the energies in the $+$ and $-$ sectors are the same. However, the electron and hole dispersion relations now differ.

At this point, we have constructed eigenstates of the free Hamiltonian with definite continuum momentum and with definite spin projection on the direction of the staggered magnetization. However, unlike the eigenstates of the underlying microscopic Hamiltonian, the states of the effective theory do not have a definite lattice momentum. Still, the low-energy effective theory defined in the continuum knows about the underlying lattice structure through the realization of the displacement symmetries D and D'. Since the symmetry D is spontaneously broken, neither the vacuum nor the single particle states are eigenstates of D. Operating twice with D acts trivially on the fields, i.e. ${}^{DD}P(x) = P(x)$, ${}^{DD}\Psi_\pm^{A,B} = \Psi_\pm^{A,B}$, and hence does not reveal any useful information. It is more useful to operate with the unbroken displacement symmetry D'. In particular, the vacuum state $P(x) = \frac{1}{2}(\mathbb{1} + \sigma_3)$ is invariant under D'. Still, in the way we constructed them, the electron or hole states of the effective theory are not eigenstates of D'. However, since states with spin parallel and antiparallel to the staggered magnetization are degenerate with each other, one can form appropriate linear combinations that are eigenstates of the displacement symmetry D'. Applying D' twice one obtains

$${}^{D'D'}\psi_\pm^{A,B}(x) = \pm\, {}^{D'}\psi_\mp^{B,A}(x) = -\psi_\pm^{A,B}(x), \tag{8.54}$$

which implies that the corresponding eigenvalue $\lambda = \exp(ika)$ of D' obeys

$$\lambda^2 = \exp(2ika) = -1 \quad \Rightarrow \quad ka = \pm\frac{\pi}{2}. \tag{8.55}$$

This is reminiscent of the result, mentioned in the introduction, that low-energy hole states are located at lattice momenta $(\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$ [6, 7, 17, 34, 37–40]. However, the comparison

with these findings is subtle. In particular, the results of the exact diagonalization study on small [17] and of the Monte Carlo study on larger volumes [34] must be interpreted carefully. In a finite volume (with periodic boundary conditions), in analogy to QCD [92], both the $SU(2)_s$ spin symmetry and the displacement symmetry D are restored and the staggered magnetization acts as a quantum rotor [49]. As a result, in contrast to the infinite volume limit, the single particle states in a finite volume can be constructed as eigenstates of D. It is interesting to note that the finite volume effects that lead to the restoration of the spontaneously broken symmetries $SU(2)_s$ and D can be understood in the framework of the effective theory. This requires a non-perturbative quantum mechanical treatment along the lines of [49, 92].

9 Systems with holes only

In this chapter we consider the t-J model as well as its low-energy effective theory. In the t-J model holes are the only charge carriers which leads to substantial simplifications in the effective theory.

9.1 The t-J model

The t-J model is defined by the Hamilton operator

$$H = P\Big\{ -t\sum_{x,i}(c_x^\dagger c_{x+\hat{i}} + c_{x+\hat{i}}^\dagger c_x) + J\sum_{x,i}\vec{S}_x\cdot\vec{S}_{x+\hat{i}} - \mu\sum_x(n_x-1)\Big\}P, \tag{9.1}$$

with

$$c_x = \begin{pmatrix} c_{x\uparrow} \\ c_{x\downarrow} \end{pmatrix}, \qquad S_x = c_x^\dagger\frac{\vec{\sigma}}{2}c_x, \qquad n_x = c_x^\dagger c_x. \tag{9.2}$$

In contrast to the Hubbard model, in the t-J model the operators act in a restricted Hilbert space of empty or at most singly occupied sites. In particular, states with doubly occupied sites are exiled from the physical Hilbert space by the projection operator P. Hence, by definition, the t-J model does not allow the addition of electrons to a half-filled state. Consequently, the only charge carriers are holes.

It is straightforward to show that the t-J model has the same symmetries as the Hubbard model. The only exception is the $SU(2)_Q$ symmetry which in the Hubbard model relates electrons to holes. Consequently, due to the lack of electrons, the $SU(2)_Q$ symmetry is absent in the t-J model. Still, the Abelian fermion number symmetry $U(1)_Q$ remains exact in the t-J model.

9.2 Effective theory for magnons and holes only

Since, up to the $SU(2)_Q$ symmetry, the t-J model has the same symmetries as the Hubbard model, the effective theory of the previous chapter also applies in this case. Of course, the values of the low-energy parameters will be different than for the Hubbard model. Still, the absence of electrons beyond half-filling leads to drastic simplifications. In particular, in the effective theory the absence of electrons manifests itself by an infinite electron rest mass. Consequently, with a finite amount of energy these excitations cannot be generated. As discussed in Section 8.7, the diagonalization of the mass matrices of electrons and holes yields

$$U_\pm \begin{pmatrix} m \pm M_2 & M_1 \\ M_1 & m \mp M_2 \end{pmatrix} U_\pm^\dagger = \begin{pmatrix} m \pm \sqrt{M_1^2 + M_2^2} & 0 \\ 0 & m \mp \sqrt{M_1^2 + M_2^2} \end{pmatrix}, \tag{9.3}$$

where the unitary matrices $U_\pm$ can be written in the form

$$U_\pm = \begin{pmatrix} X & \pm Y \\ \mp Y & X \end{pmatrix}, \qquad X, Y \in \mathbb{R}. \tag{9.4}$$

The eigenvectors corresponding to the eigenvalue $m + \sqrt{M_1^2 + M_2^2}$ describe electrons, while the ones corresponding to $m - \sqrt{M_1^2 + M_2^2}$ describe holes. When the electron rest mass $m + \sqrt{M_1^2 + M_2^2}$ goes to infinity, the corresponding eigenvector fields

$$X\psi_+^A(x) + Y\psi_+^B(x) = 0, \qquad Y\psi_-^A(x) + X\psi_-^B(x) = 0, \tag{9.5}$$

which describe the electrons, must be put to zero. The orthogonal combinations

$$\psi_+(x) = -Y\psi_+^A(x) + X\psi_+^B(x), \qquad \psi_-(x) = X\psi_-^A(x) - Y\psi_-^B(x), \tag{9.6}$$

describe the holes $\psi_\pm(x)$ and must be kept. As a result, the number of degrees of freedom is reduced by a factor of two. In complete analogy to the discussion in Appendix D one can show that the hole fields $\psi_\pm(x)$ transform as follows under the various symmetry operations

$$\begin{aligned}
SU(2)_s: \quad & \psi_\pm(x)' = \exp(\pm i\alpha(x))\psi_\pm(x), \\
U(1)_Q: \quad & {}^Q\psi_\pm(x) = \exp(i\omega)\psi_\pm(x), \\
D: \quad & {}^D\psi_\pm(x) = \mp\exp(\mp i\varphi(x))\psi_\mp(x), \\
D': \quad & {}^{D'}\psi_\pm(x) = \pm\psi_\mp(x), \\
O: \quad & {}^O\psi_\pm(x) = \psi_\pm(Ox), \\
R: \quad & {}^R\psi_\pm(x) = \psi_\pm(Rx), \\
T: \quad & {}^T\psi_\pm(x) = \exp(\mp i\varphi(Tx))\psi_\pm^\dagger(Tx), \\
& {}^T\psi_\pm^\dagger(x) = -\exp(\pm i\varphi(Tx))\psi_\pm(Tx), \\
T': \quad & {}^{T'}\psi_\pm(x) = -\psi_\pm^\dagger(Tx), \\
& {}^{T'}\psi_\pm^\dagger(x) = \psi_\pm(Tx).
\end{aligned} \tag{9.7}$$

Hence, except for the $SU(2)_Q$ symmetry, all symmetries can also be implemented on the hole fields alone. Note that the transformation laws for $\psi_\pm(x)$ result from those for $\psi_\pm^{A,B}(x)$ simply by dropping the sublattice indices A and B.

The absence of electron fields also drastically reduces the number of terms one can write down in the low-energy effective theory. In particular, the leading terms in the effective action now take the form

$$\begin{aligned} S[\psi_\pm^\dagger, \psi_\pm, P] = \int d^2x\, dt\, \Big\{ & \rho_s \mathrm{Tr}[\partial_i P \partial_i P + \frac{1}{c^2}\partial_t P \partial_t P] + M(\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_-) \\ & + \psi_+^\dagger D_t \psi_+ + \psi_-^\dagger D_t \psi_- + \frac{1}{2M'}(D_i \psi_+^\dagger D_i \psi_+ + D_i \psi_-^\dagger D_i \psi_-) \\ & + \Lambda(\psi_+^\dagger v_t^+ \psi_- + \psi_-^\dagger v_t^- \psi_+) \\ & + iK(D_i \psi_+^\dagger v_i^+ \psi_- - \psi_-^\dagger v_i^- D_i \psi_+ + D_i \psi_-^\dagger v_i^- \psi_+ - \psi_+^\dagger v_i^+ D_i \psi_-) \\ & + N(\psi_+^\dagger v_i^+ v_i^- \psi_+ + \psi_-^\dagger v_i^- v_i^+ \psi_-) + G \psi_+^\dagger \psi_+ \psi_-^\dagger \psi_- \Big\}. \end{aligned} \tag{9.8}$$

This form of the effective action is similar to (but not identical with) the ones of [11, 13, 25, 32, 36]. In particular, in some of those works spin-charge separation was invoked and spinless fermions were considered. Also the role of the sublattice indices (which have at this stage disappeared from our description) is different in those approaches. Furthermore, the dynamical role attributed to the composite gauge field in some of those works is different than in our effective theory. It should be pointed out that the above effective Lagrangian correctly describes the low-energy dynamics of holes only if electrons are completely absent beyond half-filling (as it is indeed the case in the t-J model). Otherwise the general effective theory of the previous chapter with a larger number of low-energy constants (and thus with a somewhat reduced predictive power) must be employed.

10 Conclusions

In this thesis we have constructed a systematic low-energy effective field theory for magnons and charge carriers doped into an antiferromagnet. The effective theory describes the interactions of electrons and holes through magnon exchange. Furthermore, we have also constructed a simplified version of the effective theory which describes only magnons and holes but no electrons. The construction of the effective theory is based on a detailed symmetry analysis of the corresponding underlying microscopic system—the Hubbard and t-J model, respectively. In general, the effective theory makes universal predictions for a whole class of systems, since its construction is insensitive to the microscopic details of the specific underlying microscopic system they are based on.

Possible basic applications of the effective theory constructed in this thesis include magnon-hole and magnon-electron scattering and the determination of the resulting long-range effective potential between the charge carriers. These would be highly non-trivial

and non-perturbative issues in the microscopic models. On the other hand, in the effective theory they can be understood quantitatively by perturbative analytic calculations. This is due to the fact that the effective charge carrier fields are only weakly coupled in contrast to the strongly correlated microscopic degrees of freedom. More ambitious, non-perturbative applications could comprise the reduction of the staggered magnetization upon doping, and, in view of high-temperature superconductivity, a systematic investigation of dynamical mechanisms for the preformation of electron or hole pairs in the antiferromagnetic phase.

Before one can do loop calculations in the effective theory, a consistent power-counting scheme must be established. Since the charge carriers are massive they set a new scale in the theory which highly complicates this issue. As described in Section 1.2, for (baryon) chiral perturbation theory of QCD consistent power-counting schemes are already established in the baryon number sectors $B = 0$ and $B = 1$. One can expect that this scheme can be adapted to the effective theory for magnons and charge carriers developed in this thesis. However, a systematic power-counting in the sectors with $B \geq 2$ is still controversial. It is therefore expected that the power-counting in the sectors with two or more charge carriers is non-trivial as well in the effective theory for magnons and charge carriers.

Let us review the low-energy parameters we had to introduce during the construction of the effective Lagrangian. To this end we have summarized in Table 10.1 all low-energy parameters according to the class they originate from. For the Hubbard model based effective theory, which describes magnons, holes and electrons, we distinguish in addition between $SU(2)_Q$ invariant and $SU(2)_Q$ breaking (but still $U(1)_Q$ invariant) expressions. The corresponding coupling constants are listed in the second and third column of Table 10.1, respectively. On the other hand, there is no $SU(2)_Q$ symmetry in the t-J model based effective theory, which describes magnons and holes only. Hence, the corresponding coupling constants are solely listed in the fourth column of Table 10.1.

In the purely magnonic sector of the classes $(2, 0, 0)$ and $(0, 2, 0)$ there are only 2 low-energy parameters. This small number of low-energy parameters leads to a very predictive effective theory, however, only for the limited question of the purely magnonic behaviour. In contrast, if one considers the fermions as well, which is the primary goal of this thesis, there is a whole zoo of additional low-energy parameters.

Let us first examine the effective theory for holes and electrons. Here, there are a total of 18 additional low-energy parameters that originate from the $SU(2)_Q$ invariant expressions only. If one includes as well the expressions that break the $SU(2)_Q$ symmetry, this number even increases to 28. This large number of a priori undetermined low-energy parameters is the price one has to pay for the universality and model-independence of the effective theory. Only in this way the low-energy physics of a whole class of actual materials can be captured by the effective theory. Since the predictive power of an effective theory diminishes with an increasing number of low-energy parameters, these large numbers of parameters do not look very encouraging at first. However, a closer look reveals that about half of these low-energy parameters arise from the fermion contact interactions of the classes $(0, 0, n_\psi)$ with $n_\psi = 4, 6, 8$, which describe the short-range physics of the fermions only.

Class (n_t, n_i, n_ψ)	$SU(2)_Q$ invariant	$SU(2)_Q$ breaking	Holes only	Description of the low-energy parameter
020	ρ_s	—	ρ_s	Spin stiffness
200	ρ_s/c^2	—	ρ_s/c^2	Spin wave velocity
002	M_1, M_2	m	M	Fermion rest mass
1_D02	1	—	1	Set to 1 by convention
1_V02	Λ_1, Λ_2	λ	Λ	Fermion-one-magnon vertex
$02_{DD}2$	$(2M_1')^{-1}$, $(2M_2')^{-1}$	$(2m')^{-1}$	$(2M')^{-1}$	Fermion kinetic mass
$02_{DV}2$	K_1, K_2	κ_1, κ_2	K	Fermion-one-magnon vertex
$02_{VV}2$	N_1, N_2	ν	N	Fermion-two-magnon vertex
004	G_1, G_2, G_3, G_4, G_5	g_1, g_2, g_3	G	4-fermion contact interaction
006	H_1, H_2	h	—	6-fermion contact interaction
008	I	—	—	8-fermion contact interaction

Table 10.1: *Low-energy parameters and couplings of the constructed effective Lagrangians. The coupling constants of the Hubbard model based effective theory, which describes magnons, holes, and electrons are listed in the second and third column. On the other hand, the coupling constants of the t-J model based effective theory, which describes magnons and holes only are listed in the fourth column.*

Hence, if one is interested in the long-range interactions between the fermions only, these parameters can be neglected. For this limited question of the long-range forces between the charge carriers, the number of relevant additional parameters reduces to 10 or 16 for the $SU(2)_Q$ invariant or breaking configuration, respectively. Still these numbers are large and it seems unlikely to be able to make general meaningful predictions. However, at least in some relevant physical quantities only a small number of low-energy parameters enter. One important example is the one-magnon exchange potential between the charge carriers, which is expected to be relevant for potential mechanisms for preforming charge carrier pairs in the antiferromagnetic phase. As its name implies, the potential depends solely on the fermion-one-magnon vertex couplings, which correspond to only 4 or 7 parameters for the $SU(2)_Q$ invariant or breaking configuration, respectively.

On the other hand, the effective theory for holes but no electrons has a significantly smaller number of low-energy parameters. The full set involves only 6 parameters for the fermionic sector in addition to the 2 parameters of the purely magnonic sector. Also the number of contact interactions is drastically reduced, i.e. there is only 1 parameter for a 4-fermion contact interaction, whereas the 6- and 8-fermion contact interactions are completely inexistent due to the elimination of the electron field. Furthermore, the one-magnon exchange potential depends here only on 2 parameters.

The reason why the effective theory for holes and electrons has by far more than twice as many parameters than the one for holes only is quite obvious. In addition to the corresponding vertices for electrons there are also vertices that mix electrons and holes. In

the actual doped materials, however, only one charge carrier type is dominant, while the other is drastically suppressed. Hence, all the vertices which involve the suppressed charge carrier type are suppressed as well. For such applications it seems therefore to be more economical to engage the effective theory that supports solely the dominant charge carrier type (e.g. holes). Another question which arises in this context is how the parameters of the two effective theories map to each other. Since the effective theory for holes only is a subset of the one for holes and electrons, one can express the parameters of the effective theory for holes in terms of the parameters of the effective theory for holes and electrons.

It would be interesting to perform numerical simulations of the Hubbard or t-J model in order to determine the values of the corresponding low-energy parameters by comparison with calculations in the effective theory. For example, with simulations of the t-J model in the one-hole sector, one can determine all low-energy parameters of the corresponding effective theory, with the one exception of G, which requires computations in the two-hole sector. It may also be useful to formulate the effective theory for magnons and charge carriers on the lattice. For example, it would be interesting to investigate if the effective theory is more easily solvable by numerical simulation than the underlying microscopic Hubbard-type models.

It should be pointed out that the effective theory presented in this thesis runs into troubles for sufficiently large doping μ. First, an increasing amount of doping destroys the antiferromagnetic order until the antiferromagnetic phase finally breaks down. From this point on the magnons get more and more massive, until they eventually no longer exist among the lightest excitations and the effective theory breaks down. Second, a large μ sets a new scale in the theory that has to be taken into account in the power-counting. Hence, the effective theory presented here is well suited for questions at low doping, i.e. in the antiferromagnetic phase like e.g. the study of the long-range forces between electrons or holes in the antiferromagnetic phase. It should also allow a quantitative investigation of the reduction of the staggered magnetization upon doping. However, at sufficiently high doping, like the one needed to enter the high-temperature superconducting phase, it is questionable if the effective theory is still usable.

As already mentioned in Section 1.3, it is also possible to consider phonons in addition to magnons within the framework of the effective theory, in which case the Galilean (or even Poincaré) symmetry would be non-linearly realized. Such an effective theory would allow to investigate the interplay of magnons and phonons, e.g. in the context of more complicated potential mechanisms for Cooper pair preformation.

APPENDIX

A Electron and hole representation of the Hubbard model operators

For $U \gg |t|$ the Hubbard model at half-filling reduces to the antiferromagnetic quantum Heisenberg model. In contrast to the Heisenberg ferromagnet, the ground state of the antiferromagnet is not known analytically. In particular, the naive Néel state

$$|N\rangle = \prod_{x\in A} c^\dagger_{x\downarrow} \prod_{y\in B} c^\dagger_{y\uparrow}|0\rangle, \tag{A.1}$$

with all spins down on the even sublattice A and all spins up on the odd sublattice B is not an eigenstate of the Hubbard model Hamiltonian. Still, we use this state in order to define electron and hole operators. For even sites we then find

$$c_{x\uparrow}|N\rangle = 0, \qquad c^\dagger_{x\downarrow}|N\rangle = 0, \qquad x \in A. \tag{A.2}$$

Correspondingly, $c^\dagger_{x\uparrow}$ creates an electron, while $c_{x\downarrow}$ creates a hole. Hence, just like a relativistic Dirac spinor, the $SU(2)_s$ spinor

$$c_x = \begin{pmatrix} c_{x\uparrow} \\ c_{x\downarrow} \end{pmatrix} = \begin{pmatrix} a_{x\uparrow} \\ b^\dagger_{x\uparrow} \end{pmatrix}, \qquad x \in A, \tag{A.3}$$

consists of an electron (particle) annihilation operator $a_{x\uparrow}$ in the upper component and a hole (antiparticle) creation operator $b^\dagger_{x\uparrow}$ in the lower component. Note that the annihilation of an electron with spin down via $c_{x\downarrow}$ corresponds to the creation of a hole with spin up via $b^\dagger_{x\uparrow}$. Similarly, on the odd sites one has

$$c_{y\downarrow}|N\rangle = 0, \qquad c^\dagger_{y\uparrow}|N\rangle = 0, \qquad y \in B. \tag{A.4}$$

In this case, $c^\dagger_{y\downarrow}$ creates an electron, while $c_{y\uparrow}$ creates a hole and we write

$$c_y = \begin{pmatrix} c_{y\uparrow} \\ c_{y\downarrow} \end{pmatrix} = \begin{pmatrix} b^\dagger_{y\downarrow} \\ a_{y\downarrow} \end{pmatrix}, \qquad y \in B. \tag{A.5}$$

B Explicit construction of the leading expressions in component notation

In this appendix we want to construct at full length all linearly independent leading expressions in component notation of all the classes we consider. Some parts of this project were for didactical reasons already discussed in the main text and hence will not be repeated here again. For the same reason we will not consider the classes $(2,0,0)$ and $(0,2,0)$ of the purely magnonic sector here, since the corresponding expressions were already found in Section 6.1.

B.1 Classes $(0, 0, n_\psi)$ with $n_\psi = 4, 6, 8$

First we want to handle the remaining expressions with fermion fields but no derivatives. The construction of the $U(1)_Q$ and $SU(2)_s$ invariant terms for these classes is covered in the main text in Section 8.4. Here we just adopt the result from there and impose the displacement and time reversal symmetries in addition. For the class $(0, 0, 2)$ this is covered in Section 8.5 of the main text.

B.1.1 Class $(0, 0, 4)$

After imposing the $U(1)_Q$ and $SU(2)_s$ symmetries, there are according to Eq. (8.19) still eighteen remaining terms A_r^{004}, namely

$$
\begin{aligned}
A_1^{004} &= \psi_+^{A\dagger}\psi_-^{A\dagger}\psi_+^A\psi_-^A &&= -\psi_+^{A\dagger}\psi_+^A\psi_-^{A\dagger}\psi_-^A, && (4),\\
A_2^{004} &= \psi_+^{A\dagger}\psi_-^{A\dagger}\psi_+^A\psi_-^B &&= -\psi_+^{A\dagger}\psi_+^A\psi_-^{A\dagger}\psi_-^B, && (3),\\
A_3^{004} &= \psi_+^{A\dagger}\psi_-^{A\dagger}\psi_-^A\psi_+^B &&= +\psi_+^{A\dagger}\psi_+^B\psi_-^{A\dagger}\psi_-^A, && (3),\\
A_4^{004} &= \psi_+^{A\dagger}\psi_-^{A\dagger}\psi_+^B\psi_-^B &&= -\psi_+^{A\dagger}\psi_+^B\psi_-^{A\dagger}\psi_-^B, && (2),\\
A_5^{004} &= \psi_+^{A\dagger}\psi_+^{B\dagger}\psi_+^A\psi_+^B &&= -\psi_+^{A\dagger}\psi_+^A\psi_+^{B\dagger}\psi_+^B, && (2),\\
A_6^{004} &= \psi_+^{A\dagger}\psi_-^{B\dagger}\psi_+^A\psi_-^A &&= -\psi_+^{A\dagger}\psi_+^A\psi_-^{B\dagger}\psi_-^A, && (3),\\
A_7^{004} &= \psi_+^{A\dagger}\psi_-^{B\dagger}\psi_+^A\psi_-^B &&= -\psi_+^{A\dagger}\psi_+^A\psi_-^{B\dagger}\psi_-^B, && (2),\\
A_8^{004} &= \psi_+^{A\dagger}\psi_-^{B\dagger}\psi_-^A\psi_+^B &&= -\psi_+^{A\dagger}\psi_-^A\psi_-^{B\dagger}\psi_+^B, && (2),\\
A_9^{004} &= \psi_+^{A\dagger}\psi_-^{B\dagger}\psi_+^B\psi_-^B &&= -\psi_+^{A\dagger}\psi_+^B\psi_-^{B\dagger}\psi_-^B, && (3),\\
A_{10}^{004} &= \psi_-^{A\dagger}\psi_+^{B\dagger}\psi_+^A\psi_-^A &&= +\psi_+^{B\dagger}\psi_+^A\psi_-^{A\dagger}\psi_-^A, && (3),\\
A_{11}^{004} &= \psi_-^{A\dagger}\psi_+^{B\dagger}\psi_+^A\psi_-^B &&= -\psi_-^{A\dagger}\psi_+^A\psi_+^{B\dagger}\psi_-^B, && (2),\\
A_{12}^{004} &= \psi_-^{A\dagger}\psi_+^{B\dagger}\psi_-^A\psi_+^B &&= -\psi_-^{A\dagger}\psi_-^A\psi_+^{B\dagger}\psi_+^B, && (2),\\
A_{13}^{004} &= \psi_-^{A\dagger}\psi_+^{B\dagger}\psi_+^B\psi_-^B &&= +\psi_+^{B\dagger}\psi_+^B\psi_-^{A\dagger}\psi_-^B, && (3),\\
A_{14}^{004} &= \psi_-^{A\dagger}\psi_-^{B\dagger}\psi_-^A\psi_-^B &&= -\psi_-^{A\dagger}\psi_-^A\psi_-^{B\dagger}\psi_-^B, && (2),\\
A_{15}^{004} &= \psi_+^{B\dagger}\psi_-^{B\dagger}\psi_+^A\psi_-^A &&= -\psi_+^{B\dagger}\psi_+^A\psi_-^{B\dagger}\psi_-^A, && (2),\\
A_{16}^{004} &= \psi_+^{B\dagger}\psi_-^{B\dagger}\psi_+^A\psi_-^B &&= -\psi_+^{B\dagger}\psi_+^A\psi_-^{B\dagger}\psi_-^B, && (3),\\
A_{17}^{004} &= \psi_+^{B\dagger}\psi_-^{B\dagger}\psi_-^A\psi_+^B &&= +\psi_+^{B\dagger}\psi_+^B\psi_-^{B\dagger}\psi_-^A, && (3),\\
A_{18}^{004} &= \psi_+^{B\dagger}\psi_-^{B\dagger}\psi_+^B\psi_-^B &&= -\psi_+^{B\dagger}\psi_+^B\psi_-^{B\dagger}\psi_-^B, && (4).
\end{aligned}
\tag{B.1}
$$

For convenience we also show here a second, more popular permutation of the fermion fields, the one we will use to state the action. Again the number in parentheses after each term is n_{AB} as defined in Eq. (8.9). Let us first examine the properties of these terms under

Hermitean conjugation. Thereby we find the relations

$$\begin{aligned}
A_1^{004\dagger} &= A_1^{004}, \quad (4), & A_8^{004\dagger} &= A_{11}^{004}, \quad (2),\\
A_2^{004\dagger} &= A_6^{004}, \quad (3), & A_9^{004\dagger} &= A_{16}^{004}, \quad (3),\\
A_3^{004\dagger} &= A_{10}^{004}, \quad (3), & A_{12}^{004\dagger} &= A_{12}^{004}, \quad (2),\\
A_4^{004\dagger} &= A_{15}^{004}, \quad (2), & A_{13}^{004\dagger} &= A_{17}^{004}, \quad (3),\\
A_5^{004\dagger} &= A_5^{004}, \quad (2), & A_{14}^{004\dagger} &= A_{14}^{004}, \quad (2),\\
A_7^{004\dagger} &= A_7^{004}, \quad (2), & A_{18}^{004\dagger} &= A_{18}^{004}, \quad (4).
\end{aligned} \tag{B.2}$$

Next we impose the displacement symmetry D. In complete analogy to the class $(0,0,2)$ we now transform each A_r^{004} in turn to figure out which terms have to be combined together in order to build invariant expressions. Here we find that the combinations

$$\begin{aligned}
B_1^{004} &= A_1^{004} + A_{18}^{004}, \quad (4), & B_7^{004} &= A_7^{004}, & (2),\\
B_2^{004} &= A_2^{004} + A_{16}^{004}, \quad (3), & B_8^{004} &= A_8^{004}, & (2),\\
B_3^{004} &= A_3^{004} + A_{17}^{004}, \quad (3), & B_9^{004} &= A_{10}^{004} + A_{13}^{004}, & (3),\\
B_4^{004} &= A_4^{004} + A_{15}^{004}, \quad (2), & B_{10}^{004} &= A_{11}^{004}, & (2),\\
B_5^{004} &= A_5^{004} + A_{14}^{004}, \quad (2), & B_{11}^{004} &= A_{12}^{004}, & (2),\\
B_6^{004} &= A_6^{004} + A_9^{004}, \quad (3), & & &
\end{aligned} \tag{B.3}$$

are invariant under the displacement symmetry D. Note that e.g. A_7^{004} does not need to be combined, since it is already invariant under this symmetry.

After that we impose the time reversal symmetry T following the same pattern as before. We find that further combinations have to be made in order to render the expressions invariant, i.e.

$$\begin{aligned}
C_1^{004} &= B_1^{004} &&= A_1^{004} + A_{18}^{004}, && (4),\\
C_2^{004} &= B_2^{004} + B_6^{004} &&= A_2^{004} + A_6^{004} + A_9^{004} + A_{16}^{004}, && (3),\\
C_3^{004} &= B_3^{004} + B_9^{004} &&= A_3^{004} + A_{10}^{004} + A_{13}^{004} + A_{17}^{004}, && (3),\\
C_4^{004} &= B_4^{004} &&= A_4^{004} + A_{15}^{004}, && (2),\\
C_5^{004} &= B_5^{004} &&= A_5^{004} + A_{14}^{004}, && (2),\\
C_6^{004} &= B_7^{004} &&= A_7^{004}, && (2),\\
C_7^{004} &= B_8^{004} + B_{10}^{004} &&= A_8^{004} + A_{11}^{004}, && (2),\\
C_8^{004} &= B_{11}^{004} &&= A_{12}^{004}, && (2).
\end{aligned} \tag{B.4}$$

This result uncovers that the $U(1)_Q$, $SU(2)_s$, D, and T invariant vector space is 8-dimensional. Note that, as already seen in Section 8.5, no further terms had to be dropped in order to make invariant expressions.

As before the most difficult part of the whole story is to handle the $SU(2)_Q$ symmetry correctly. Here we find that only C_1^{004} and C_7^{004} are already $SU(2)_Q$ invariant. Therefore we have to examine the other 6 expressions carefully in order to detect further potential $SU(2)_Q$ invariant linear combinations among them. Since terms with different n_{AB} do not mix, we only have to linearly combine expressions with identical n_{AB}. Luckily, this splits the problem into simpler parts and leads us to

$$\begin{aligned} &\beta_2 C_2^{004} + \beta_3 C_3^{004}, && (3),\\ &\beta_4 C_4^{004} + \beta_5 C_5^{004} + \beta_6 C_6^{004} + \beta_8 C_8^{004}, && (2). \end{aligned} \tag{B.5}$$

Again we operate with an arbitrary $SU(2)_Q$ symmetry transformation Ω^T on each of these general linear combinations, which results in

$$\beta_3 C_2^{004} + \beta_2 C_3^{004} + (\beta_2 - \beta_3) \sum_{1\le k<l<m<n\le 8} f_1^{004}(\Omega^T)_{klmn}\psi_k\psi_l\psi_m\psi_n, \quad (3),$$

$$\begin{aligned} \beta_4 C_4^{004} + \beta_5 C_5^{004} + \beta_8 C_6^{004} + \beta_6 C_8^{004} &\\ + (\beta_4 + \beta_5 + \beta_6) \sum_{1\le k<l<m<n\le 8} & f_2^{004}(\Omega^T)_{klmn}\psi_k\psi_l\psi_m\psi_n\\ + (\beta_4 + \beta_5 + \beta_8) \sum_{1\le k<l<m<n\le 8} & f_3^{004}(\Omega^T)_{klmn}\psi_k\psi_l\psi_m\psi_n, \quad (2). \end{aligned} \tag{B.6}$$

As before the $f_a^{004}(\Omega^T)_{klmn}$ are polynomials in the matrix elements of Ω^T and as such absorb the $SU(2)_Q$ breaking part of the linear combinations. Again, we subtract these expressions from their corresponding original linear combinations (B.5) and we get

$$(\beta_2 - \beta_3)C_2^{004} - (\beta_2 - \beta_3)C_3^{004} - (\beta_2 - \beta_3) \sum_{1\le k<l<m<n\le 8} f_1^{004}(\Omega^T)_{klmn}\psi_k\psi_l\psi_m\psi_n, \quad (3),$$

$$\begin{aligned} (\beta_6 - \beta_8)C_6^{004} - (\beta_6 - \beta_8)C_8^{004} &\\ - (\beta_4 + \beta_5 + \beta_6) \sum_{1\le k<l<m<n\le 8} & f_2^{004}(\Omega^T)_{klmn}\psi_k\psi_l\psi_m\psi_n\\ - (\beta_4 + \beta_5 + \beta_8) \sum_{1\le k<l<m<n\le 8} & f_3^{004}(\Omega^T)_{klmn}\psi_k\psi_l\psi_m\psi_n, \quad (2). \end{aligned} \tag{B.7}$$

The first equation vanishes only if $\beta_3 = \beta_2$. We can therefore choose e.g. $\beta_2 = \beta_3 = 1$ to build an additional $SU(2)_Q$ invariant expression and e.g. $\beta_2 = 1, \beta_3 = -1$ for a linearly independent, $SU(2)_Q$ breaking one. The second equation vanishes for $\beta_8 = \beta_6 = -(\beta_4+\beta_5)$. Here two linearly independent choices are possible: We can choose either e.g. $\beta_4 = 1, \beta_5 = 0$ or e.g. $\beta_4 = 0, \beta_5 = 1$ to build two additional $SU(2)_Q$ invariant expressions. Consequently, the $SU(2)_Q$ invariant subspace is $2 + 1 + 2 = 5$-dimensional. In addition we need two linearly independent $SU(2)_Q$ breaking expressions, such that we have indeed $1 + 2 = 3$

$SU(2)_Q$ breaking expressions. Summarizing all this we can write

$$\begin{aligned}
D_1^{004} = C_1^{004} &= A_1^{004} + A_{18}^{004}, && (4),\\
D_2^{004} = C_2^{004} + C_3^{004} &= A_2^{004} + A_3^{004} + A_6^{004} + A_9^{004} \\
&\quad + A_{10}^{004} + A_{13}^{004} + A_{16}^{004} + A_{17}^{004}, && (3),\\
D_3^{004} = C_4^{004} - C_6^{004} - C_8^{004} &= A_4^{004} - A_7^{004} - A_{12}^{004} + A_{15}^{004}, && (2),\\
D_4^{004} = C_5^{004} - C_6^{004} - C_8^{004} &= A_5^{004} - A_7^{004} - A_{12}^{004} + A_{14}^{004}, && (2),\\
D_5^{004} = C_7^{004} &= A_8^{004} + A_{11}^{004}, && (2),\\
\widetilde{D}_1^{004} = C_2^{004} - C_3^{004} &= A_2^{004} - A_3^{004} + A_6^{004} + A_9^{004} \\
&\quad - A_{10}^{004} - A_{13}^{004} + A_{16}^{004} - A_{17}^{004}, && (3),\\
\widetilde{D}_2^{004} = C_4^{004} + C_5^{004} &= A_4^{004} + A_5^{004} + A_{14}^{004} + A_{15}^{004}, && (2),\\
\widetilde{D}_3^{004} = C_6^{004} - C_8^{004} &= A_7^{004} - A_{12}^{004}, && (2).
\end{aligned} \tag{B.8}$$

As before the $SU(2)_Q$ breaking expressions are designated with a tilde. Note that $D_3^{004} - D_4^{004} = C_4^{004} - C_5^{004}$. Therefore we choose $C_4^{004} + C_5^{004}$ as a linearly independent, $SU(2)_Q$ breaking contribution. Similarly $D_3^{004} + D_4^{004} - \widetilde{D}_2^{004} = -2(C_6^{004} + C_8^{004})$, which leads us to choose $C_6^{004} - C_8^{004}$ as the second linearly independent, $SU(2)_Q$ breaking expression. According to Eq. (B.2) all the above expressions are Hermitean.

In the end we want to formulate the above result in trace notation, i.e.

$$\begin{aligned}
E_1^{004} &= \frac{1}{12}\mathrm{Tr}[\Psi^{A\dagger}\Psi^A\Psi^{A\dagger}\Psi^A + \Psi^{B\dagger}\Psi^B\Psi^{B\dagger}\Psi^B] && = -D_1^{004}, && (4),\\
E_2^{004} &= \frac{1}{2}\mathrm{Tr}[\Psi^{A\dagger}\Psi^A\Psi^{B\dagger}\Psi^B] && = 2D_3^{004} - D_4^{004} \\
&= 2A_4^{004} - A_5^{004} - A_7^{004} - A_{12}^{004} - A_{14}^{004} + 2A_{15}^{004}, && && (2),\\
E_3^{004} &= \frac{1}{2}\mathrm{Tr}[\Psi^{A\dagger}\Psi^B\Psi^{B\dagger}\Psi^A] && = D_4^{004} + 2D_5^{004} \\
&= A_5^{004} - A_7^{004} + 2A_8^{004} + 2A_{11}^{004} - A_{12}^{004} + A_{14}^{004}, && && (2),\\
E_4^{004} &= \frac{1}{2}\mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^A\Psi^{B\dagger}\sigma_3\Psi^B] && = -D_4^{004}, && (2),\\
E_5^{004} &= \mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^A\Psi^{A\dagger}\Psi^B - \Psi^{B\dagger}\sigma_3\Psi^B\Psi^{B\dagger}\Psi^A] && = -D_2^{004}, && (3),\\
\widetilde{E}_1^{004} &= \frac{1}{4}\mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^A\Psi^{B\dagger}\Psi^B\sigma_3 - \Psi^{B\dagger}\sigma_3\Psi^B\Psi^{A\dagger}\Psi^A\sigma_3] && = -\widetilde{D}_3^{004}, && (2),\\
\widetilde{E}_2^{004} &= \frac{1}{2}\mathrm{Tr}[\Psi^{A\dagger}\Psi^A\sigma_3\Psi^{B\dagger}\Psi^B\sigma_3] && = D_4^{004} - 2\widetilde{D}_2^{004} \\
&= -2A_4^{004} - A_5^{004} - A_7^{004} - A_{12}^{004} - A_{14}^{004} - 2A_{15}^{004}, && && (2),\\
\widetilde{E}_3^{004} &= \mathrm{Tr}[\Psi^{A\dagger}\Psi^A\sigma_3\Psi^{A\dagger}\Psi^B + \Psi^{B\dagger}\Psi^B\sigma_3\Psi^{B\dagger}\Psi^A] && = \widetilde{D}_1^{004}, && (3).
\end{aligned} \tag{B.9}$$

Again we present here just the findings of Appendix C and their relations to the results in component notation of this subsection.

B.1.2 Class $(0,0,6)$

According to Eq. (8.19) there are eight remaining terms A_r^{006} after imposing the $U(1)_Q$ and $SU(2)_s$ symmetries, i.e.

$$\begin{aligned}
A_1^{006} &= \psi_+^{A\dagger}\psi_-^{A\dagger}\psi_+^{B\dagger}\psi_+^A\psi_-^A\psi_+^B = -\psi_+^{A\dagger}\psi_+^A\psi_-^{A\dagger}\psi_-^A\psi_+^{B\dagger}\psi_+^B, \quad (4),\\
A_2^{006} &= \psi_+^{A\dagger}\psi_-^{A\dagger}\psi_+^{B\dagger}\psi_+^A\psi_+^B\psi_-^B = +\psi_+^{A\dagger}\psi_+^A\psi_+^{B\dagger}\psi_+^B\psi_-^{A\dagger}\psi_-^B, \quad (3),\\
A_3^{006} &= \psi_+^{A\dagger}\psi_-^{A\dagger}\psi_-^{B\dagger}\psi_+^A\psi_-^A\psi_-^B = -\psi_+^{A\dagger}\psi_+^A\psi_-^{A\dagger}\psi_-^A\psi_-^{B\dagger}\psi_-^B, \quad (4),\\
A_4^{006} &= \psi_+^{A\dagger}\psi_-^{A\dagger}\psi_-^{B\dagger}\psi_-^A\psi_+^B\psi_-^B = +\psi_-^{A\dagger}\psi_-^A\psi_-^{B\dagger}\psi_-^B\psi_+^{A\dagger}\psi_+^B, \quad (3),\\
A_5^{006} &= \psi_+^{A\dagger}\psi_+^{B\dagger}\psi_-^{B\dagger}\psi_+^A\psi_-^A\psi_+^B = +\psi_+^{A\dagger}\psi_+^A\psi_+^{B\dagger}\psi_+^B\psi_-^{B\dagger}\psi_-^A, \quad (3),\\
A_6^{006} &= \psi_+^{A\dagger}\psi_+^{B\dagger}\psi_-^{B\dagger}\psi_+^A\psi_+^B\psi_-^B = -\psi_+^{A\dagger}\psi_+^A\psi_+^{B\dagger}\psi_+^B\psi_-^{B\dagger}\psi_-^B, \quad (4),\\
A_7^{006} &= \psi_-^{A\dagger}\psi_+^{B\dagger}\psi_-^{B\dagger}\psi_+^A\psi_-^A\psi_-^B = +\psi_-^{A\dagger}\psi_-^A\psi_-^{B\dagger}\psi_-^B\psi_+^{B\dagger}\psi_+^A, \quad (3),\\
A_8^{006} &= \psi_-^{A\dagger}\psi_+^{B\dagger}\psi_-^{B\dagger}\psi_-^A\psi_+^B\psi_-^B = -\psi_-^{A\dagger}\psi_-^A\psi_+^{B\dagger}\psi_+^B\psi_-^{B\dagger}\psi_-^B, \quad (4).
\end{aligned} \tag{B.10}$$

As before we present here at the same time a second, more popular permutation of the fermion fields, the one we will use later to state the action. Again let us first analyze the properties of these terms under Hermitean conjugation. Here we find

$$\begin{aligned}
A_1^{006\dagger} &= A_1^{006}, \quad (4), & A_4^{006\dagger} &= A_7^{006}, \quad (3),\\
A_2^{006\dagger} &= A_5^{006}, \quad (3), & A_6^{006\dagger} &= A_6^{006}, \quad (4),\\
A_3^{006\dagger} &= A_3^{006}, \quad (4), & A_8^{006\dagger} &= A_8^{006}, \quad (4).
\end{aligned} \tag{B.11}$$

As we have already seen before, now it is the time to impose the displacement symmetry D. After probing each A_r^{006} with the D transformation law, we realize that the combinations

$$\begin{aligned}
B_1^{006} &= A_1^{006} + A_8^{006}, \quad (4), & B_3^{006} &= A_3^{006} + A_6^{006}, \quad (4),\\
B_2^{006} &= A_2^{006} + A_7^{006}, \quad (3), & B_4^{006} &= A_4^{006} + A_5^{006}, \quad (3),
\end{aligned} \tag{B.12}$$

have to be made in order to comply with the displacement symmetry D. Subsequently we conduct the same step again, now with the time reversal symmetry T acting on the B_s^{006}. Here we find that the further combinations

$$\begin{aligned}
C_1^{006} &= B_1^{006} &&= A_1^{006} + A_8^{006}, && (4),\\
C_2^{006} &= B_2^{006} + B_4^{006} &&= A_2^{006} + A_4^{006} + A_5^{006} + A_7^{006}, && (3),\\
C_3^{006} &= B_3^{006} &&= A_3^{006} + A_6^{006}, && (4),
\end{aligned} \tag{B.13}$$

are required to render the expressions invariant. This result indicates that the $U(1)_Q$, $SU(2)_s$, D and T invariant vector space is 3-dimensional.

Now we have to handle again the $SU(2)_Q$ symmetry. We find that only C_2^{006} is already invariant under this extended symmetry. However, the other two C_q^{006} can be combined together to form another $SU(2)_Q$ invariant expression, such that the $SU(2)_Q$ invariant subspace is 2-dimensional. The final result is therefore

$$\begin{aligned} D_1^{006} &= C_1^{006} - C_3^{006} = A_1^{006} - A_3^{006} - A_6^{006} + A_8^{006}, \quad (4), \\ D_2^{006} &= C_2^{006} \qquad\qquad = A_2^{006} + A_4^{006} + A_5^{006} + A_7^{006}, \quad (3), \\ \widetilde{D}_1^{006} &= C_1^{006} + C_3^{006} = A_1^{006} + A_3^{006} + A_6^{006} + A_8^{006}, \quad (4). \end{aligned} \tag{B.14}$$

The made choice for the $SU(2)_Q$ breaking expression $\widetilde{D}_1^{006}$ is now quite obvious. According to Eq. (B.11) all these expressions are again Hermitean.

Finally, we want to state the above results in the trace notation, which according to Appendix C yields

$$\begin{aligned} E_1^{006} &= \frac{1}{4}\mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^A\Psi^{A\dagger}\sigma_3\Psi^A\Psi^{B\dagger}\sigma_3\Psi^B - \Psi^{B\dagger}\sigma_3\Psi^B\Psi^{B\dagger}\sigma_3\Psi^B\Psi^{A\dagger}\sigma_3\Psi^A] = D_1^{006}, \quad (4), \\ E_2^{006} &= \frac{1}{3}\mathrm{Tr}[\Psi^{A\dagger}\Psi^B\Psi^{A\dagger}\Psi^B\Psi^{A\dagger}\Psi^B] \qquad\qquad = D_2^{006}, \quad (3), \\ \widetilde{E}_1^{006} &= \frac{1}{4}\mathrm{Tr}[\Psi^{A\dagger}\Psi^A\sigma_3\Psi^{A\dagger}\Psi^A\sigma_3\Psi^{B\dagger}\Psi^B\sigma_3 + \Psi^{B\dagger}\Psi^B\sigma_3\Psi^{B\dagger}\Psi^B\sigma_3\Psi^{A\dagger}\Psi^A\sigma_3] = \widetilde{D}_1^{006}, \quad (4). \end{aligned} \tag{B.15}$$

B.1.3 Class $(0, 0, 8)$

In accordance with Eq. (8.19) after imposing the $U(1)_Q$ and $SU(2)_s$ symmetries there is only one remaining term, namely

$$A_1^{008} = \psi_+^{A\dagger}\psi_-^{A\dagger}\psi_+^{B\dagger}\psi_-^{B\dagger}\psi_+^A\psi_-^A\psi_+^B\psi_-^B = \psi_+^{A\dagger}\psi_+^A\psi_+^{B\dagger}\psi_+^B\psi_-^{A\dagger}\psi_-^A\psi_-^{B\dagger}\psi_-^B, \quad (4). \tag{B.16}$$

Again, we present a second more popular representation. It turns out that this single term is Hermitean and invariant under the displacement symmetry D, the time reversal symmetry T as well as under the $SU(2)_Q$ symmetry. Therefore we can write

$$D_1^{008} = C_1^{008} = B_1^{008} = A_1^{008} = A_1^{008\dagger}, \quad (4). \tag{B.17}$$

Consequently there are no $SU(2)_Q$ breaking contributions. In trace notation this appears as

$$E_1^{008} = \frac{1}{24}\mathrm{Tr}[\Psi^{A\dagger}\Psi^A\Psi^{B\dagger}\Psi^B\Psi^{A\dagger}\Psi^A\Psi^{B\dagger}\Psi^B] = -D_1^{008}, \quad (4). \tag{B.18}$$

B.2 Class $(1, 0, 2)$

Now we step on to the terms with one temporal derivative. To begin with, we realize that the 'derivative' can be implemented in two different ways. The first variety simply uses the temporal derivative ∂_t (or more precisely the covariant derivative $D_t = \partial_t \pm i v_t^3$) itself. The second variety, on the other hand, uses the magnon field $v_t^\pm$ instead. This actually makes sense since its definition $v_t = u\partial_t u^\dagger$ contains a derivative as well, which has to be counted on equal footing. In the following we handle the first case as class $(1_D, 0, 2)$ and the second case as class $(1_V, 0, 2)$.

B.2.1 Class $(1_D, 0, 2)$

First, we want to create a unique list of all possible terms, irrespective of the symmetries. Here, because of the derivative a product of two equal fermion fields does not vanish and hence this case has to be included as well. Therefore our list looks like

$$1_D02: \quad \psi_k \partial_t \psi_l, \quad 1 \leq k \leq l \leq 8, \quad n = 8 + 7 + \ldots + 1 = 36. \tag{B.19}$$

Note that there is no need to consider terms in which the derivative acts on the first fermion field, since it is always possible to partially integrate it onto the other fermion field.

Next we want to impose the $U(1)_Q$ symmetry. Since this is a global symmetry (i.e. independent of the coordinate x) this works despite the derivative exactly as before, i.e. every term that has remaining factors of $\exp(\pm i\omega)$ after transformation has to be dropped. Upon completion of this step we end up with

$$1_D02: \quad \psi_k \partial_t \psi_l, \quad \text{unique, } U(1)_Q \text{ invariant}, \quad n = 16. \tag{B.20}$$

Now we advance to the $SU(2)_s$ symmetry. Very similarly, every term with remaining factors of $\exp(\pm i\alpha(x))$ after transformation is incompatible with the $SU(2)_s$ symmetry and hence has to be dropped. However, in contrast to the $U(1)_Q$ symmetry the global $SU(2)_s$ symmetry is realized non-linearly and as such depends on the coordinate x. Consequently, the $SU(2)_s$ transformation is sensitive to the derivative ∂_t, i.e.

$$\begin{aligned} 1_D02: &\quad [\psi_+^{A\dagger} \partial_t \psi_+^A]' = e^{-i\alpha(x)} \psi_+^{A\dagger}\, \partial_t (e^{+i\alpha(x)} \psi_+^A) = \psi_+^{A\dagger}\, (\partial_t + i\partial_t\alpha(x)) \psi_+^A, \\ 1_D02: &\quad [\psi_-^{A\dagger} \partial_t \psi_-^B]' = e^{+i\alpha(x)} \psi_-^{A\dagger}\, \partial_t (e^{-i\alpha(x)} \psi_-^B) = \psi_-^{A\dagger}\, (\partial_t - i\partial_t\alpha(x)) \psi_-^B. \end{aligned} \tag{B.21}$$

Of course we can correct this misbehavior if we replace the ordinary derivative ∂_t with the appropriate covariant derivative $D_t = \partial_t \pm i v_t^3$, i.e.

$$\begin{aligned} 1_D02: &\quad [\psi_+^{A\dagger}\, (\partial_t + i v_t^3) \psi_+^A]' = \psi_+^{A\dagger} \left(\partial_t + i\partial_t\alpha(x) + i(v_t^3 - \partial_t\alpha(x))\right) \psi_+^A, \\ 1_D02: &\quad [\psi_-^{A\dagger}\, (\partial_t - i v_t^3) \psi_-^B]' = \psi_-^{A\dagger} \left(\partial_t - i\partial_t\alpha(x) - i(v_t^3 - \partial_t\alpha(x))\right) \psi_-^B, \end{aligned} \tag{B.22}$$

such that afterwards the terms are really $SU(2)_s$ invariant. After augmenting all derivatives accordingly we find that the following eight terms remain

$$\begin{aligned}
A_1^{1_D02} &= \psi_+^{A\dagger}\,(\partial_t + iv_t^3)\psi_+^A, \quad (2), & A_5^{1_D02} &= \psi_+^{B\dagger}\,(\partial_t + iv_t^3)\psi_+^A, \quad (1),\\
A_2^{1_D02} &= \psi_+^{A\dagger}\,(\partial_t + iv_t^3)\psi_+^B, \quad (1), & A_6^{1_D02} &= \psi_+^{B\dagger}\,(\partial_t + iv_t^3)\psi_+^B, \quad (2),\\
A_3^{1_D02} &= \psi_-^{A\dagger}\,(\partial_t - iv_t^3)\psi_-^A, \quad (2), & A_7^{1_D02} &= \psi_-^{B\dagger}\,(\partial_t - iv_t^3)\psi_-^A, \quad (1),\\
A_4^{1_D02} &= \psi_-^{A\dagger}\,(\partial_t - iv_t^3)\psi_-^B, \quad (1), & A_8^{1_D02} &= \psi_-^{B\dagger}\,(\partial_t - iv_t^3)\psi_-^B, \quad (2),
\end{aligned} \tag{B.23}$$

which are now all invariant under the $U(1)_Q$ and $SU(2)_s$ symmetries. Again we first focus on the properties under Hermitean conjugation, whereby we obtain

$$\begin{aligned}
A_1^{1_D02\dagger} &= -A_1^{1_D02}, \quad (2), & A_4^{1_D02\dagger} &= -A_7^{1_D02}, \quad (1),\\
A_2^{1_D02\dagger} &= -A_5^{1_D02}, \quad (1), & A_6^{1_D02\dagger} &= -A_6^{1_D02}, \quad (2),\\
A_3^{1_D02\dagger} &= -A_3^{1_D02}, \quad (2), & A_8^{1_D02\dagger} &= -A_8^{1_D02}, \quad (2).
\end{aligned} \tag{B.24}$$

Note that we assume here that the terms are part of the action integral, since we have to perform a partial integration in order to derive the above relations. There is also the origin of the minus sign.

From here on the story runs as supplied before. First we impose the displacement symmetry D, which forces us to make the combinations

$$\begin{aligned}
B_1^{1_D02} &= A_1^{1_D02} + A_8^{1_D02}, \quad (2), & B_3^{1_D02} &= A_3^{1_D02} + A_6^{1_D02}, \quad (2),\\
B_2^{1_D02} &= A_2^{1_D02} + A_7^{1_D02}, \quad (1), & B_4^{1_D02} &= A_4^{1_D02} + A_5^{1_D02}, \quad (1).
\end{aligned} \tag{B.25}$$

The same procedure for the time reversal symmetry T reveals that

$$\begin{aligned}
C_1^{1_D02} &= B_1^{1_D02} &&= A_1^{1_D02} + A_8^{1_D02}, && (2),\\
C_2^{1_D02} &= B_2^{1_D02} + B_4^{1_D02} &&= A_2^{1_D02} + A_4^{1_D02} + A_5^{1_D02} + A_7^{1_D02}, && (1),\\
C_3^{1_D02} &= B_3^{1_D02} &&= A_3^{1_D02} + A_6^{1_D02}, && (2),
\end{aligned} \tag{B.26}$$

have to be combined together and that the total, $U(1)_Q$ invariant vector space is 3-dimensional.

Here no $C_q^{1_D02}$ is invariant under the $SU(2)_Q$ symmetry right from start, but $C_1^{1_D02}$ and $C_3^{1_D02}$ can be combined to form an $SU(2)_Q$ invariant expression, i.e.

$$\begin{aligned}
D_1^{1_D02} &= C_1^{1_D02} + C_3^{1_D02} = A_1^{1_D02} + A_3^{1_D02} + A_6^{1_D02} + A_8^{1_D02}, \quad (2),\\
\widetilde{D}_1^{1_D02} &= C_2^{1_D02} \qquad\qquad = A_2^{1_D02} + A_4^{1_D02} + A_5^{1_D02} + A_7^{1_D02}, \quad (1),\\
\widetilde{D}_2^{1_D02} &= C_1^{1_D02} - C_3^{1_D02} = A_1^{1_D02} - A_3^{1_D02} - A_6^{1_D02} + A_8^{1_D02}, \quad (2).
\end{aligned} \tag{B.27}$$

Consequently the $SU(2)_Q$ invariant subspace is only 1-dimensional. Consulting Eq. (B.24) we see that all these expressions are anti-Hermitean, and according to Appendix C they can be written in trace notation as

$$\begin{aligned}
E_1^{1_D02} &= \frac{1}{2}\mathrm{Tr}[\Psi^{A\dagger} D_t \Psi^A + \Psi^{B\dagger} D_t \Psi^B] &&= D_1^{1_D02}, \quad (2),\\
\widetilde{E}_1^{1_D02} &= \mathrm{Tr}[\Psi^{A\dagger} D_t \Psi^B \sigma_3] &&= \widetilde{D}_1^{1_D02}, \quad (1),\\
\widetilde{E}_2^{1_D02} &= \frac{1}{2}\mathrm{Tr}[\Psi^{A\dagger}\sigma_3 D_t \Psi^A \sigma_3 - \Psi^{B\dagger}\sigma_3 D_t \Psi^B \sigma_3] &&= \widetilde{D}_2^{1_D02}, \quad (2).
\end{aligned} \tag{B.28}$$

Note that $\widetilde{E}_1^{1_D02}$ and $\widetilde{E}_2^{1_D02}$ will not be contained in the effective action, since they can be absorbed by an appropriate field redefinition as shown in Appendix D.

B.2.2 Class $(1_V, 0, 2)$

As before we should begin with a unique list of all possible terms, whereby we again neglect the symmetries at first. In contrast to the previous class, here the 'derivative' is implemented by a magnon field $v_t^\pm$. This has two consequences: First, terms with two equal fermion fields vanish again, and second, there are now two possibilities for the 'derivative' to choose from, namely v_t^+ or v_t^-. Therefore our list now reads

$$1_V02: \quad \psi_k v_t^+ \psi_l, \;\; \psi_k v_t^- \psi_l, \quad 1 \le k < l \le 8, \quad n = 2(7+6+\ldots+1) = 56. \tag{B.29}$$

Since the magnon field $v_t^\pm$ does not transform under the $U(1)_Q$ symmetry, imposing this symmetry works exactly as before. After dropping the terms with remaining factors of $\exp(\pm i\omega)$ we obtain the shortened list

$$1_V02: \quad \psi_k v_t^+ \psi_l, \;\; \psi_k v_t^- \psi_l, \quad \text{unique, } U(1)_Q \text{ invariant}, \quad n = 32. \tag{B.30}$$

On the other hand the magnon field $v_t^\pm$ does transform under the $SU(2)_s$ symmetry. As a result we find that e.g.

$$\begin{aligned}
1_V02: &\quad [\psi_+^{A\dagger} v_t^+ \psi_+^A]' = e^{-i\alpha(x)} \psi_+^{A\dagger} e^{+2i\alpha(x)} v_t^+ e^{+i\alpha(x)} \psi_+^A,\\
1_V02: &\quad [\psi_+^{A\dagger} v_t^- \psi_+^A]' = e^{-i\alpha(x)} \psi_+^{A\dagger} e^{-2i\alpha(x)} v_t^- e^{+i\alpha(x)} \psi_+^A,\\
1_V02: &\quad [\psi_+^{A\dagger} v_t^- \psi_-^B]' = e^{-i\alpha(x)} \psi_+^{A\dagger} e^{-2i\alpha(x)} v_t^- e^{-i\alpha(x)} \psi_-^B,
\end{aligned} \tag{B.31}$$

are incompatible with the $SU(2)_s$ symmetry and hence must be dropped, whereas e.g.

$$1_V02: \quad [\psi_+^{A\dagger} v_t^+ \psi_-^B]' = e^{-i\alpha(x)} \psi_+^{A\dagger} e^{+2i\alpha(x)} v_t^+ e^{-i\alpha(x)} \psi_-^B, \tag{B.32}$$

has no remaining factors of $e^{\pm i\alpha(x)}$ and hence must be kept. Doing so for all terms we find that only the following eight terms

$$\begin{aligned}
A_1^{1_V02} &= \psi_+^{A\dagger} v_t^+ \psi_-^A, \quad (2), & A_5^{1_V02} &= \psi_+^{B\dagger} v_t^+ \psi_-^A, \quad (1),\\
A_2^{1_V02} &= \psi_+^{A\dagger} v_t^+ \psi_-^B, \quad (1), & A_6^{1_V02} &= \psi_+^{B\dagger} v_t^+ \psi_-^B, \quad (2),\\
A_3^{1_V02} &= \psi_-^{A\dagger} v_t^- \psi_+^A, \quad (2), & A_7^{1_V02} &= \psi_-^{B\dagger} v_t^- \psi_+^A, \quad (1),\\
A_4^{1_V02} &= \psi_-^{A\dagger} v_t^- \psi_+^B, \quad (1), & A_8^{1_V02} &= \psi_-^{B\dagger} v_t^- \psi_+^B, \quad (2),
\end{aligned} \tag{B.33}$$

are invariant under the $U(1)_Q$ and $SU(2)_s$ symmetries. As usual we now first examine the properties under Hermitean conjugation, whereby we obtain

$$\begin{aligned} A_1^{1V02\dagger} &= A_3^{1V02}, \quad (2), & A_4^{1V02\dagger} &= A_5^{1V02}, \quad (1), \\ A_2^{1V02\dagger} &= A_7^{1V02}, \quad (1), & A_6^{1V02\dagger} &= A_8^{1V02}, \quad (2). \end{aligned} \tag{B.34}$$

Note that $v_t^{\pm\dagger} = v_t^{\mp}$.

Imposing the displacement symmetry D forces us to make the combinations

$$\begin{aligned} B_1^{1V02} &= A_1^{1V02} + A_8^{1V02}, \quad (2), & B_3^{1V02} &= A_3^{1V02} + A_6^{1V02}, \quad (2), \\ B_2^{1V02} &= A_2^{1V02} + A_7^{1V02}, \quad (1), & B_4^{1V02} &= A_4^{1V02} + A_5^{1V02}, \quad (1). \end{aligned} \tag{B.35}$$

The same target for the time reversal symmetry T leads us to

$$\begin{aligned} C_1^{1V02} &= B_1^{1V02} + B_3^{1V02} = A_1^{1V02} + A_3^{1V02} + A_6^{1V02} + A_8^{1V02}, \quad (2), \\ C_2^{1V02} &= B_2^{1V02} \qquad\qquad = A_2^{1V02} + A_7^{1V02}, \quad (1), \\ C_3^{1V02} &= B_4^{1V02} \qquad\qquad = A_4^{1V02} + A_5^{1V02}, \quad (1), \end{aligned} \tag{B.36}$$

which uncovers that the total, $U(1)_Q$ invariant vector space is 3-dimensional.

Here only C_1^{1V02} is already invariant under the $SU(2)_Q$ symmetry, but the other two C_q^{1V02} can be combined together to form another $SU(2)_Q$ invariant contribution, such that the $SU(2)_Q$ invariant subspace is 2-dimensional, i.e.

$$\begin{aligned} D_1^{1V02} &= C_1^{1V02} \qquad\qquad = A_1^{1V02} + A_3^{1V02} + A_6^{1V02} + A_8^{1V02}, \quad (2), \\ D_2^{1V02} &= C_2^{1V02} - C_3^{1V02} = A_2^{1V02} - A_4^{1V02} - A_5^{1V02} + A_7^{1V02}, \quad (1), \\ \widetilde{D}_1^{1V02} &= C_2^{1V02} + C_3^{1V02} = A_2^{1V02} + A_4^{1V02} + A_5^{1V02} + A_7^{1V02}, \quad (1). \end{aligned} \tag{B.37}$$

According to Eq. (B.34) all these expressions are Hermitean. Finally the above result can be formulated in trace notation as

$$\begin{aligned} E_1^{1V02} &= \frac{1}{2}\mathrm{Tr}[\Psi^{A\dagger} V_t \Psi^A + \Psi^{B\dagger} V_t \Psi^B] = D_1^{1V02}, \quad (2), \\ E_2^{1V02} &= \mathrm{Tr}[\Psi^{A\dagger} \sigma_3 V_t \Psi^B] \qquad\qquad = D_2^{1V02}, \quad (1), \\ \widetilde{E}_1^{1V02} &= \mathrm{Tr}[\Psi^{A\dagger} V_t \Psi^B \sigma_3] \qquad\qquad = \widetilde{D}_1^{1V02}, \quad (1). \end{aligned} \tag{B.38}$$

B.3 Class $(0, 2, 2)$

Now we step on to the terms with two spatial derivatives. As already seen in the previous section the 'derivative' can be implemented in two different ways. Here we can use either the spatial derivative ∂_i (or more precisely the covariant derivative $D_i = \partial_i \pm i v_i^3$) itself or alternatively the magnon field $v_i^{\pm}$. But in contrast to the previous section we now have two 'derivatives' to implement and for each one we can either choose ∂_i or $v_i^{\pm}$. Consequently there are now 3 possible cases to distinguish, which we will handle in the following subsections as classes $(0, 2_{DD}, 2)$, $(0, 2_{DV}, 2)$ and $(0, 2_{VV}, 2)$.

B.3.1 Class $(0, 2_{DD}, 2)$

In complete analogy to the class $(1_D, 0, 2)$ the unique list of all possible terms reads

$$02_{DD}2: \quad \partial_i\psi_k\partial_i\psi_l, \quad 1 \leq k \leq l \leq 8, \quad n = 8 + 7 + \ldots + 1 = 36, \tag{B.39}$$

where we have again neglected the symmetries so far. Note that there is no need to consider terms in which the two derivatives act on the same fermion field (i.e. $\psi_k\partial_i\partial_i\psi_l$ or $\partial_i\partial_i\psi_k\psi_l$), since these can be reached by partial integration. After imposing the $U(1)_Q$ symmetry the above list reduces to

$$02_{DD}2: \quad \partial_i\psi_k\partial_i\psi_l, \quad \text{unique, } U(1)_Q \text{ invariant}, \quad n = 16. \tag{B.40}$$

While imposing the $SU(2)_s$ symmetry we again recognize that the ordinary derivative ∂_i has to be replaced with the appropriate covariant derivative $D_i = \partial_i \pm iv_i^3$ in order to render the terms $SU(2)_s$ invariant. After doing so, we end up with eight $U(1)_Q$ and $SU(2)_s$ invariant terms, i.e.

$$\begin{aligned}
A_1^{02_{DD}2} &= (\partial_i - iv_i^3)\psi_+^{A\dagger}(\partial_i + iv_i^3)\psi_+^A, \quad (2), & A_5^{02_{DD}2} &= (\partial_i - iv_i^3)\psi_+^{B\dagger}(\partial_i + iv_i^3)\psi_+^A, \quad (1),\\
A_2^{02_{DD}2} &= (\partial_i - iv_i^3)\psi_+^{A\dagger}(\partial_i + iv_i^3)\psi_+^B, \quad (1), & A_6^{02_{DD}2} &= (\partial_i - iv_i^3)\psi_+^{B\dagger}(\partial_i + iv_i^3)\psi_+^B, \quad (2),\\
A_3^{02_{DD}2} &= (\partial_i + iv_i^3)\psi_-^{A\dagger}(\partial_i - iv_i^3)\psi_-^A, \quad (2), & A_7^{02_{DD}2} &= (\partial_i + iv_i^3)\psi_-^{B\dagger}(\partial_i - iv_i^3)\psi_-^A, \quad (1),\\
A_4^{02_{DD}2} &= (\partial_i + iv_i^3)\psi_-^{A\dagger}(\partial_i - iv_i^3)\psi_-^B, \quad (1), & A_8^{02_{DD}2} &= (\partial_i + iv_i^3)\psi_-^{B\dagger}(\partial_i - iv_i^3)\psi_-^B, \quad (2).
\end{aligned} \tag{B.41}$$

Exploring the properties under Hermitean conjugation of these terms shows that

$$\begin{aligned}
A_1^{02_{DD}2\dagger} &= A_1^{02_{DD}2}, \quad (2), & A_4^{02_{DD}2\dagger} &= A_7^{02_{DD}2}, \quad (1),\\
A_2^{02_{DD}2\dagger} &= A_5^{02_{DD}2}, \quad (1), & A_6^{02_{DD}2\dagger} &= A_6^{02_{DD}2}, \quad (2),\\
A_3^{02_{DD}2\dagger} &= A_3^{02_{DD}2}, \quad (2), & A_8^{02_{DD}2\dagger} &= A_8^{02_{DD}2}, \quad (2).
\end{aligned} \tag{B.42}$$

Note that here in contrast to the class $(1_D, 0, 2)$ no partial integration is required to obtain these relations.

Imposing the displacement symmetry D forces us to build the combinations

$$\begin{aligned}
B_1^{02_{DD}2} &= A_1^{02_{DD}2} + A_8^{02_{DD}2}, \quad (2), & B_3^{02_{DD}2} &= A_3^{02_{DD}2} + A_6^{02_{DD}2}, \quad (2),\\
B_2^{02_{DD}2} &= A_2^{02_{DD}2} + A_7^{02_{DD}2}, \quad (1), & B_4^{02_{DD}2} &= A_4^{02_{DD}2} + A_5^{02_{DD}2}, \quad (1).
\end{aligned} \tag{B.43}$$

The same operation for the time reversal symmetry T leads us to

$$\begin{aligned}
C_1^{02_{DD}2} &= B_1^{02_{DD}2} &&= A_1^{02_{DD}2} + A_8^{02_{DD}2}, && (2),\\
C_2^{02_{DD}2} &= B_2^{02_{DD}2} + B_4^{02_{DD}2} &&= A_2^{02_{DD}2} + A_4^{02_{DD}2} + A_5^{02_{DD}2} + A_7^{02_{DD}2}, && (1),\\
C_3^{02_{DD}2} &= B_3^{02_{DD}2} &&= A_3^{02_{DD}2} + A_6^{02_{DD}2}, && (2).
\end{aligned} \tag{B.44}$$

Obviously the total, $U(1)_Q$ invariant vector space is again 3-dimensional. Here $C_2^{02_{DD}2}$ as well as a combination of $C_1^{02_{DD}2}$ and $C_3^{02_{DD}2}$ are $SU(2)_Q$ invariant, i.e.

$$\begin{aligned} D_1^{02_{DD}2} &= C_1^{02_{DD}2} - C_3^{02_{DD}2} = A_1^{02_{DD}2} - A_3^{02_{DD}2} - A_6^{02_{DD}2} + A_8^{02_{DD}2}, \quad (2), \\ D_2^{02_{DD}2} &= C_2^{02_{DD}2} \qquad\qquad = A_2^{02_{DD}2} + A_4^{02_{DD}2} + A_5^{02_{DD}2} + A_7^{02_{DD}2}, \quad (1), \\ \widetilde{D}_1^{02_{DD}2} &= C_1^{02_{DD}2} + C_3^{02_{DD}2} = A_1^{02_{DD}2} + A_3^{02_{DD}2} + A_6^{02_{DD}2} + A_8^{02_{DD}2}, \quad (2). \end{aligned} \tag{B.45}$$

Consequently the $SU(2)_Q$ invariant subspace is after all 2-dimensional. According to Eq. (B.42) all these expressions are again Hermitean. Consulting Appendix C we find that the above result can be stated in trace notation as

$$\begin{aligned} E_1^{02_{DD}2} &= \frac{1}{2}\mathrm{Tr}[D_i\Psi^{A\dagger}\sigma_3 D_i\Psi^A - D_i\Psi^{B\dagger}\sigma_3 D_i\Psi^B] = D_1^{02_{DD}2}, \quad (2), \\ E_2^{02_{DD}2} &= \mathrm{Tr}[D_i\Psi^{A\dagger} D_i\Psi^B] \qquad\qquad = D_2^{02_{DD}2}, \quad (1), \\ \widetilde{E}_1^{02_{DD}2} &= \frac{1}{2}\mathrm{Tr}[D_i\Psi^{A\dagger} D_i\Psi^A\sigma_3 + D_i\Psi^{B\dagger} D_i\Psi^B\sigma_3] = \widetilde{D}_1^{02_{DD}2}, \quad (2). \end{aligned} \tag{B.46}$$

B.3.2 Class $(0, 2_{DV}, 2)$

Again we have to build the unique list of all possible terms in advance. Here some more options exist, resulting in a somewhat larger number of terms we have to examine. Now the derivative can act either on the first or the second fermion field and in addition we can again choose v_i^+ or v_i^- for the magnon field. However, there is no need to consider terms in which the derivative acts on the magnon field $v_i^\pm$, since it is possible to partial integrate it back onto the fermion fields. Therefore our list reads

$$\begin{aligned} 02_{DV}2: \quad & \psi_k v_i^+ \partial_i \psi_l, \;\; \psi_k v_i^- \partial_i \psi_l, \;\; \partial_i\psi_k v_i^+ \psi_l, \;\; \partial_i\psi_k v_i^- \psi_l, \\ & 1 \le k \le l \le 8, \quad n = 4(8 + 7 + \ldots + 1) = 144. \end{aligned} \tag{B.47}$$

After imposing the $U(1)_Q$ symmetry this reduces to

$$\begin{aligned} 02_{DV}2: \quad & \psi_k v_i^+ \partial_i \psi_l, \;\; \psi_k v_i^- \partial_i \psi_l, \;\; \partial_i\psi_k v_i^+ \psi_l, \;\; \partial_i\psi_k v_i^- \psi_l, \\ & \text{unique, } U(1)_Q \text{ invariant}, \quad n = 64. \end{aligned} \tag{B.48}$$

Imposing the $SU(2)_s$ symmetry again forces us to introduce the covariant derivative and we end up with only sixteen remaining terms, namely

$$\begin{aligned}
A_1^{02_{DV}2} &= (\partial_i - iv_i^3)\psi_+^{A\dagger}\, v_i^+ \psi_-^A, \quad (2), & A_9^{02_{DV}2} &= (\partial_i - iv_i^3)\psi_+^{B\dagger}\, v_i^+ \psi_-^A, \quad (1),\\
A_2^{02_{DV}2} &= \psi_+^{A\dagger} v_i^+ \,(\partial_i - iv_i^3)\psi_-^A, \quad (2), & A_{10}^{02_{DV}2} &= \psi_+^{B\dagger} v_i^+ \,(\partial_i - iv_i^3)\psi_-^A, \quad (1),\\
A_3^{02_{DV}2} &= (\partial_i - iv_i^3)\psi_+^{A\dagger}\, v_i^+ \psi_-^B, \quad (1), & A_{11}^{02_{DV}2} &= (\partial_i - iv_i^3)\psi_+^{B\dagger}\, v_i^+ \psi_-^B, \quad (2),\\
A_4^{02_{DV}2} &= \psi_+^{A\dagger} v_i^+ \,(\partial_i - iv_i^3)\psi_-^B, \quad (1), & A_{12}^{02_{DV}2} &= \psi_+^{B\dagger} v_i^+ \,(\partial_i - iv_i^3)\psi_-^B, \quad (2),\\
A_5^{02_{DV}2} &= (\partial_i + iv_i^3)\psi_-^{A\dagger}\, v_i^- \psi_+^A, \quad (2), & A_{13}^{02_{DV}2} &= (\partial_i + iv_i^3)\psi_-^{B\dagger}\, v_i^- \psi_+^A, \quad (1),\\
A_6^{02_{DV}2} &= \psi_-^{A\dagger} v_i^- \,(\partial_i + iv_i^3)\psi_+^A, \quad (2), & A_{14}^{02_{DV}2} &= \psi_-^{B\dagger} v_i^- \,(\partial_i + iv_i^3)\psi_+^A, \quad (1),\\
A_7^{02_{DV}2} &= (\partial_i + iv_i^3)\psi_-^{A\dagger}\, v_i^- \psi_+^B, \quad (1), & A_{15}^{02_{DV}2} &= (\partial_i + iv_i^3)\psi_-^{B\dagger}\, v_i^- \psi_+^B, \quad (2),\\
A_8^{02_{DV}2} &= \psi_-^{A\dagger} v_i^- \,(\partial_i + iv_i^3)\psi_+^B, \quad (1), & A_{16}^{02_{DV}2} &= \psi_-^{B\dagger} v_i^- \,(\partial_i + iv_i^3)\psi_+^B, \quad (2).
\end{aligned} \tag{B.49}$$

As usual we now analyze the properties under Hermitean conjugation, whereby we find

$$\begin{aligned}
A_1^{02_{DV}2\dagger} &= A_6^{02_{DV}2}, \quad (2), & A_7^{02_{DV}2\dagger} &= A_{10}^{02_{DV}2}, \quad (1),\\
A_2^{02_{DV}2\dagger} &= A_5^{02_{DV}2}, \quad (2), & A_8^{02_{DV}2\dagger} &= A_9^{02_{DV}2}, \quad (1),\\
A_3^{02_{DV}2\dagger} &= A_{14}^{02_{DV}2}, \quad (1), & A_{11}^{02_{DV}2\dagger} &= A_{16}^{02_{DV}2}, \quad (2),\\
A_4^{02_{DV}2\dagger} &= A_{13}^{02_{DV}2}, \quad (1), & A_{12}^{02_{DV}2\dagger} &= A_{15}^{02_{DV}2}, \quad (2).
\end{aligned} \tag{B.50}$$

Note that again no partial integration is required in order to obtain these relations.

Imposing the displacement symmetry D leads us to build the combinations

$$\begin{aligned}
B_1^{02_{DV}2} &= A_1^{02_{DV}2} + A_{15}^{02_{DV}2}, \quad (2), & B_5^{02_{DV}2} &= A_5^{02_{DV}2} + A_{11}^{02_{DV}2}, \quad (2),\\
B_2^{02_{DV}2} &= A_2^{02_{DV}2} + A_{16}^{02_{DV}2}, \quad (2), & B_6^{02_{DV}2} &= A_6^{02_{DV}2} + A_{12}^{02_{DV}2}, \quad (2),\\
B_3^{02_{DV}2} &= A_3^{02_{DV}2} + A_{13}^{02_{DV}2}, \quad (1), & B_7^{02_{DV}2} &= A_7^{02_{DV}2} + A_9^{02_{DV}2}, \quad (1),\\
B_4^{02_{DV}2} &= A_4^{02_{DV}2} + A_{14}^{02_{DV}2}, \quad (1), & B_8^{02_{DV}2} &= A_8^{02_{DV}2} + A_{10}^{02_{DV}2}, \quad (1).
\end{aligned} \tag{B.51}$$

Next we realize that further combinations are required in order to comply with the time reversal symmetry T, i.e.

$$\begin{aligned}
C_1^{02_{DV}2} &= B_1^{02_{DV}2} - B_6^{02_{DV}2} = A_1^{02_{DV}2} - A_6^{02_{DV}2} - A_{12}^{02_{DV}2} + A_{15}^{02_{DV}2}, \quad (2),\\
C_2^{02_{DV}2} &= B_2^{02_{DV}2} - B_5^{02_{DV}2} = A_2^{02_{DV}2} - A_5^{02_{DV}2} - A_{11}^{02_{DV}2} + A_{16}^{02_{DV}2}, \quad (2),\\
C_3^{02_{DV}2} &= B_3^{02_{DV}2} - B_4^{02_{DV}2} = A_3^{02_{DV}2} - A_4^{02_{DV}2} + A_{13}^{02_{DV}2} - A_{14}^{02_{DV}2}, \quad (1),\\
C_4^{02_{DV}2} &= B_7^{02_{DV}2} - B_8^{02_{DV}2} = A_7^{02_{DV}2} - A_8^{02_{DV}2} + A_9^{02_{DV}2} - A_{10}^{02_{DV}2}, \quad (1),
\end{aligned} \tag{B.52}$$

such that the total, $U(1)_Q$ invariant vector space is 4-dimensional.

Here no $C_q^{02_{DV}2}$ is invariant under $SU(2)_Q$ from scratch. But they can be combined to form a 2-dimensional $SU(2)_Q$ invariant subspace, i.e.

$$\begin{aligned}
D_1^{02_{DV}2} &= C_1^{02_{DV}2} + C_2^{02_{DV}2} = \quad A_1^{02_{DV}2} + A_2^{02_{DV}2} - A_5^{02_{DV}2} - A_6^{02_{DV}2} \\
&\qquad - A_{11}^{02_{DV}2} - A_{12}^{02_{DV}2} + A_{15}^{02_{DV}2} + A_{16}^{02_{DV}2}, \quad (2), \\
D_2^{02_{DV}2} &= C_3^{02_{DV}2} + C_4^{02_{DV}2} = \quad A_3^{02_{DV}2} - A_4^{02_{DV}2} + A_7^{02_{DV}2} - A_8^{02_{DV}2} \\
&\qquad + A_9^{02_{DV}2} - A_{10}^{02_{DV}2} + A_{13}^{02_{DV}2} - A_{14}^{02_{DV}2}, \quad (1), \\
\widetilde{D}_1^{02_{DV}2} &= C_1^{02_{DV}2} - C_2^{02_{DV}2} = \quad A_1^{02_{DV}2} - A_2^{02_{DV}2} + A_5^{02_{DV}2} - A_6^{02_{DV}2} \\
&\qquad + A_{11}^{02_{DV}2} - A_{12}^{02_{DV}2} + A_{15}^{02_{DV}2} - A_{16}^{02_{DV}2}, \quad (2), \\
\widetilde{D}_2^{02_{DV}2} &= C_3^{02_{DV}2} - C_4^{02_{DV}2} = \quad A_3^{02_{DV}2} - A_4^{02_{DV}2} - A_7^{02_{DV}2} + A_8^{02_{DV}2} \\
&\qquad - A_9^{02_{DV}2} + A_{10}^{02_{DV}2} + A_{13}^{02_{DV}2} - A_{14}^{02_{DV}2}, \quad (1).
\end{aligned} \tag{B.53}$$

According to Eq. (B.50) all these expressions are anti-Hermitean. Finally this result can be expressed in trace notation as

$$\begin{aligned}
E_1^{02_{DV}2} &= \mathrm{Tr}[D_i\Psi^{A\dagger}\sigma_3 V_i\Psi^A - D_i\Psi^{B\dagger}\sigma_3 V_i\Psi^B] &&= D_1^{02_{DV}2}, \quad (2), \\
E_2^{02_{DV}2} &= \mathrm{Tr}[D_i\Psi^{A\dagger}V_i\Psi^B + D_i\Psi^{B\dagger}V_i\Psi^A] &&= D_2^{02_{DV}2}, \quad (1), \\
\widetilde{E}_1^{02_{DV}2} &= \mathrm{Tr}[D_i\Psi^{A\dagger}V_i\Psi^A\sigma_3 + D_i\Psi^{B\dagger}V_i\Psi^B\sigma_3] &&= \widetilde{D}_1^{02_{DV}2}, \quad (2), \\
\widetilde{E}_2^{02_{DV}2} &= \mathrm{Tr}[D_i\Psi^{A\dagger}\sigma_3 V_i\Psi^B\sigma_3 + D_i\Psi^{B\dagger}V_i\sigma_3\Psi^A\sigma_3] &&= \widetilde{D}_2^{02_{DV}2}, \quad (1).
\end{aligned} \tag{B.54}$$

B.3.3 Class $(0, 2_{VV}, 2)$

As always we should first set up the unique list of all possible terms. Now we have two magnon fields $v_i^\pm$ to implement, and for each one we can either choose v_i^+ or v_i^-. Naively thinking this results in 3 possibilities (i.e. $v_i^+v_i^+$, $v_i^+v_i^-$ and $v_i^-v_i^-$). But if we take a look ahead on the $SU(2)_s$ transformation law of $v_i^\pm$ we find that $[v_i^\pm v_i^\pm]' = \exp(\pm 4i\alpha(x))v_i^\pm v_i^\pm$. Clearly this prefactor is out of reach to cancel for two fermion fields, which at most can deliver a prefactor of $\exp(\pm 2i\alpha(x))$. Consequently we can confidently omit these two possibilities right from start and therefore our list shrinks to

$$02_{VV}2: \quad \psi_k v_i^+ v_i^- \psi_l, \quad 1 \le k < l \le 8, \quad n = 7 + 6 + \ldots + 1 = 28. \tag{B.55}$$

After imposing the $U(1)_Q$ symmetry this reduces even more to

$$02_{VV}2: \quad \psi_k v_i^+ v_i^- \psi_l, \quad \text{unique, } U(1)_Q \text{ invariant}, \quad n = 16. \tag{B.56}$$

Playing the same game for the $SU(2)_s$ symmetry we have to drop further terms and finally we end up with eight remaining terms, i.e.

$$\begin{aligned}
A_1^{02_{VV}2} &= \psi_+^{A\dagger} v_i^+ v_i^- \psi_+^A, \quad (2), & A_5^{02_{VV}2} &= \psi_+^{B\dagger} v_i^+ v_i^- \psi_+^A, \quad (1), \\
A_2^{02_{VV}2} &= \psi_+^{A\dagger} v_i^+ v_i^- \psi_+^B, \quad (1), & A_6^{02_{VV}2} &= \psi_+^{B\dagger} v_i^+ v_i^- \psi_+^B, \quad (2), \\
A_3^{02_{VV}2} &= \psi_-^{A\dagger} v_i^- v_i^+ \psi_-^A, \quad (2), & A_7^{02_{VV}2} &= \psi_-^{B\dagger} v_i^- v_i^+ \psi_-^A, \quad (1), \\
A_4^{02_{VV}2} &= \psi_-^{A\dagger} v_i^- v_i^+ \psi_-^B, \quad (1), & A_8^{02_{VV}2} &= \psi_-^{B\dagger} v_i^- v_i^+ \psi_-^B, \quad (2).
\end{aligned} \tag{B.57}$$

The examination of the properties under Hermitean conjugation reveals here that

$$\begin{aligned}
A_1^{02_{VV}2\dagger} &= A_1^{02_{VV}2}, \quad (2), & A_4^{02_{VV}2\dagger} &= A_7^{02_{VV}2}, \quad (1),\\
A_2^{02_{VV}2\dagger} &= A_5^{02_{VV}2}, \quad (1), & A_6^{02_{VV}2\dagger} &= A_6^{02_{VV}2}, \quad (2),\\
A_3^{02_{VV}2\dagger} &= A_3^{02_{VV}2}, \quad (2), & A_8^{02_{VV}2\dagger} &= A_8^{02_{VV}2}, \quad (2).
\end{aligned} \tag{B.58}$$

Again the above terms have to be combined in order to be compatible with the displacement symmetry D, i.e.

$$\begin{aligned}
B_1^{02_{VV}2} &= A_1^{02_{VV}2} + A_8^{02_{VV}2}, \quad (2), & B_3^{02_{VV}2} &= A_3^{02_{VV}2} + A_6^{02_{VV}2}, \quad (2),\\
B_2^{02_{VV}2} &= A_2^{02_{VV}2} + A_7^{02_{VV}2}, \quad (1), & B_4^{02_{VV}2} &= A_4^{02_{VV}2} + A_5^{02_{VV}2}, \quad (1).
\end{aligned} \tag{B.59}$$

The same step for the time reversal symmetry T leads us to

$$\begin{aligned}
C_1^{02_{VV}2} &= B_1^{02_{VV}2} &&= A_1^{02_{VV}2} + A_8^{02_{VV}2}, && (2),\\
C_2^{02_{VV}2} &= B_2^{02_{VV}2} + B_4^{02_{VV}2} &&= A_2^{02_{VV}2} + A_4^{02_{VV}2} + A_5^{02_{VV}2} + A_7^{02_{VV}2}, && (1),\\
C_3^{02_{VV}2} &= B_3^{02_{VV}2} &&= A_3^{02_{VV}2} + A_6^{02_{VV}2}, && (2).
\end{aligned} \tag{B.60}$$

Apparently the total, $U(1)_Q$ invariant vector space is 3-dimensional. Furthermore the $SU(2)_Q$ invariant subspace is 2-dimensional, i.e.

$$\begin{aligned}
D_1^{02_{VV}2} &= C_1^{02_{VV}2} - C_3^{02_{VV}2} &&= A_1^{02_{VV}2} - A_3^{02_{VV}2} - A_6^{02_{VV}2} + A_8^{02_{VV}2}, && (2),\\
D_2^{02_{VV}2} &= C_2^{02_{VV}2} &&= A_2^{02_{VV}2} + A_4^{02_{VV}2} + A_5^{02_{VV}2} + A_7^{02_{VV}2}, && (1),\\
\widetilde{D}_1^{02_{VV}2} &= C_1^{02_{VV}2} + C_3^{02_{VV}2} &&= A_1^{02_{VV}2} + A_3^{02_{VV}2} + A_6^{02_{VV}2} + A_8^{02_{VV}2}, && (2).
\end{aligned} \tag{B.61}$$

Again this result can be restated in trace notation as

$$\begin{aligned}
E_1^{02_{VV}2} &= \frac{1}{2}\mathrm{Tr}[\Psi^{A\dagger} V_i \sigma_3 V_i \Psi^A - \Psi^{B\dagger} V_i \sigma_3 V_i \Psi^B] = -D_1^{02_{VV}2}, && (2),\\
E_2^{02_{VV}2} &= \mathrm{Tr}[\Psi^{A\dagger} V_i V_i \Psi^B] = D_2^{02_{VV}2}, && (1),\\
\widetilde{E}_1^{02_{VV}2} &= \frac{1}{2}\mathrm{Tr}[\Psi^{A\dagger} V_i V_i \Psi^A \sigma_3 + \Psi^{B\dagger} V_i V_i \Psi^B \sigma_3] = \widetilde{D}_1^{02_{VV}2}, && (2).
\end{aligned} \tag{B.62}$$

According to Eq. (B.58) all these expressions are Hermitean.

C Search of the leading expressions in trace notation

In this appendix we want to rewrite the found expressions from component notation to the trace notation. As already mentioned in the main text, this is unfortunately not a straightforward procedure. For this reason we have to try several promising trace expressions in turn until we eventually find the appropriate ones. To give the reader a better

understanding for this process, we will also show cases, where we are thereby on the wrong track. Note that the list of trace expressions presented during this process is by far not exhaustive, its main purpose is just to illustrate the arising problems.

We will find that some traces simply vanish, while others are incompatible with the symmetries. In addition there exist several linear dependencies among the various trace expressions. To detect all these conditions we have to decompose the traces to component notation first. Furthermore since the displacement symmetry D and the time reversal symmetry T are not manifest in the trace notation, we have to explicitly check these symmetries for every trace expression under consideration. Again, this must be done in the component notation.

As before, we use the notation (n_t, n_i, n_ψ) to classify the expressions according to the number of temporal derivatives n_t, the number of spatial derivatives n_i, and the number of fermion fields n_ψ they contain. In addition we split up our search with respect to $SU(2)_Q$ invariant and $SU(2)_Q$ breaking (but $U(1)_Q$ invariant) expressions. This is easy, since in trace notation this property is manifest, i.e. traces containing factors of $\Psi_\pm^{A,B}\sigma_3$ or $\sigma_3\Psi_\pm^{A,B\dagger}$ break the $SU(2)_Q$ symmetry (but not the $U(1)_Q$ symmetry, as required). Finally we can use n_{AB} as defined in Eq. (8.9) to split up our search even more.

C.1 Classes $(2,0,0)$ and $(0,2,0)$

$SU(2)_Q$ invariant

First we want to handle the purely magnonic sector. Here we just find the well known expressions for a 'relativistic' field, i.e.

$$E_1^{200} = \mathrm{Tr}[\partial_t P \partial_t P], \quad (0), \qquad\qquad E_1^{020} = \mathrm{Tr}[\partial_i P \partial_i P], \quad (0). \tag{C.1}$$

Note that these expressions are invariant under the $SU(2)_Q$ symmetry in a trivial way, since the magnon field P does not transform under this symmetry. Consequently no $SU(2)_Q$ breaking contributions exist. Again the number in parentheses after each expression is n_{AB}.

C.2 Class $(0,0,2)$

$SU(2)_Q$ invariant

Now we step on to the expressions with fermion fields but no derivatives. From the analysis in component notation, we already know that the $SU(2)_Q$ invariant subspace is 2-dimensional. Furthermore we know that one of these corresponding expressions has $n_{AB} = 1$, whereas the other has $n_{AB} = 2$.

For $n_{AB} = 1$ we find that the simplest expression one can think of, i.e.

$$E_1^{002} = \mathrm{Tr}[\Psi^{A\dagger}\Psi^B] = \mathrm{Tr}[\Psi^{B\dagger}\Psi^A], \quad (1), \tag{C.2}$$

is already the wanted trace expression. Note that the reordering of the fields in this relation is non-trivial despite the cyclic property of the trace, which only holds for commutative matrix elements. In fact all such reordering relations result from the anticommutative behavior of the fermion fields and the explicit content of the Ψ-matrices. More complicated examples that illustrate this will follow. Further potential contributions for $n_{AB} = 1$ are incompatible with the symmetries, i.e.

$$\mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^B] = -\mathrm{Tr}[\Psi^{B\dagger}\sigma_3\Psi^A], \quad (T \text{ breaking}), \quad (1). \tag{C.3}$$

For $n_{AB} = 2$ the simplest possible expression simply vanishes, i.e.

$$\mathrm{Tr}[\Psi^{A\dagger}\Psi^A] = \mathrm{Tr}[\Psi^{B\dagger}\Psi^B] = 0, \quad (2), \tag{C.4}$$

but our next try

$$2E_2^{002} = \mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^A] - \mathrm{Tr}[\Psi^{B\dagger}\sigma_3\Psi^B], \quad (2), \tag{C.5}$$

turns out to be the expression we are looking for. Note that we have to combine here two trace expressions, one for each sublattice index A and B. This is a necessity in order to build expressions that comply with the displacement symmetry D, since this symmetry interchanges the two sublattice indices. The correct relative sign is determined by applying a displacement symmetry transformation D after decomposing the trace expression to component notation.

According to our preceding dimension analysis we are finished here, since we have found all expressions in demand so far. Hence we can step on to the $SU(2)_Q$ breaking expressions.

$SU(2)_Q$ breaking but $U(1)_Q$ invariant

From the analysis in component notation, we know that the total, $U(1)_Q$ invariant vector space is 3-dimensional. Since the $SU(2)_Q$ invariant subspace is 2-dimensional there must be exactly one $SU(2)_Q$ breaking expression. Furthermore we know that this one has $n_{AB} = 2$. Consequently we should be unable to find a valid trace expression with $n_{AB} = 1$.

Indeed for $n_{AB} = 1$ all expressions under consideration do not match the symmetry requirements, i.e.

$$\begin{aligned} \mathrm{Tr}[\Psi^{A\dagger}\Psi^B\sigma_3] &= -\mathrm{Tr}[\Psi^{B\dagger}\Psi^A\sigma_3], \quad (D,\, T \text{ breaking}), \quad (1), \\ \mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^B\sigma_3] &= +\mathrm{Tr}[\Psi^{B\dagger}\sigma_3\Psi^A\sigma_3], \quad (D \text{ breaking}), \quad (1). \end{aligned} \tag{C.6}$$

On the other hand for $n_{AB} = 2$ we actually find the required expression at first go, i.e.

$$2\widetilde{E}_1^{002} = \mathrm{Tr}[\Psi^{A\dagger}\Psi^A\sigma_3] + \mathrm{Tr}[\Psi^{B\dagger}\Psi^B\sigma_3], \quad (2). \tag{C.7}$$

In addition further potential contributions effectively vanish, i.e.

$$\mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^A\sigma_3] = \mathrm{Tr}[\Psi^{B\dagger}\sigma_3\Psi^B\sigma_3] = 0, \quad (2), \tag{C.8}$$

in accordance to the above dimension analysis. Again we have found what we need and hence can move on to the next class.

C.3 Class $(0, 0, 4)$

$SU(2)_Q$ invariant

Again from the analysis in component notation it is clear that there must be five linearly independent $SU(2)_Q$ invariant expressions in the trace notation as well. Furthermore we also know from there that three of them have $n_{AB} = 2$, while the other two have $n_{AB} = 3$ and $n_{AB} = 4$, respectively.

For $n_{AB} = 2$ it is quite easy to find expressions that meet all symmetries, i.e.

$$\begin{aligned}
2E_2^{004} = T_1^{004} &= \mathrm{Tr}[\Psi^{A\dagger}\Psi^A\Psi^{B\dagger}\Psi^B] &&= \mathrm{Tr}[\Psi^{B\dagger}\Psi^B\Psi^{A\dagger}\Psi^A], && (2),\\
2E_3^{004} = T_2^{004} &= \mathrm{Tr}[\Psi^{A\dagger}\Psi^B\Psi^{B\dagger}\Psi^A] &&= \mathrm{Tr}[\Psi^{B\dagger}\Psi^A\Psi^{A\dagger}\Psi^B], && (2),\\
T_3^{004} &= \mathrm{Tr}[\Psi^{A\dagger}\Psi^B\Psi^{A\dagger}\Psi^B] &&= \mathrm{Tr}[\Psi^{B\dagger}\Psi^A\Psi^{B\dagger}\Psi^A], && (2),\\
T_4^{004} &= \mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^B\Psi^{B\dagger}\sigma_3\Psi^A] &&= \mathrm{Tr}[\Psi^{B\dagger}\sigma_3\Psi^A\Psi^{A\dagger}\sigma_3\Psi^B], && (2),\\
2E_4^{004} = T_5^{004} &= \mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^A\Psi^{B\dagger}\sigma_3\Psi^B] &&= \mathrm{Tr}[\Psi^{B\dagger}\sigma_3\Psi^B\Psi^{A\dagger}\sigma_3\Psi^A], && (2),\\
T_6^{004} &= \mathrm{Tr}[\Psi^{A\dagger}\Psi^B]\mathrm{Tr}[\Psi^{A\dagger}\Psi^B] &&= \mathrm{Tr}[\Psi^{A\dagger}\Psi^B]\mathrm{Tr}[\Psi^{B\dagger}\Psi^A], && (2).
\end{aligned} \tag{C.9}$$

The reason why we find here even six expressions instead of only the required three is simply because these expressions are not linearly independent. Indeed we find the linearly independent relations

$$\begin{aligned}
T_1^{004} + T_2^{004} &= -2T_3^{004}, \quad (2),\\
T_2^{004} + T_4^{004} &= -2T_5^{004}, \quad (2),\\
T_1^{004} - T_2^{004} &= -2T_6^{004}, \quad (2),
\end{aligned} \tag{C.10}$$

which cancel again three out of the above six expressions, such that we actually end up with three linearly independent expressions as expected.

For $n_{AB} = 3$ our first naive guesses

$$\begin{aligned}
T_7^{004} &= \mathrm{Tr}[\Psi^{A\dagger}\Psi^A\Psi^{A\dagger}\Psi^B] &&= -\mathrm{Tr}[\Psi^{B\dagger}\Psi^A\Psi^{A\dagger}\Psi^A], && (D,\, T \text{ breaking}), && (3),\\
T_8^{004} &= \mathrm{Tr}[\Psi^{B\dagger}\Psi^B\Psi^{B\dagger}\Psi^A] &&= -\mathrm{Tr}[\Psi^{A\dagger}\Psi^B\Psi^{B\dagger}\Psi^B], && (D,\, T \text{ breaking}), && (3),\\
T_9^{004} &= \mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^A\Psi^{A\dagger}\sigma_3\Psi^B] &&= -\mathrm{Tr}[\Psi^{B\dagger}\sigma_3\Psi^A\Psi^{A\dagger}\sigma_3\Psi^A], && (D,\, T \text{ breaking}), && (3),\\
T_{10}^{004} &= \mathrm{Tr}[\Psi^{B\dagger}\sigma_3\Psi^B\Psi^{B\dagger}\sigma_3\Psi^A] &&= -\mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^B\Psi^{B\dagger}\sigma_3\Psi^B], && (D,\, T \text{ breaking}), && (3),
\end{aligned} \tag{C.11}$$

all violate both the displacement and the time reversal symmetries. Clearly, since the displacement symmetry D interchanges the sublattice indices A and B it is evident that an expression containing e.g. three A and one B sublattice indices cannot be invariant under this symmetry. To get rid of this symmetry breaking behavior, we have to combine the expressions appropriately. Actually we can impose the displacement symmetry D in this way, but we still get stuck with the time reversal symmetry T breaking behavior, i.e.

$$\begin{aligned} &T_7^{004} + T_8^{004}, \quad (T \text{ breaking}), \quad (3), \\ &T_9^{004} + T_{10}^{004}, \quad (T \text{ breaking}), \quad (3). \end{aligned} \tag{C.12}$$

Worse, these two combinations are even linearly dependent, i.e.

$$T_7^{004} + T_8^{004} = -3(T_9^{004} + T_{10}^{004}), \quad (T \text{ breaking}), \quad (3). \tag{C.13}$$

So after all this approach turns out to be a dead end and we have to watch out for a better one. After further investigation we finally find the expressions

$$\begin{aligned} E_5^{004} = T_{11}^{004} &= \mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^A\Psi^{A\dagger}\Psi^B] - \mathrm{Tr}[\Psi^{B\dagger}\sigma_3\Psi^B\Psi^{B\dagger}\Psi^A], \quad (3), \\ T_{12}^{004} &= \mathrm{Tr}[\Psi^{A\dagger}\Psi^B]\left(\mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^A] - \mathrm{Tr}[\Psi^{B\dagger}\sigma_3\Psi^B]\right), \quad (3), \end{aligned} \tag{C.14}$$

which are now compatible with all symmetries. Again these two expressions are linearly dependent, i.e.

$$T_{12}^{004} = 2T_{11}^{004}, \quad (3), \tag{C.15}$$

such that we indeed close with one linearly independent expression for $n_{AB} = 3$ as expected.

For $n_{AB} = 4$ we readily find

$$12E_1^{004} = \mathrm{Tr}[\Psi^{A\dagger}\Psi^A\Psi^{A\dagger}\Psi^A] + \mathrm{Tr}[\Psi^{B\dagger}\Psi^B\Psi^{B\dagger}\Psi^B], \quad (4), \tag{C.16}$$

to be the sole expression in demand. Note that again we have to combine here A and B sublattice index expressions in order to comply with the displacement symmetry D.

$SU(2)_Q$ breaking but $U(1)_Q$ invariant

According to the prior work in component notation we already know that there are a total of three $SU(2)_Q$ breaking expressions, namely two for $n_{AB} = 2$ and one for $n_{AB} = 3$.

For $n_{AB} = 2$ our first attempt yields

$$\begin{aligned} \mathrm{Tr}[\Psi^{A\dagger}\Psi^A\Psi^{B\dagger}\Psi^B\sigma_3] \quad &= -\mathrm{Tr}[\Psi^{B\dagger}\Psi^B\Psi^{A\dagger}\Psi^A\sigma_3], \quad (D,\, T \text{ breaking}), \quad (2), \\ \widetilde{T}_1^{004} = \mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^A\Psi^{B\dagger}\Psi^B\sigma_3] &= +\mathrm{Tr}[\Psi^{B\dagger}\Psi^B\Psi^{A\dagger}\sigma_3\Psi^A\sigma_3], \quad (D \text{ breaking}), \quad (2), \\ \widetilde{T}_2^{004} = \mathrm{Tr}[\Psi^{A\dagger}\Psi^A\Psi^{B\dagger}\sigma_3\Psi^B\sigma_3] &= +\mathrm{Tr}[\Psi^{B\dagger}\sigma_3\Psi^B\Psi^{A\dagger}\Psi^A\sigma_3], \quad (D \text{ breaking}), \quad (2). \end{aligned} \tag{C.17}$$

Since the first expression breaks both symmetries, we leave it alone and focus on the other two, which only break the displacement symmetry D. Actually we can combine these two to an invariant expression, i.e.

$$4\widetilde{E}_1^{004} = \widetilde{T}_3^{004} = \widetilde{T}_1^{004} - \widetilde{T}_2^{004}, \quad (2). \tag{C.18}$$

Therewith we have found the first of the two expressions in demand for $n_{AB} = 2$. To find the second one we try for example

$$\begin{aligned} 2\widetilde{E}_2^{004} = \widetilde{T}_4^{004} &= \mathrm{Tr}[\Psi^{A\dagger}\Psi^A\sigma_3\Psi^{B\dagger}\Psi^B\sigma_3] = \mathrm{Tr}[\Psi^{B\dagger}\Psi^B\sigma_3\Psi^{A\dagger}\Psi^A\sigma_3], && (2), \\ \widetilde{T}_5^{004} &= \mathrm{Tr}[\Psi^{A\dagger}\Psi^B\sigma_3\Psi^{B\dagger}\Psi^A\sigma_3] = \mathrm{Tr}[\Psi^{B\dagger}\Psi^A\sigma_3\Psi^{A\dagger}\Psi^B\sigma_3], && (2), \\ \widetilde{T}_6^{004} &= \left(\mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^A] - \mathrm{Tr}[\Psi^{B\dagger}\sigma_3\Psi^B]\right)\left(\mathrm{Tr}[\Psi^{A\dagger}\Psi^A\sigma_3] + \mathrm{Tr}[\Psi^{B\dagger}\Psi^B\sigma_3]\right), && (2). \end{aligned} \tag{C.19}$$

Again these three invariant expressions are linearly dependent, i.e.

$$\begin{aligned} T_1^{004} + \widetilde{T}_4^{004} &= -2\widetilde{T}_5^{004}, \quad (2), \\ \widetilde{T}_6^{004} &= +2\widetilde{T}_3^{004}, \quad (2), \end{aligned} \tag{C.20}$$

such that we indeed end up with only one additional linearly independent expression as expected. Note that, surprisingly, the first term in the first relation here is $SU(2)_Q$ invariant, whereas the other two in the same relation are not. This shows that non-obvious cancellations take place in the trace notation.

For $n_{AB} = 3$ we track down

$$\begin{aligned} \widetilde{E}_3^{004} = \widetilde{T}_7^{004} &= \mathrm{Tr}[\Psi^{A\dagger}\Psi^A\sigma_3\Psi^{A\dagger}\Psi^B] + \mathrm{Tr}[\Psi^{B\dagger}\Psi^B\sigma_3\Psi^{B\dagger}\Psi^A], \quad (3), \\ \widetilde{T}_8^{004} &= \left(\mathrm{Tr}[\Psi^{A\dagger}\Psi^A\sigma_3] + \mathrm{Tr}[\Psi^{B\dagger}\Psi^B\sigma_3]\right)\mathrm{Tr}[\Psi^{A\dagger}\Psi^B], \quad (3), \end{aligned} \tag{C.21}$$

whereat the two expressions are up to a prefactor identical, i.e.

$$\widetilde{T}_8^{004} = -2\widetilde{T}_7^{004}, \quad (3). \tag{C.22}$$

C.4 Class $(0, 0, 6)$

$SU(2)_Q$ invariant

From the earlier work we know that we have to locate one trace expression for $n_{AB} = 3$ and one for $n_{AB} = 4$.

For $n_{AB} = 3$ we readily get

$$3E_2^{006} = \mathrm{Tr}[\Psi^{A\dagger}\Psi^B\Psi^{A\dagger}\Psi^B\Psi^{A\dagger}\Psi^B] = \mathrm{Tr}[\Psi^{B\dagger}\Psi^A\Psi^{B\dagger}\Psi^A\Psi^{B\dagger}\Psi^A], \quad (3). \tag{C.23}$$

On the other hand for $n_{AB} = 4$ our first attempts fail, since the expressions vanish, i.e.

$$\begin{aligned} &\mathrm{Tr}[\Psi^{A\dagger}\Psi^{A}\Psi^{A\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{B}] = \mathrm{Tr}[\Psi^{A\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{B}\Psi^{B\dagger}\Psi^{B}] = 0, \quad (4), \\ &\mathrm{Tr}[\Psi^{A\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{B}\Psi^{A\dagger}\Psi^{A}] = \mathrm{Tr}[\Psi^{B\dagger}\Psi^{B}\Psi^{A\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{B}] = 0, \quad (4). \end{aligned} \tag{C.24}$$

Further efforts reveal

$$\begin{aligned} 4E_1^{006} = T_1^{006} &= \mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^{A}\Psi^{A\dagger}\sigma_3\Psi^{A}\Psi^{B\dagger}\sigma_3\Psi^{B}] - \mathrm{Tr}[\Psi^{B\dagger}\sigma_3\Psi^{B}\Psi^{B\dagger}\sigma_3\Psi^{B}\Psi^{A\dagger}\sigma_3\Psi^{A}], \quad (4), \\ T_2^{006} &= \big(\mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^{A}] - \mathrm{Tr}[\Psi^{B\dagger}\sigma_3\Psi^{B}]\big)\,\mathrm{Tr}[\Psi^{A\dagger}\sigma_3\Psi^{A}\Psi^{B\dagger}\sigma_3\Psi^{B}], \quad (4). \end{aligned} \tag{C.25}$$

Again these two expressions are related as

$$T_2^{006} = 2T_1^{006}, \quad (4), \tag{C.26}$$

such that only one linearly independent expression survives.

$SU(2)_Q$ breaking but $U(1)_Q$ invariant

The only $SU(2)_Q$ breaking trace expression we have to find has $n_{AB} = 4$. After some work we encounter

$$4\widetilde{E}_1^{006} = \mathrm{Tr}[\Psi^{A\dagger}\Psi^{A}\sigma_3\Psi^{A\dagger}\Psi^{A}\sigma_3\Psi^{B\dagger}\Psi^{B}\sigma_3] + \mathrm{Tr}[\Psi^{B\dagger}\Psi^{B}\sigma_3\Psi^{B\dagger}\Psi^{B}\sigma_3\Psi^{A\dagger}\Psi^{A}\sigma_3], \quad (4), \tag{C.27}$$

as one possible representative for it.

C.5 Class $(0, 0, 8)$

$SU(2)_Q$ invariant

Here the only existing expression has $n_{AB} = 4$. Our first guess fails, i.e.

$$\mathrm{Tr}[\Psi^{A\dagger}\Psi^{B}\Psi^{A\dagger}\Psi^{B}\Psi^{A\dagger}\Psi^{B}\Psi^{A\dagger}\Psi^{B}] = \mathrm{Tr}[\Psi^{B\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{A}] = 0, \quad (4), \tag{C.28}$$

but after a while we find

$$\begin{aligned} T_1^{008} &= \mathrm{Tr}[\Psi^{A\dagger}\Psi^{A}\Psi^{A\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{B}\Psi^{B\dagger}\Psi^{B}] = \mathrm{Tr}[\Psi^{B\dagger}\Psi^{B}\Psi^{B\dagger}\Psi^{B}\Psi^{A\dagger}\Psi^{A}\Psi^{A\dagger}\Psi^{A}], \quad (4), \\ 24E_1^{008} = T_2^{008} &= \mathrm{Tr}[\Psi^{A\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{B}\Psi^{A\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{B}] = \mathrm{Tr}[\Psi^{B\dagger}\Psi^{B}\Psi^{A\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{B}\Psi^{A\dagger}\Psi^{A}], \quad (4), \\ T_3^{008} &= \mathrm{Tr}[\Psi^{A\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{B}]\mathrm{Tr}[\Psi^{A\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{B}], \quad (4). \end{aligned} \tag{C.29}$$

Again all these expressions are up to a prefactor identical, i.e.

$$\begin{aligned} T_1^{008} &= -3T_2^{008}, \quad (4), \\ T_3^{008} &= -2T_2^{008}, \quad (4). \end{aligned} \tag{C.30}$$

As already mentioned no $SU(2)_Q$ breaking expressions exist here.

C.6 Class $(1_D, 0, 2)$

$SU(2)_Q$ invariant

For $n_{AB} = 1$ no terms that are compatible with all the symmetries can be found, i.e.

$$\begin{aligned} \mathrm{Tr}[\Psi^{A\dagger} D_t \Psi^B] &= -\mathrm{Tr}[\Psi^{B\dagger} D_t \Psi^A], \quad (D,\, T \text{ breaking}), \quad (1), \\ \mathrm{Tr}[\Psi^{A\dagger} \sigma_3 D_t \Psi^B] &= +\mathrm{Tr}[\Psi^{B\dagger} \sigma_3 D_t \Psi^A], \quad (D \text{ breaking}), \quad (1). \end{aligned} \tag{C.31}$$

This is exactly what the analysis in the previous appendix predicts.

For $n_{AB} = 2$ on the other hand we directly find the sole expression in demand, i.e.

$$2E_1^{1_D 02} = \mathrm{Tr}[\Psi^{A\dagger} D_t \Psi^A] + \mathrm{Tr}[\Psi^{B\dagger} D_t \Psi^B], \quad (2). \tag{C.32}$$

Further potential contributions indeed vanish, i.e.

$$\mathrm{Tr}[\Psi^{A\dagger} \sigma_3 D_t \Psi^A] = \mathrm{Tr}[\Psi^{B\dagger} \sigma_3 D_t \Psi^B] = 0, \quad (2). \tag{C.33}$$

$SU(2)_Q$ breaking but $U(1)_Q$ invariant

Here we must discover two $SU(2)_Q$ breaking expressions, namely one for $n_{AB} = 1$ and one for $n_{AB} = 2$.

For $n_{AB} = 1$ our first try already yields the solution

$$\widetilde{E}_1^{1_D 02} = \mathrm{Tr}[\Psi^{A\dagger} D_t \Psi^B \sigma_3] = \mathrm{Tr}[\Psi^{B\dagger} D_t \Psi^A \sigma_3], \quad (1). \tag{C.34}$$

As anticipated, further potential expressions are again unusable, here because they violate the time reversal symmetry T, i.e.

$$\mathrm{Tr}[\Psi^{A\dagger} \sigma_3 D_t \Psi^B \sigma_3] = -\mathrm{Tr}[\Psi^{B\dagger} \sigma_3 D_t \Psi^A \sigma_3], \quad (T \text{ breaking}), \quad (1). \tag{C.35}$$

For $n_{AB} = 2$ our first approach unfortunately vanishes, i.e.

$$\mathrm{Tr}[\Psi^{A\dagger} D_t \Psi^A \sigma_3] = \mathrm{Tr}[\Psi^{B\dagger} D_t \Psi^B \sigma_3] = 0, \quad (2), \tag{C.36}$$

but the second yields the desired result, i.e.

$$2\widetilde{E}_2^{1_D 02} = \mathrm{Tr}[\Psi^{A\dagger} \sigma_3 D_t \Psi^A \sigma_3] - \mathrm{Tr}[\Psi^{B\dagger} \sigma_3 D_t \Psi^B \sigma_3], \quad (2). \tag{C.37}$$

C.7 Class $(1_V, 0, 2)$

$SU(2)_Q$ invariant

As before we have to uncover one expression for $n_{AB} = 1$ and one for $n_{AB} = 2$ here.

For $n_{AB} = 1$ our first guess is incompatible with the symmetries, i.e.

$$\mathrm{Tr}[\Psi^{A\dagger} V_t \Psi^B] = -\mathrm{Tr}[\Psi^{B\dagger} V_t \Psi^A], \quad (D,\, T \text{ breaking}), \quad (1), \tag{C.38}$$

however, the next is what we are looking for, i.e.

$$E_2^{1_V 02} = \mathrm{Tr}[\Psi^{A\dagger} \sigma_3 V_t \Psi^B] = -\mathrm{Tr}[\Psi^{B\dagger} \sigma_3 V_t \Psi^A], \quad (1). \tag{C.39}$$

For $n_{AB} = 2$ we instantly find

$$2E_1^{1_V 02} = \mathrm{Tr}[\Psi^{A\dagger} V_t \Psi^A] + \mathrm{Tr}[\Psi^{B\dagger} V_t \Psi^B], \quad (2), \tag{C.40}$$

which is the wanted expression. Other potential contributions cannot be made invariant under all symmetries, i.e.

$$\mathrm{Tr}[\Psi^{A\dagger} \sigma_3 V_t \Psi^A] - \mathrm{Tr}[\Psi^{B\dagger} \sigma_3 V_t \Psi^B], \quad (T \text{ breaking}), \quad (2). \tag{C.41}$$

$SU(2)_Q$ breaking but $U(1)_Q$ invariant

Here the only $SU(2)_Q$ breaking expression has $n_{AB} = 1$. This one is rapidly identified as

$$\widetilde{E}_1^{1_V 02} = \mathrm{Tr}[\Psi^{A\dagger} V_t \Psi^B \sigma_3] = \mathrm{Tr}[\Psi^{B\dagger} V_t \Psi^A \sigma_3], \quad (1). \tag{C.42}$$

Again other potential contributions are unusable, since they are incompatible with the symmetries, i.e.

$$\mathrm{Tr}[\Psi^{A\dagger} \sigma_3 V_t \Psi^B \sigma_3] = \mathrm{Tr}[\Psi^{B\dagger} \sigma_3 V_t \Psi^A \sigma_3], \quad (D,\, T \text{ breaking}), \quad (1). \tag{C.43}$$

Furthermore all potential contributions with $n_{AB} = 2$ vanish, i.e.

$$\begin{aligned} \mathrm{Tr}[\Psi^{A\dagger} V_t \Psi^A \sigma_3] = \mathrm{Tr}[\Psi^{B\dagger} V_t \Psi^B \sigma_3] \quad &= 0, \quad (2), \\ \mathrm{Tr}[\Psi^{A\dagger} \sigma_3 V_t \Psi^A \sigma_3] = \mathrm{Tr}[\Psi^{B\dagger} \sigma_3 V_t \Psi^B \sigma_3] &= 0, \quad (2). \end{aligned} \tag{C.44}$$

Once more this result is in complete agreement with the findings in component notation.

C.8 Class $(0, 2_{DD}, 2)$

$SU(2)_Q$ invariant

Here two $SU(2)_Q$ invariant expressions exist. The one for $n_{AB} = 1$ is

$$E_2^{02_{DD}2} = \mathrm{Tr}[D_i \Psi^{A\dagger} D_i \Psi^B] = \mathrm{Tr}[D_i \Psi^{B\dagger} D_i \Psi^A], \quad (1), \tag{C.45}$$

and the other for $n_{AB} = 2$ is

$$2E_1^{02_{DD}2} = \mathrm{Tr}[D_i \Psi^{A\dagger} \sigma_3 D_i \Psi^A] - \mathrm{Tr}[D_i \Psi^{B\dagger} \sigma_3 D_i \Psi^B], \quad (2). \tag{C.46}$$

$SU(2)_Q$ breaking but $U(1)_Q$ invariant

The sole $SU(2)_Q$ breaking expression here has $n_{AB} = 2$ and reads

$$2\widetilde{E}_1^{02_{DD}2} = \mathrm{Tr}[D_i\Psi^{A\dagger}D_i\Psi^A\sigma_3] + \mathrm{Tr}[D_i\Psi^{B\dagger}D_i\Psi^B\sigma_3], \quad (2). \tag{C.47}$$

C.9 Class $(0, 2_{DV}, 2)$

$SU(2)_Q$ invariant

According to the analysis in component notation we have to discover two trace expressions here, one for $n_{AB} = 1$ and one for $n_{AB} = 2$.

Our first attempt for $n_{AB} = 1$ still lacks compliance with the displacement symmetry D, i.e.

$$\begin{aligned} T_1^{02_{DV}2} &= \mathrm{Tr}[D_i\Psi^{A\dagger}V_i\Psi^B] = -\mathrm{Tr}[\Psi^{B\dagger}V_iD_i\Psi^A], \quad (D \text{ breaking}), \quad (1), \\ T_2^{02_{DV}2} &= \mathrm{Tr}[D_i\Psi^{B\dagger}V_i\Psi^A] = -\mathrm{Tr}[\Psi^{A\dagger}V_iD_i\Psi^B], \quad (D \text{ breaking}), \quad (1). \end{aligned} \tag{C.48}$$

Fortunately this is easy to cure if we build the combination

$$E_2^{02_{DV}2} = T_1^{02_{DV}2} + T_2^{02_{DV}2}, \quad (1). \tag{C.49}$$

Similarly for $n_{AB} = 2$ we find the combination

$$E_1^{02_{DV}2} = \mathrm{Tr}[D_i\Psi^{A\dagger}\sigma_3V_i\Psi^A] - \mathrm{Tr}[D_i\Psi^{B\dagger}\sigma_3V_i\Psi^B], \quad (2), \tag{C.50}$$

to be the invariant expression in demand.

$SU(2)_Q$ breaking but $U(1)_Q$ invariant

Since here the total, $U(1)_Q$ invariant vector space is even 4-dimensional there are now still two $SU(2)_Q$ breaking expressions left to find, namely one for $n_{AB} = 1$ and one for $n_{AB} = 2$.

As before we have to combine for $n_{AB} = 1$ two traces in order to build an expression that is invariant under all symmetries, i.e.

$$\widetilde{E}_2^{02_{DV}2} = \mathrm{Tr}[D_i\Psi^{A\dagger}\sigma_3V_i\Psi^B\sigma_3] + \mathrm{Tr}[D_i\Psi^{B\dagger}V_i\sigma_3\Psi^A\sigma_3], \quad (1). \tag{C.51}$$

Likewise for $n_{AB} = 2$ we obtain

$$\widetilde{E}_1^{02_{DV}2} = \mathrm{Tr}[D_i\Psi^{A\dagger}V_i\Psi^A\sigma_3] + \mathrm{Tr}[D_i\Psi^{B\dagger}V_i\Psi^B\sigma_3], \quad (2). \tag{C.52}$$

C.10 Class $(0, 2_{VV}, 2)$

$SU(2)_Q$ invariant

Finally here over again two expressions exist, one for $n_{AB} = 1$ and one for $n_{AB} = 2$.

For $n_{AB} = 1$ we instantly find

$$E_2^{02_{VV}2} = \mathrm{Tr}[\Psi^{A\dagger} V_i V_i \Psi^B] = \mathrm{Tr}[\Psi^{B\dagger} V_i V_i \Psi^A], \quad (1), \tag{C.53}$$

while for $n_{AB} = 2$ we have to build the combination

$$2E_1^{02_{VV}2} = \mathrm{Tr}[\Psi^{A\dagger} V_i \sigma_3 V_i \Psi^A] - \mathrm{Tr}[\Psi^{B\dagger} V_i \sigma_3 V_i \Psi^B], \quad (2). \tag{C.54}$$

$SU(2)_Q$ breaking but $U(1)_Q$ invariant

Here the sole $SU(2)_Q$ breaking expression has $n_{AB} = 2$ and reads

$$2\widetilde{E}_1^{02_{VV}2} = \mathrm{Tr}[\Psi^{A\dagger} V_i V_i \Psi^A \sigma_3] + \mathrm{Tr}[\Psi^{B\dagger} V_i V_i \Psi^B \sigma_3], \quad (2). \tag{C.55}$$

D Removal of the non-canonical terms by a field redefinition

The most general $SU(2)_Q$ breaking but $U(1)_Q$ invariant expressions containing one covariant temporal derivative are given by

$$\begin{aligned} aE_1^{1_D 02} + c\widetilde{E}_1^{1_D 02} + b\widetilde{E}_2^{1_D 02} = & \quad \frac{a}{2}\mathrm{Tr}[\Psi^{A\dagger} D_t \Psi^A + \Psi^{B\dagger} D_t \Psi^B] + c\mathrm{Tr}[\Psi^{A\dagger} D_t \Psi^B \sigma_3] \\ & + \frac{b}{2}\mathrm{Tr}[\Psi^{A\dagger}\sigma_3 D_t \Psi^A \sigma_3 - \Psi^{B\dagger}\sigma_3 D_t \Psi^B \sigma_3] \\ = & \quad (\psi_+^{A\dagger}, \psi_+^{B\dagger}) \begin{pmatrix} a+b & c \\ c & a-b \end{pmatrix} \begin{pmatrix} D_t\psi_+^A \\ D_t\psi_+^B \end{pmatrix} \\ & + (\psi_-^{A\dagger}, \psi_-^{B\dagger}) \begin{pmatrix} a-b & c \\ c & a+b \end{pmatrix} \begin{pmatrix} D_t\psi_-^A \\ D_t\psi_-^B \end{pmatrix} \\ = & \quad (\widetilde{\psi}_+^{A\dagger}, \widetilde{\psi}_+^{B\dagger}) \begin{pmatrix} D_t\widetilde{\psi}_+^A \\ D_t\widetilde{\psi}_+^B \end{pmatrix} + (\widetilde{\psi}_-^{A\dagger}, \widetilde{\psi}_-^{B\dagger}) \begin{pmatrix} D_t\widetilde{\psi}_-^A \\ D_t\widetilde{\psi}_-^B \end{pmatrix}. \end{aligned} \tag{D.1}$$

Here the new fields $\widetilde{\psi}_\pm^{A,B}(x)$ result from a field redefinition that diagonalizes the matrices in the previous expression. Only the term proportional to a contains the standard form $\psi_+^{A\dagger}\partial_t\psi_+^A + \psi_-^{A\dagger}\partial_t\psi_-^A + \psi_+^{B\dagger}\partial_t\psi_+^B + \psi_-^{B\dagger}\partial_t\psi_-^B$ which implies the canonical anticommutation relations between fermionic creation and annihilation operators in the Hamiltonian

formulation. The non-canonical terms (proportional to b and c) can be removed by an appropriate field redefinition

$$\begin{pmatrix} \widetilde{\psi}^A_\pm(x) \\ \widetilde{\psi}^B_\pm(x) \end{pmatrix} = \begin{pmatrix} \sqrt{\lambda_\pm} & 0 \\ 0 & \sqrt{\lambda_\mp} \end{pmatrix} U_\pm \begin{pmatrix} \psi^A_\pm(x) \\ \psi^B_\pm(x) \end{pmatrix},$$
$$\lambda_\pm = a \pm \sqrt{b^2+c^2}, \qquad U_\pm = \begin{pmatrix} X & \pm Y \\ \mp Y & X \end{pmatrix}. \tag{D.2}$$

Here $U_\pm$ are unitary matrices with $X, Y \in \mathbb{R}$ which obey

$$U_\pm \begin{pmatrix} a \pm b & c \\ c & a \mp b \end{pmatrix} U_\pm^\dagger = \begin{pmatrix} \lambda_\pm & 0 \\ 0 & \lambda_\mp \end{pmatrix}. \tag{D.3}$$

It is straightforward to show that the redefined fields $\widetilde{\psi}^{A,B}_\pm(x)$ have the same symmetry properties of Eqs. (8.4) and (8.6) as the original fields $\psi^{A,B}_\pm(x)$. Under the $SU(2)_s$ symmetry the original fields transform as

$$\psi^{A,B}_\pm(x)' = \exp(\pm i\alpha(x))\psi^{A,B}_\pm(x), \tag{D.4}$$

and after the field redefinition again

$$\begin{aligned}
\widetilde{\psi}^A_\pm(x)' &= \sqrt{\lambda_\pm}[X\psi^A_\pm(x)' \pm Y\psi^B_\pm(x)'] \\
&= \exp(\pm i\alpha(x))\sqrt{\lambda_\pm}[X\psi^A_\pm(x) \pm Y\psi^B_\pm(x)] \quad = \exp(\pm i\alpha(x))\widetilde{\psi}^A_\pm(x), \\
\widetilde{\psi}^B_\pm(x)' &= \sqrt{\lambda_\mp}[\mp Y\psi^A_\pm(x)' + X\psi^B_\pm(x)'] \\
&= \exp(\pm i\alpha(x))\sqrt{\lambda_\mp}[\mp Y\psi^A_\pm(x) + X\psi^B_\pm(x)] = \exp(\pm i\alpha(x))\widetilde{\psi}^B_\pm(x).
\end{aligned} \tag{D.5}$$

Similarly, under the $U(1)_Q$ symmetry the original fields transform as

$${}^Q\psi^{A,B}_\pm(x) = \exp(i\omega)\psi^{A,B}_\pm(x), \tag{D.6}$$

and again

$$\begin{aligned}
{}^Q\widetilde{\psi}^A_\pm(x) &= \sqrt{\lambda_\pm}[X\ {}^Q\psi^A_\pm(x) \pm Y\ {}^Q\psi^B_\pm(x)] \\
&= \exp(i\omega)\sqrt{\lambda_\pm}[X\psi^A_\pm(x) \pm Y\psi^B_\pm(x)] \quad = \exp(i\omega)\widetilde{\psi}^A_\pm(x), \\
{}^Q\widetilde{\psi}^B_\pm(x) &= \sqrt{\lambda_\mp}[\mp Y\ {}^Q\psi^A_\pm(x) + X\ {}^Q\psi^B_\pm(x)] \\
&= \exp(i\omega)\sqrt{\lambda_\mp}[\mp Y\psi^A_\pm(x) + X\psi^B_\pm(x)] = \exp(i\omega)\widetilde{\psi}^B_\pm(x).
\end{aligned} \tag{D.7}$$

Under the modified displacement symmetry D' one has

$${}^{D'}\psi^{A,B}_\pm(x) = \pm\psi^{B,A}_\mp(x), \tag{D.8}$$

and after the field redefinition one again obtains

$$\begin{aligned}
{}^{D'}\widetilde{\psi}^{A}_{\pm}(x) &= \sqrt{\lambda_{\pm}}[X\ {}^{D'}\psi^{A}_{\pm}(x) \pm Y\ {}^{D'}\psi^{B}_{\pm}(x)] \\
&= \pm\sqrt{\lambda_{\pm}}[X\psi^{B}_{\mp}(x) \pm Y\psi^{A}_{\mp}(x)] \quad = \pm\widetilde{\psi}^{B}_{\mp}(x), \\
{}^{D'}\widetilde{\psi}^{B}_{\pm}(x) &= \sqrt{\lambda_{\mp}}[\mp Y\ {}^{D'}\psi^{A}_{\pm}(x) + X\ {}^{D'}\psi^{B}_{\pm}(x)] \\
&= \pm\sqrt{\lambda_{\mp}}[\mp Y\psi^{B}_{\mp}(x) + X\psi^{A}_{\mp}(x)] = \pm\widetilde{\psi}^{A}_{\mp}(x).
\end{aligned} \tag{D.9}$$

Since the displacement symmetry D is a combination of D' and $SU(2)_s$ it also maintains its original form. The same is true for the discrete symmetries O and R. Finally, under the modified time-reversal T' the original fields transform as

$${}^{T'}\psi^{A,B}_{\pm}(x) = -\psi^{A,B\dagger}_{\pm}(Tx), \tag{D.10}$$

such that

$$\begin{aligned}
{}^{T'}\widetilde{\psi}^{A}_{\pm}(x) &= \sqrt{\lambda_{\pm}}[X\ {}^{T'}\psi^{A}_{\pm}(x) \pm Y\ {}^{T'}\psi^{B}_{\pm}(x)] \\
&= -\sqrt{\lambda_{\pm}}[X\psi^{A\dagger}_{\pm}(Tx) \pm Y\psi^{B\dagger}_{\pm}(Tx)] \quad = -\widetilde{\psi}^{A\dagger}_{\pm}(Tx), \\
{}^{T'}\widetilde{\psi}^{B}_{\pm}(x) &= \sqrt{\lambda_{\mp}}[\mp Y\ {}^{T'}\psi^{A}_{\pm}(x) + X\ {}^{T'}\psi^{B}_{\pm}(x)] \\
&= -\sqrt{\lambda_{\mp}}[\mp Y\psi^{A\dagger}_{\pm}(Tx) + X\psi^{B\dagger}_{\pm}(Tx)] = -\widetilde{\psi}^{B\dagger}_{\pm}(Tx).
\end{aligned} \tag{D.11}$$

As a combination of T' and $SU(2)_s$ the time-reversal symmetry T also maintains its original form after the field redefinition. The only symmetry that does not maintain its original form is $SU(2)_Q$. This is no problem since the non-canonical terms can arise only when the $SU(2)_Q$ symmetry is explicitly broken down to $U(1)_Q$ and is hence no longer a symmetry of the theory.

Since the redefined fields transform exactly like the original ones, the terms in the effective Lagrangian take exactly the same form as before. Hence, it is indeed justified not to include the non-canonical terms in the effective Lagrangian.

E WZW term for magnons

It is possible to construct a symmetry invariant magnon action term which contains only a single temporal derivative, i.e.

$$S_{\mathrm{WZW}}[\vec{e}] = -iM \int d^2x \int_{H^2} dt\ d\tau\ \vec{e}\cdot(\partial_t\vec{e}\times\partial_\tau\vec{e})\,. \tag{E.1}$$

This so-called WZW term (Wess, Zumino, Witten) is of topological nature and its construction involves the extension of the $(2+1)$-dimensional space-time into a fourth direction with a coordinate $\tau \in [0,1]$. The integration runs over the 2-dimensional hemisphere H^2 with the compactified Euclidean time S^1 (corresponding to $\tau = 0$) as its boundary.

The requirement that the $(2+1)$-dimensional physics at $\tau = 0$ must be independent of the magnon field in the bulk of the hemisphere in the fourth direction (corresponding to $\tau \in (0, 1]$) constraints the prefactor of the topological term to be quantized and correspond to the magnetization M (i.e. the total spin) of the system. Since for antiferromagnetic magnons the magnetization M is zero the WZW term vanishes and the dispersion relation remains indeed relativistic. On the other hand, for ferromagnetic magnons the WZW term is present and renders the dispersion relation non-relativistic. The construction of the WZW term is discussed in more detail in [79].

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Curriculum vitae

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1989-1992	Studies in electrical engineering at School of Engineering and Information Technology in Biel
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