University of Bern

Institute for Theoretical Physics,
Albert Einstein Center for Fundamental Physics

Bachelor Thesis

Dimensional Regularisation and Asymptotic Freedom of the δ-Function Potential in Relativistic Quantum Mechanics

Author:
Dario Müller

Supervisor:
Prof. Dr. U.-J. Wiese
Institute for Theoretical Physics,
University of Bern

September 17, 2014

Abstract

In this thesis the Hamiltonian for a relativistic particle in a δ-potential is investigated. Thereby I work with the concept of renormalization, an important instrument in quantum field theory. As a final result, I obtain the wave function for the bound and the scattering states in position space. Furthermore I examine asymptotic freedom and I introduce the β-function of the coupling constant.

Contents

Chapter 1

Introduction

When I started to study physics, several times I was shown the diagram in fig.(1.1). Slow and big objects

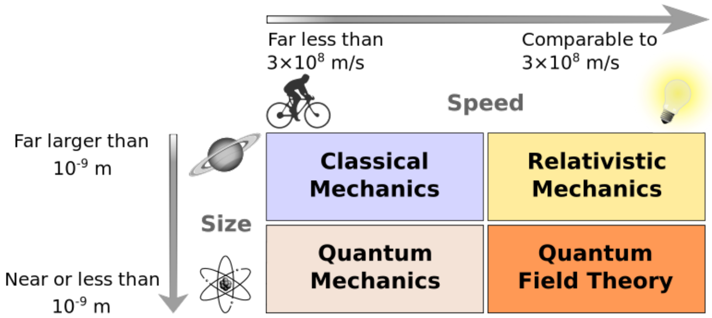

Figure 1.1: Divisions of physics for which each theory gives accurate results [1].

can be described physically well with classical mechanics. For big objects with a speed comparable to the speed of light we use relativistic mechanics. Small and slow particles are treated with quantum mechanics. Small, relativistic objects are described up today with high accuracy by quantum field theory.
However, this leads to the question why we have to make use of an even more unintuitive quantum field theory, which is no longer based on particles itself but on fields? Alternatively, one could attempt to extend the known quantum theory such that relativistic effects are also respected. In this context we have to respect the non interaction theorem [5].
In this thesis we use quantum mechanical as well as quantum fieldtheoretical concepts. Therefore we can consider it as an in-between of these two theories.
The thesis covers come issues contained in a paper published by Al-Hashimi, Shalaby, and Wiese [2] but was developed independently.

1.1 Quantum mechanical overview

When we consider a particle with mass m in quantum mechanics, its physical characteristics are described by the wave function $\Psi(\vec{x},t)$. The probability for finding this particle in an interval $[a,b]$ can be calculated as follows

$$P_{a\leq x\leq b}(t) = \int_a^b |\Psi(\vec{x},t)|^2 dx \ . \tag{1.1}$$

We can determine the wave function by solving the Schrödinger equation

$$i\hbar\frac{\partial\Psi(\vec{x},t)}{\partial t} = \hat{H}\Psi(\vec{x},t) \ . \tag{1.2}$$

$\hat{H}$ is the Hamiltonian, where the hat illustrates its function as a quantum mechanical operator. The Hamiltonian itself consists of the particle's (kinetic) energy and the potential, to which it is exposed. The bracket indicates that in classical mechanics only a kinetic term occurs, contrary to the relativistic

situation where the rest energy enters as well.
As an example, in the non-relativistic case the Hamiltonian usually has the following form (possibly obtained by the Legendre transformation of the Lagrangian)

$$\mathcal{H} = \frac{\vec{p}}{2m} + V(\vec{x}, t) \ . \tag{1.3}$$

If we replace the classical expressions by quantum mechanical operators

$$\vec{p} \to \hat{\vec{p}} = -i\hbar\vec{\nabla} \quad , \quad \vec{x} \to \hat{\vec{x}} = \vec{x} \quad , \quad E \to \hat{E} = i\hbar\frac{\partial}{\partial t} \ , \tag{1.4}$$

we can rewrite the Hamiltonian

$$\hat{H} = \frac{-\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{x}, t) \ , \tag{1.5}$$

insert this in eq.(1.2) and obtain the relation

$$i\hbar\frac{\partial\Psi(\vec{x}, t)}{\partial t} = \left[\frac{-\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{x}, t)\right]\Psi(\vec{x}, t) \tag{1.6}$$

(remember that $\vec{\nabla}^2 = \Delta$).
From now on, we only consider time-independent situations (where this independence refers particularly to the potential)

$$\Psi(\vec{x}, t) \to \Psi(\vec{x}) \ . \tag{1.7}$$

We can write the time-independent Schrödinger equation in a brief and general form

$$\hat{H}\Psi(\vec{x}) = E\Psi(\vec{x}) \ . \tag{1.8}$$

If we act with the Hamiltonian on $\Psi(\vec{x})$ and the result is proportional to the wave function, we can directly read off the energy of the particle. In more mathematical terms: $\Psi(\vec{x})$ is the eigenvector of the Hamiltonian with the eigenvalue E that corresponds to the energy of the particle.
As a further step we reduce the problem to one dimension $\Psi(\vec{x}) \to \mathit{\Psi}(x)$. This is the final situation, for which we will solve the relativistic Schrödinger equation.
If one is interested in a more detailed to non-relativistic classical quantum mechanics and perturbation theory, I would recommend *Griffiths, Quantenmechanik* [3].

In relativistic quantum mechanics the Hamiltonian for a time independent and one-dimensional situation takes the form

$$\hat{H} = \sqrt{m^2c^2 - c^2\hbar^2\frac{\partial^2}{\partial x^2}} + V(x) \ . \tag{1.9}$$

One may be sceptical about the first term, which contains a square root of a second derivative. What does this mean mathematically? Either one takes a derivative or not, but how does a so-called pseudo-differential operator act? We could expand the square root as a series, but the higher-order terms do not necessarily converge such that we have to deal with derivatives of infinite order, which are no longer point like operations. In this thesis we will avoid such troubles.
From now on, we will work in natural units. This means $c = \hbar = 1$.

1.2 No-interaction theorem

In 1963, Currie, Jordan and Sudarshan proved that two particles can not interact without violating the principles of relativity [4]. This was extended by H. Leutwyler (a retired professor at the university of Bern) to an arbitrary number of particles [5]. However, particle interactions play a fundamental role in physics and we need to treat them. Thus, in general quantum field theory needs to be introduced. I like to give a very short overview of the problem and the exceptions. I recommend the thesis of Daniel Klauser [6], who examined this topic in more detail.
In one dimension, under the use of the composition law for infinitesimal transformations the Hamiltonian $\hat{H}$, the momentum operator $\hat{P}$, and the boost generator $\hat{K}$ (the generators of the Poincaré group) obey

$$[\hat{P}, \hat{H}] = 0 \quad , \quad [\hat{K}, \hat{H}] = i\hat{P} \quad , \quad [\hat{K}, \hat{P}] = i\hat{H} \ . \tag{1.10}$$

They form the so-called Poincaré algebra.
Leutwyler was able to show that at the classical level no potential exists such that these conditions are fulfilled. Therefore no particle interactions occur. This is the no-interaction theorem.
There is one exception: For the one-dimensional linear potential eq.(1.10) is fulfilled.

1.3 The δ-potential

In this thesis we work with the δ-potential

$$V(x) = \lambda\delta(x) \ , \tag{1.11}$$

where λ is the coupling constant and $\delta(x)$ the Dirac delta. It is defined as follows

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} \tag{1.12}$$

and

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a) \ . \tag{1.13}$$

Especially, we have for the constant function $f(x) = 1$

$$\int_{-\infty}^{\infty} \delta(x-a) dx = 1 \ . \tag{1.14}$$

It is no potential in the classical sense. Therefore the δ-potential was not considered by Leutwyler for the no-interaction theorem. It is possible that it fulfils eq.(1.10). In this thesis we do not address this issue, because we work with a system that is not invariant under translations.

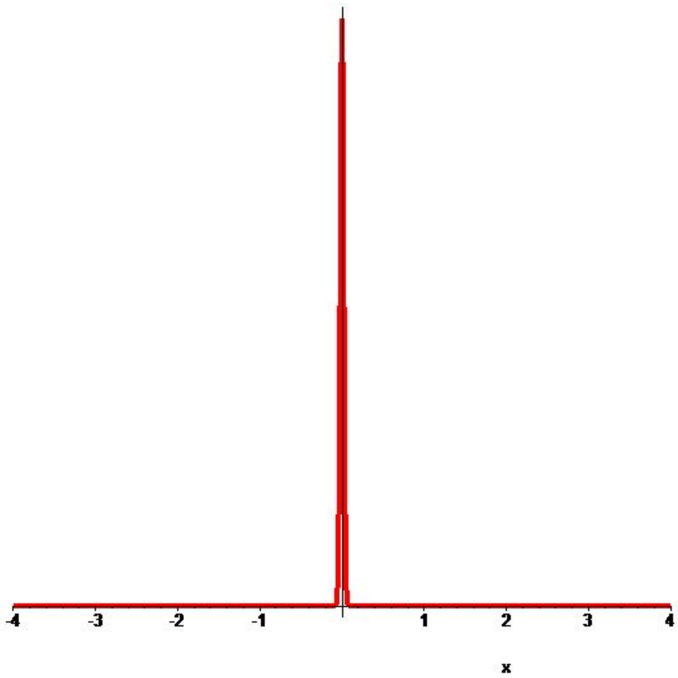

Figure 1.2: The potential $V(x) = \lambda\delta(x)$ with a positive coupling constant $\lambda > 0$.

1.4 Dimensional Regularisation

Before we deal with the regularisation of ultraviolet divergences in a mathematical way, I would like to give a short overview. The historical informations are taken from a very interesting article by W. Bietenholz and L. Prado in *Physics Today* [7].
In physics we use regularisation and renormalisation to deal with divergent expressions. In dimensional regularisation the dimension of space-time is analytically continued to complex values in order to identify poles associated with integer dimensions. The (dimensional) regularisation is used primarily in quantum field theory, but also in this thesis.
We can distinguish between the minimal subtraction scheme MS, where we only reduce the divergent

part and the modified minimal subtraction or MS-bar scheme $\overline{MS}$, where also a constant term is affected. The $\overline{MS}$ scheme is used more commonly and is also used in this thesis.
In 1971 a new method to handle divergences was found by the Argentinian theoretical physicists C. G. Bollini and J. J. Giambiagi. The situation under which the two professors had to do research is remarkable. From 1966 to 1973 a repressive military dictatorship complicated the work on such an important discovery, that changed and advanced particle physics. For a long time, all this was not widely attributed to them.
Giambiagi and Bollini wrote an article and sent it to the Dutch journal *Physics Letter B* for publishing. It was refused with the explanation of being too strange. They wrote a second article presenting their new approach and sent it to the Italian journal *Il Nuovo Cimento B* where it was released half a year later. Meanwhile the Dutch physicists G. 't Hooft and M. Veltman included the idea of dimensional regularisation into their more extensive work. Hooft and Veltman proved the renormalisability of Yang-Mills theory, the description of the strong and weak force. These results were a basis for the introduction of the standard model.
G. 't Hooft and M. Veltman were awarded the Nobel prize in 1999.

1.5 Additional calculations

In this thesis we will meet several rather complicated integrals. Long and non-trivial calculations are described in the appendix. This will be announced in each case.
All non-analytical calculations as well as the plots of the wave functions are realised using *MAPLE 8*. The sketches in the appendix are drawn by *GeoGebra*. To plot the β-function, I wrote a program in C and plotted with *ROOT*.

Chapter 2

Solution of the non-relativistic problem

At first we start with the solution of the non-relativistic problem. Hence we solve the Schrödinger equation of the form

$$\frac{-1}{2m}\frac{\partial^2}{\partial x^2}\Psi(x) + \lambda\delta(x)\Psi(x) = E\Psi(x) \; . \tag{2.1}$$

As already mentioned, λ can be positive or negative. We distinguish between the bound and the scattering states of the particle.

2.1 $\lambda < 0$

2.1.1 Solution for the bound state

For the bound state the particle's energy E_B is negative, such that energy has to be added to obtain an unbound particle. This can occur only if the potential peak is negative. This means $\lambda < 0$.
To solve eq.(2.1) we consider the two regions $x < 0$ and $x > 0$ separately and finally connect these two solutions.
In the case $x < 0$ we have the differential equation

$$\frac{\partial^2 \Psi_I}{\partial x^2} = -2mE_B\Psi_I = \kappa^2\Psi_I \tag{2.2}$$

with

$$\kappa = \pm\sqrt{-2mE_B} \; . \tag{2.3}$$

Because $E_B < 0$, κ is a real number.
The general solution of this problem is easy to find and looks as follows

$$\Psi_I(x) = Ae^{-\kappa x} + Be^{\kappa x} \; . \tag{2.4}$$

An analogous calculation for $x > 0$ gives

$$\Psi_{II}(x) = Ce^{-\kappa x} + De^{\kappa x} \; . \tag{2.5}$$

The wave function still must be normalised, which is possible only if we set $A = D = 0$. A further condition is the continuity of the wave function, which implies

$$\Psi_I(x = 0) = \Psi_{II}(x = 0) \Rightarrow B = C \tag{2.6}$$

and it follows that

$$\Psi_B(x) = Be^{-\kappa|x|} \; . \tag{2.7}$$

The index B refers to the bound state.
Let's now consider the first derivative of the wave function. It should be continuous in every location except at the position of the potential peak. We analyse the behaviour of the function around $x = 0$ by

integrating the Schrödinger equation (2.1) from $-\epsilon$ to ϵ and examine it in the limit $\epsilon \to 0$

$$\lim_{\epsilon\to 0}\left(\frac{-1}{2m}\int_{-\epsilon}^{\epsilon}\frac{\partial^2\Psi_B}{\partial x^2}dx + \lambda\int_{-\epsilon}^{\epsilon}\delta(x)\Psi_B(x)dx = E_B\int_{-\epsilon}^{\epsilon}\Psi_B(x)dx\right) \ . \tag{2.8}$$

The first term contains an integration of a second derivative and therefore becomes a derivative of first order, executed at the integration boundaries. The second term can be computed with the definition of the δ-distribution. The term on the right-hand side becomes 0 when ϵ tends to 0

$$\lim_{\epsilon\to 0}\frac{-1}{2m}\left(\left.\frac{d\Psi_B}{dx}\right|_{\epsilon} - \left.\frac{d\Psi_B}{dx}\right|_{-\epsilon}\right) + \lambda\Psi_B(0) = 0 \ . \tag{2.9}$$

Finally we obtain

$$-2\kappa B = 2m\lambda\Psi_B(0) \ . \tag{2.10}$$

Using

$$\Psi_B(0) = B \tag{2.11}$$

and the definition of κ in eq.(2.3) this leads to the binding energy

$$E_B = -\frac{\lambda^2 m}{2} \ . \tag{2.12}$$

As a final step we normalise the wave function to obtain the still unknown factor B. This is a simple integral

$$\int_{-\infty}^{\infty}|\Psi_B(x)|^2dx = 2\int_0^{\infty}|B|^2e^{-2\kappa x}dx = \frac{|B|^2}{\kappa} \overset{!}{=} 1 \Rightarrow B = \sqrt{\kappa} \ . \tag{2.13}$$

Hence we obtain the wave function for a non-relativistic bound state (remember that λ is negative)

$$\Psi_B(x) = \sqrt{\kappa}e^{-\kappa|x|} = \sqrt{-m\lambda}e^{m\lambda|x|} \ . \tag{2.14}$$

2.1.2 Solution for the scattering states

Now we consider the other case, where the particle's energy is bigger than zero. As we did in the bound state case, we divide the system into two parts, one where x is smaller than zero and the other one where x is bigger than zero. Finally we connect the results using the same conditions as for the bound state.
Let us consider the Schrödinger equation (2.1). However, as already said, with an energy $E > 0$. Therefore we have to solve for $x < 0$

$$\frac{\partial^2\Psi_I}{\partial x^2} = -2mE\Psi_I = -k^2\Psi_I \ , \tag{2.15}$$

with

$$k = \sqrt{2Em} \tag{2.16}$$

as a real number.
The solution of this problem has the well-known form

$$\Psi_I = Ae^{ikx} + Be^{-ikx} \ . \tag{2.17}$$

In an analogous approach for the region $x > 0$ one obtains

$$\Psi_{II} = Ce^{ikx} + De^{-ikx} \ . \tag{2.18}$$

So we have

$$\Psi_k(x) = \begin{cases} Ce^{ikx} + De^{-ikx} & x > 0 \\ Ae^{ikx} + Be^{-ikx} & x < 0 \end{cases} \ . \tag{2.19}$$

The index refers to a scattering state with momentum k.
First of all, the wave function has to be continuous such that

$$\Psi_I(x=0) \overset{!}{=} \Psi_{II}(x=0) \Rightarrow A + B = C + D \ . \tag{2.20}$$

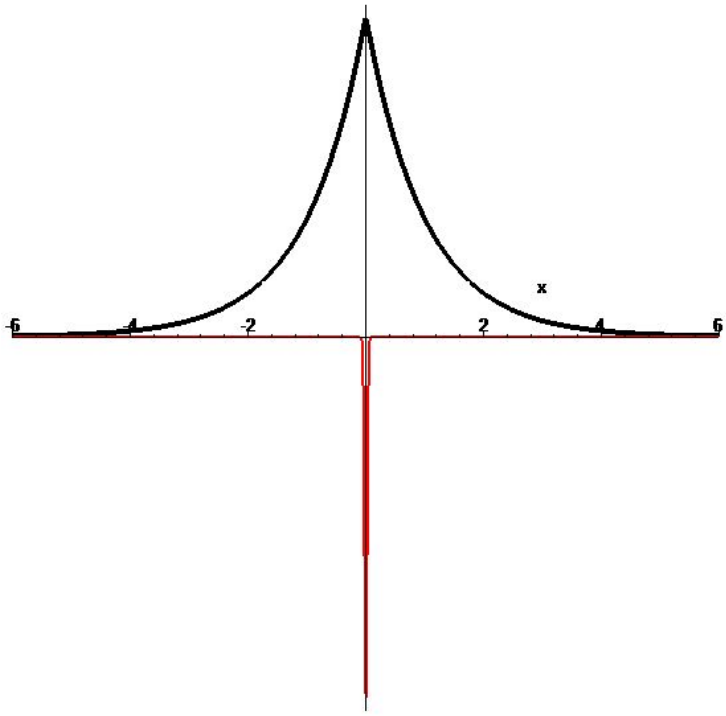

Figure 2.1: Solution of the Schrödinger equation for the bound state for $m = 1$ and $\lambda = -1$. The black line describes the wave function, the red line the potential. Remember that the probability to find a particle in a region is equal to the integration of the squared wave function over this region (see eq.(1.1)). Therefore it can be seen easily that the particle favours the region around the δ-peak. Although it can appear farther away, but with exponentially suppressed probability.

Now we consider the first derivative of the wave function and again it should be continuous everywhere except at $x = 0$ where the δ-peak is located. Therefore we integrate the solution over a range from $-\epsilon$ to ϵ and take the limit ϵ to zero. We then obtain

$$\begin{aligned} \lim_{\epsilon\to 0}\left(\frac{d\Psi_k}{dx}\bigg|_{\epsilon} - \frac{d\Psi_k}{dx}\bigg|_{-\epsilon}\right) &= 2m\lambda\Psi_k(0) \quad \text{with} \quad \Psi_k(0) = A + B \\ &\Rightarrow ik(C - D + B - A) = 2m\lambda(A + B) \\ \Rightarrow C - D &= \left(\frac{-2im\lambda}{k} + 1\right)A + \left(\frac{-2im\lambda}{k} - 1\right)B \end{aligned} \tag{2.21}$$

For simplification we set

$$\alpha = -\frac{m\lambda}{k} \tag{2.22}$$

and get the following condition

$$C - D = (2i\alpha + 1)A + (2i\alpha - 1)B \ . \tag{2.23}$$

This is already the final result.

However there is another interesting component:
If we take a particle, that starts in the region $x < 0$, its wave function is a linear combination of a left- and a right-moving stationary wave. For $x > 0$ only the right-moving part exists. Therefore we can set $D = 0$. The wave function then has the following form

$$\Psi_k(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < 0 \\ Ce^{ikx} & x > 0 \end{cases} \ . \tag{2.24}$$

Because of the continuity at $x = 0$, we obtain $A + B = C$. We insert this into the condition required by eq.(2.23). With some simple algebraic operations we obtain

$$\frac{B}{A} = \frac{-i\alpha}{i\alpha - 1} \; ,$$
$$\frac{C}{A} = \frac{1}{1 - i\alpha} \; . \tag{2.25}$$

Let us now investigate what eq.(2.25) and eq.(2.25) really mean. As we wanted the particle to start in the region $x < 0$ and let it run towards the potential peak, we can observe two possible events: Either the particle passes the potential peak or it is reflected. In the reflected case we have a back-moving wave with an amplitude B. Otherwise it passes and continues as a wave with an amplitude C. Hence B over A is simply the part of the incoming wave which is reflected, C over A the part which is transmitted. This leads to the following quantum mechanical interpretation: The square of the absolute value of eq.(2.25) gives us the reflection probability $|R|^2$. Analogously we obtain the transmission coefficient $|T|^2$ from eq.(2.25)

$$|R|^2 = \frac{|B|^2}{|A|^2} = \frac{\alpha^2}{(i\alpha - 1)(-i\alpha - 1)} = \frac{\alpha^2}{1 + \alpha^2} = \frac{m^2\lambda^2}{k^2 + m^2\lambda^2} \; , \tag{2.26}$$

$$|T|^2 = \frac{|C|^2}{|A|^2} = \frac{1}{(1 - i\alpha)(1 + i\alpha)} = \frac{1}{1 + \alpha^2} = \frac{k^2}{k^2 + m^2\lambda^2} \; . \tag{2.27}$$

It is easy to see that $|R|^2 + |T|^2 = 1$ (which has to be valid, because the particle can either be reflected or transmitted).

2.2 $\lambda > 0$

In this case no bound state exists. Therefore we only consider the scattering states.

2.2.1 Solution for the scattering states

The procedure is exactly the same as for the negative coupling constant. We solve the Schrödinger equation (2.1) by dividing the system in a region $x < 0$ and another one $x > 0$, solve them separately and then put them together. For $x < 0$ we obtain

$$\frac{\partial^2 \Psi_I}{\partial x^2} = -2mE\Psi_I = -k^2\Psi_I \; , \tag{2.28}$$

with

$$k = \sqrt{2mE} \; . \tag{2.29}$$

The solution of this differential equation is

$$\Psi_I = Ae^{ikx} + Be^{-ikx} \; . \tag{2.30}$$

The same for $x > 0$ leads to

$$\Psi_{II} = Ce^{ikx} + De^{-ikx} \; . \tag{2.31}$$

Due to continuity, it is required that

$$A + B = C + D \; . \tag{2.32}$$

Again we consider the first derivative of the wave function at $x = 0$. This is done exactly in the same way as before and finally leads to

$$ik(C - D + B - A) = 2m\lambda\Psi(0) \; ,$$
$$\Rightarrow C - D = (2i\beta + 1)A + (2i\beta - 1)B \; , \tag{2.33}$$

where we introduced

$$\beta = -\frac{m\lambda}{k} \; . \tag{2.34}$$

As in the previous chapter, we can start the particle right-moving on the negative x-axis. Therefore D becomes 0. With a little algebraic effort we obtain

$$\frac{B}{A} = \frac{i\beta}{i\beta - 1} ,$$
$$\frac{C}{A} = \frac{1}{1 - i\beta} . \tag{2.35}$$

With the same thoughts as before we conclude that the square of the absolute value gives us the reflection respectively the transmission probability of the particle

$$|R|^2 = \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2} = \frac{m^2\lambda^2}{k^2 + m^2\lambda^2} , \tag{2.36}$$

$$|T|^2 = \frac{|C|^2}{|A|^2} = \frac{1}{1+\beta^2} = \frac{k^2}{k^2 + m^2\lambda^2} . \tag{2.37}$$

Again, it is easy to check that $|R|^2 + |T|^2 = 1$.
If we compare the results for $|R|^2$ and $|T|^2$ to those with negative λ, we discover a very interesting fact: If we take a close look at the definitions of β (eq.(2.34)) and α (eq.(2.22)), we see that $\alpha = -\beta$ (because of the different sign of λ). However the transmission and reflection probabilities depend only on the square of α and β. Hence these probabilities do not care about the sign of λ. In other words: The reflection/-transmission probability is the same, independent of whether the potential is positive or negative. This is a really interesting and unexpected result.
Another case we can consider the limit of λ going to infinity. Then β (or α) tends to infinity and therefore $|R|^2$ to one and $|T|^2$ to zero.

2.3 Alternative solution with sine and cosine functions

When we solved the Schrödinger equation for the scattering states, we obtained a function consisting of linear combinations of plane waves, expressed by the exponential function. However, it is also possible to write this solution as a linear combination of parity-odd sine and parity-even cosine functions. If both ansätze solve the Schrödinger equation themselves, every linear combination is a solution as well. And hence we can construct plane waves as before using the well-known relation [8, p.31]

$$e^{ikx} = \cos(kx) + i\sin(kx) . \tag{2.38}$$

Parity-odd ansatz

We start with a simple case. Therefore we take the following ansatz for the wave function

$$\Psi_k(x) = B\sin(kx) , \tag{2.39}$$

with $k = \sqrt{2mE}$ as the quantum mechanical.
Here we do not even need to calculate anything. Sine is an odd function, this means: $\Psi(-x) = -\Psi(x)$. Notably it is $\Psi(0) = 0$. Then we have $|\Psi|^2 = 0$. So the particle is not at all influenced by the potential which only appears at $x = 0$, because the particle never stays there.

Parity-even ansatz

The case with the cosine function is a little bit more complex. As an ansatz for an even function we take

$$\Psi_k(x) = A\cos(k|x| + \varphi_0) . \tag{2.40}$$

Here φ_0 is a phase which we are going to calculate.
It is obvious that this function is continuous. Again we take a closer look at the derivative at $x = 0$

$$\lim_{\epsilon\to 0}\left(\frac{-1}{2m}\int_{-\epsilon}^{\epsilon}\frac{\partial^2\Psi_k}{\partial x^2}dx + \lambda\int_{-\epsilon}^{\epsilon}\delta(x)\Psi_k(x)dx = E\int_{-\epsilon}^{\epsilon}\Psi_k(x)dx\right) . \tag{2.41}$$

Like before, the integral with the δ-function becomes $\Psi_k(0)$ and the term on the right-hand side becomes zero. So we obtain

$$\lim_{\epsilon\to 0}\left(\frac{-1}{2m}\frac{\partial\Psi_k}{\partial x}\bigg|_{-\epsilon}^{\epsilon}+\lambda\Psi_k(0)\right)=0\ ,$$

$$\lim_{\epsilon\to 0}\left(-Ak\sin\left(k|x|+\varphi_0\right)\frac{d|x|}{dx}\bigg|_{-\epsilon}^{\epsilon}\right)=2m\lambda\Psi_k(0)=2m\lambda A\cos(\varphi_0)\ . \tag{2.42}$$

With

$$\frac{d|x|}{dx}=\begin{cases}-1 & x<0\\ 1 & x>0\end{cases}\ , \tag{2.43}$$

we finally obtain

$$\lim_{\epsilon\to 0}\left(-2Ak\sin(k\epsilon+\varphi_0)\right)=2m\lambda A\cos(\varphi_0)\ . \tag{2.44}$$

We take the limit and solve this equation for φ_0

$$\frac{\sin(\varphi_0)}{\cos(\varphi_0)}=\tan(\varphi_0)=-\frac{m\lambda}{k}\quad\Rightarrow\quad\varphi_0=\arctan(-\frac{m\lambda}{k})\ . \tag{2.45}$$

Under this additional condition for φ_0, the wave function (2.40) solves the non-relativistic Schrödinger equation, too. As a comment, we should consider the sign of λ which changes the sign of the phase and therefore the form of the wave function (compare to fig.(2.2) and fig.(2.3)). If the coupling constant is negative, the particle favours the region around $x=0$, indicated by the narrow peak in the figure (2.2), contrary to the case for $\lambda>0$.

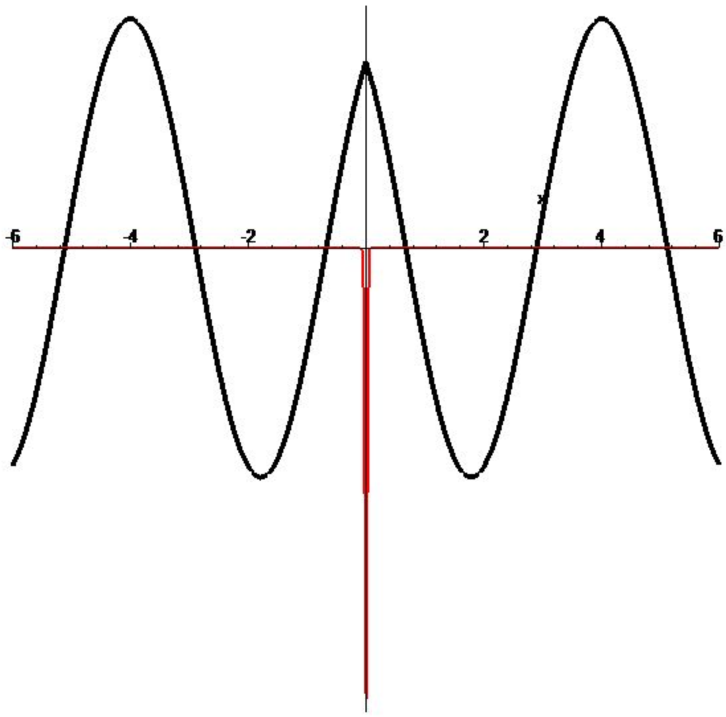

Figure 2.2: The even wave function in eq.(2.40) with $m=1$, $E=1$ and $\lambda=-1$.

2.4 Solution in momentum space

Especially for the relativistic case it is inevitable to consider the problem in momentum space. Therefore we deal with this method in the non-relativistic considerations as well.

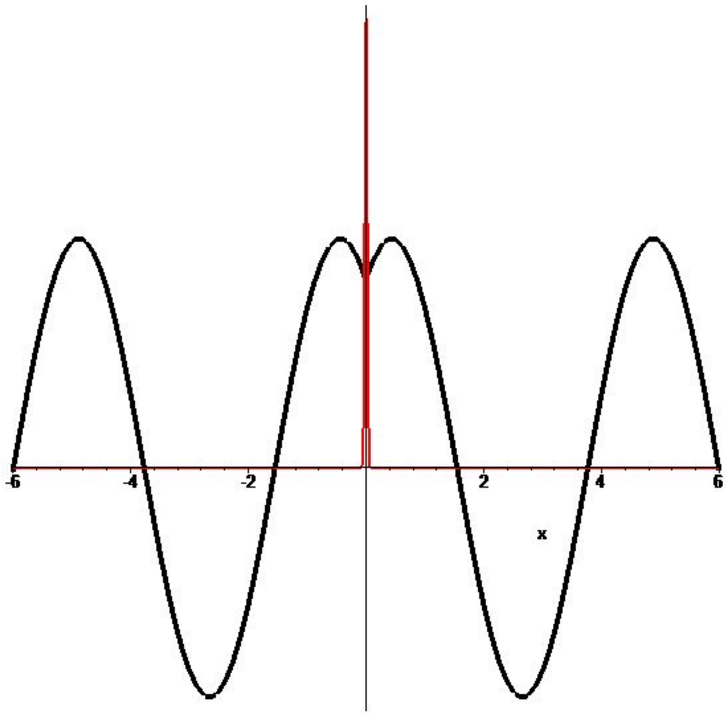

Figure 2.3: The even wave function with $m = 1$, $E = 1$ and $\lambda = 1$.

We define the Fourier transform as follows

$$\boxed{\tilde{\Psi}(p) = \int_{-\infty}^{\infty} \Psi(x) e^{-ipx} dx} \ , \tag{2.46}$$

with the inverse transform

$$\boxed{\Psi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\Psi}(p) e^{ipx} dp} \ . \tag{2.47}$$

From the inverse Fourier transform we extract another useful relation

$$\Psi(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\Psi}(p) dp \tag{2.48}$$

2.4.1 Bound state

Let us first consider the bound case $E_B < 0$. We take the Schrödinger equation in momentum space

$$\frac{p^2}{2m} \tilde{\Psi}_B(p) + \lambda \Psi_B(0) = E_B \tilde{\Psi}_B(p) \tag{2.49}$$

and solve it $\tilde{\Psi}_B(p)$. This gives

$$\tilde{\Psi}_B(p) = \frac{\lambda \Psi(0)}{E_B - \frac{p^2}{2m}} \ . \tag{2.50}$$

By using eq.(2.48) we write

$$\Psi_B(0) = \frac{1}{2\pi} \int_{\infty}^{\infty} \tilde{\Psi}_B(p) dp = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\lambda \Psi_B(0)}{E_B - \frac{p^2}{2m}} dp \ . \tag{2.51}$$

As $\Psi_B(0)$ is just a constant, we cancel it and get

$$\frac{\lambda}{2\pi} \int_{-\infty}^{\infty} \frac{1}{E_B - \frac{p^2}{2m}} dp = 1 \ . \tag{2.52}$$

This is a so-called gap equation.
Because E_B is negative there are no poles on the real axis what could cause trouble. With the help of the residue theorem it is rather easy to solve this integral. It becomes

$$-\frac{\lambda}{2\pi}\frac{\pi\sqrt{2mE_B}}{E_B} = 1 \ , \tag{2.53}$$

such that finally

$$E_B = -\frac{\lambda^2 m}{2} \ . \tag{2.54}$$

We transform $\tilde{\Psi}(p)$ back to position space

$$\Psi_B(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\Psi}_B(p)e^{ipx}dp = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\lambda\Psi_B(0)e^{ipx}}{E_B - \frac{p^2}{2m}}dp = \frac{\lambda\Psi_B(0)}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ipx}}{\frac{-\lambda^2 m}{2} - \frac{p^2}{2m}}dp \ . \tag{2.55}$$

The integrand has two poles at $p = \pm i\lambda m$ with $\lambda < 0$. We extend the function to the complex plane $p \to z$ and use the residual theorem. The integration path is composed of an integral along the real axis from $-R$ to R and semi-circle with radius R in the plane $\Im(z) > 0$. The integral over the second path disappears for $R \to \infty$ and what remains is the integral that we want to calculate

$$\int_{-\infty}^{\infty}\frac{e^{ipx}}{\frac{-\lambda^2 m}{2} - \frac{p^2}{2m}}dp = 2\pi i \ Res\left(\frac{e^{izx}}{\frac{-\lambda^2 m}{2} - \frac{z^2}{2m}}, z = -i\lambda m\right) = \frac{2\pi}{\lambda}e^{m\lambda x} \ . \tag{2.56}$$

So we have

$$\frac{\lambda\Psi_B(0)}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ipx}}{\frac{-\lambda^2 m}{2} - \frac{p^2}{2m}}dp = \frac{\lambda\Psi_B(0)}{2\pi}\frac{2\pi}{\lambda}e^{m\lambda x} = \Psi_B(0)e^{m\lambda x} \ . \tag{2.57}$$

If we use a negative value for x, we obtain $\Psi_B(0)e^{-m\lambda x}$. Therefore we can write

$$\Psi_B(x) = \Psi_B(0)e^{m\lambda|x|} \ . \tag{2.58}$$

As a final step, we still have to normalise the wave function

$$\int_{-\infty}^{\infty}|\Psi_B(x)|^2dx = \Psi_B(0)^2\int_{-\infty}^{\infty}e^{2m\lambda|x|}dx = 2\Psi_B(0)^2\int_0^{\infty}e^{2m\lambda x}dx = \frac{\Psi_B(0)^2}{-m\lambda} \stackrel{!}{=} 1 \tag{2.59}$$

and we obtain

$$\Psi_B(0) = \sqrt{-m\lambda} \quad \Longrightarrow \quad \Psi_B(x) = \sqrt{-m\lambda}e^{m\lambda|x|} \ . \tag{2.60}$$

The results for the binding energy E_B and the wave function are the same as the ones we obtained from the solution of the Schrödinger equation in position space.

2.4.2 Scattering states

Now we consider the case $E > 0$. Here we distinguish between the odd and the even wave function.

Parity-odd ansatz

As an ansatz for the odd wave function we use the sine-function in momentum space.

$$\tilde{\Psi}_k(p) = \frac{1}{2i}\left(\delta(p-k) - \delta(p+k)\right) \ , \tag{2.61}$$

where k is a fixed value. This leads to

$$\frac{p^2}{2m}\tilde{\Psi}_k(p) + \frac{\lambda}{2\pi}\int_{-\infty}^{\infty}\tilde{\Psi}_k(p)dp = E\tilde{\Psi}_k(p) \ . \tag{2.62}$$

It is easy to convince oneself that the integral is equal to zero. We also see that

$$\frac{p^2}{2m}\delta(p \pm k) = \frac{(\pm k)^2}{2m}\delta(p \pm k) = \frac{k^2}{2m}\delta(p \pm k) \ , \tag{2.63}$$

such that the Schrödinger equation simplifies to

$$\frac{k^2}{2m}\tilde{\Psi}_k(p) = E\tilde{\Psi}_k(p) \ , \tag{2.64}$$

which implies the well-known relation $k = \pm\sqrt{2Em}$.
When we transform $\tilde{\Psi}(p)$ back to position space, we obtain

$$\Psi_k(x) = \frac{1}{2\pi}\sin(kx) \ . \tag{2.65}$$

Parity-even ansatz

This case is more sophisticated. At the beginning we take an ansatz for a cosine-function in momentum space plus an additional function whose form has to be determined

$$\tilde{\Psi}_k(p) = \frac{1}{2}(\delta(p-k) + \delta(p+k)) + \tilde{\Phi}(p) \ . \tag{2.66}$$

We insert this in the momentum space Schrödinger equation (compare to eq.(2.62)). However, there is a difference to the parity-odd states: the integral no longer disappears as we can easily calculate

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{2}(\delta(p-k) + \delta(p+k)) + \tilde{\Phi}(p)\right)dp = \frac{1}{2\pi} + \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\Phi}(p)dp = \frac{1}{2\pi} + \Phi(0) \ . \tag{2.67}$$

First of all, we are only interested in the additional part. The Schrödinger equation has to be solved for this term and we obtain the following expression

$$\tilde{\Phi}(p) = \frac{\lambda\left(\frac{1}{2\pi} + \Phi(0)\right)}{E - \frac{p^2}{2m}} \ . \tag{2.68}$$

We transform $\tilde{\Phi}$ back to position space

$$\Phi(x) = \frac{\lambda\left(\frac{1}{2\pi} + \Phi(0)\right)}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ipx}}{E - \frac{p^2}{2m}}dp \ . \tag{2.69}$$

This is a non-trivial integration because $E > 0$ and therefore there are two poles on the real axis over which we integrate. We take a closer look at this problem in appendix A.1. Here we only give the result

$$\Phi(x) = \frac{\lambda\left(\frac{1}{2\pi} + \Phi(0)\right)}{2\pi}\frac{\sqrt{2mE}}{E}\pi\sin(\sqrt{(2mE)}|x|) \ . \tag{2.70}$$

For $x = 0$, Φ becomes zero and with our definition for the energy $E = \frac{k^2}{2m}$ it simplifies to

$$\Phi(x) = \frac{1}{2\pi}\frac{\lambda m}{k}\sin(k|x|) \ . \tag{2.71}$$

As already said, the first part of our ansatz is a cosine in position space and so we obtain

$$\Psi_k(x) = \frac{1}{2\pi}\left(\cos(kx) + \frac{\lambda m}{k}\sin(k|x|)\right) \ . \tag{2.72}$$

For reasons of consistency, this has to be equal to our former ansatz in eq.(2.40) which we will now examine. We equate the two solutions

$$A\cos(k|x| + \varphi_0) = \frac{1}{2\pi}\left(\cos(kx) + \frac{\lambda m}{k}\sin(k|x|)\right) \ . \tag{2.73}$$

Notably, at $x = 0$ they have to be equal such that

$$A\cos(\varphi_0) = \frac{1}{2\pi} \ . \tag{2.74}$$

For the left-hand part we use the following addition theorem [8, p.54]

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \ ,$$

which gives us

$$A(\cos(k|x|)\cos(\varphi_0) - \sin(k|x|)\sin(\varphi_0)) = \frac{1}{2\pi}\left(\cos(kx) + \frac{\lambda m}{k}\sin(k|x|)\right) \ . \tag{2.75}$$

Factorising $\cos(\varphi_0)$ on the left side, we obtain

$$A\cos(\varphi_0)\left(\cos(k|x|) - \frac{\sin(\varphi_0)}{\cos(\varphi_0)}sin(k|x|)\right) = \frac{1}{2\pi}\left(\cos(kx) + \frac{\lambda m}{k}\sin(k|x|)\right) \ . \tag{2.76}$$

The factors in front of both sides cancel because of the equality at $x = 0$. Furthermore, $\cos(k|x|) = \cos(kx)$. Therefore we can subtract them on both sides and finally divide by $\sin(k|x|)$. It results

$$\frac{\sin(\varphi_0)}{\cos(\varphi_0)} = \tan(\varphi_0) = -\frac{\lambda m}{k} \ . \tag{2.77}$$

This is exactly the same result for φ_0 that we already received in eq.(2.45). Hence we conclude that everything is consistent.

2.5 Orthogonality

The final step in our handling of the non-relativistic problem is to check whether all solutions are orthogonal to each other. We take the definition of the scalar product in a Hilbert space

$$\boxed{\langle f|g\rangle = \int_{-\infty}^{\infty} f(x)^* g(x) dx} \tag{2.78}$$

We know that if two functions are orthogonal in position space, they are orthogonal in momentum space, too. Therefore, in the different cases we always consider the simpler alternative.

Bound-Bound case

Due to normalisation, in the position space this is a trivial conclusion

$$\langle \Psi_B|\Psi_B\rangle = \int_{-\infty}^{\infty} |\Psi_B|^2 dx = 1 \ . \tag{2.79}$$

Bound-scattering case

Here we do not take the solution from chapter 2.1.2. The orthogonality can be shown more easily for the sine and cosine functions (we know that both solutions are equivalent for the scattering states). We consider two different possibilities (e stands for even and o for odd)

$$\langle \Psi_B|\Psi_k^o\rangle = \int_{-\infty}^{\infty} \sqrt{\kappa} e^{-\kappa|x|} B\sin(kx) dx = 0 \ . \tag{2.80}$$

This becomes clear because the product of an even and an odd function is again an odd function. In combination with the symmetric integration boundaries this implies that the integral is equal to zero. The second possibility requires a little bit more attention

$$\begin{aligned} \langle \Psi_B|\Psi_k^e\rangle &= \int_{-\infty}^{\infty} \sqrt{\kappa} e^{-\kappa|x|} A\cos(k|x| + \varphi_0) dx \ , \\ = 2A\sqrt{\kappa}\frac{\kappa\cos(\varphi_0) - k\sin(\varphi_0)}{k^2+\kappa^2} &= \frac{2A\sqrt{\kappa}\cos(\varphi_0)}{k^2+\kappa^2}(\kappa - k\tan(\varphi_0)) \ , \\ &\frac{2A\sqrt{\kappa}\cos(\varphi_0)}{k^2+\kappa^2}\left(\kappa + k\frac{m\lambda}{k}\right) \overset{\kappa=-m\lambda}{=} 0 \ . \end{aligned} \tag{2.81}$$

The definition of κ comes from eq.(2.3).

Scattering-Scattering case

For these considerations it is recommended to check the orthogonality in momentum space. First, we take a look at two odd wave functions

$$\tilde{\Psi}^o_{k_1}(p) = \frac{1}{2}\left(\delta(p-k_1) - \delta(p+k_1)\right) \tag{2.82}$$

$$\tilde{\Psi}^o_{k_2}(p) = \frac{1}{2}\left(\delta(p-k_2) - \delta(p+k_2)\right) \ . \tag{2.83}$$

We obtain

$$\begin{aligned}\langle\tilde{\Psi}^o_{k_1}|\tilde{\Psi}^o_{k_2}\rangle &= \int_{-\infty}^{\infty}\frac{1}{4}\left(\delta(p-k_1)-\delta(p+k_1)\right)\left(\delta(p-k_2)-\delta(p+k_2)\right)dp \\ &= \frac{1}{4}\left(\int_{-\infty}^{\infty}\delta(p-k_1)\delta(p-k_2)dp + \int_{-\infty}^{\infty}\delta(p+k_1)\delta(p+k_2)dp\right) \\ &-\frac{1}{4}\left(\int_{-\infty}^{\infty}\delta(p+k_1)\delta(p-k_2)dp + \int_{-\infty}^{\infty}\delta(p-k_1)\delta(p+k_2)dp\right) \\ &= \frac{1}{4}\left(\delta(k_2-k_1)+\delta(k_1-k_2)\right) = \frac{1}{2}\delta(k_1-k_2)\end{aligned} \tag{2.84}$$

The fact that if $k_1 = k_2$ the scalar product becomes infinite (and not one) is due to the unconventional normalisation of scattery wave functions.
Next we consider the scalar product of two even wave functions

$$\tilde{\Psi}^e_{k_1}(p) = \frac{1}{2}\left(\delta(p-k_1) + \delta(p+k_1)\right) + \tilde{\phi}_1(p) \ , \tag{2.85}$$

$$\tilde{\Psi}^e_{k_2}(p) = \frac{1}{2}\left(\delta(p-k_2) + \delta(p+k_2)\right) + \tilde{\phi}_2(p) \ . \tag{2.86}$$

We calculate the scalar product

$$\langle\tilde{\Psi}^e_{k_1}|\tilde{\Psi}^e_{k_2}\rangle = \frac{1}{4}\left(\delta(k_1-k_2)+\delta(k_2-k_1)\right) + \left(\tilde{\phi}_1(k_2)+\tilde{\phi}_2(k_1)\right) + \langle\tilde{\phi}_1|\tilde{\phi}_2\rangle \ . \tag{2.87}$$

If we use

$$\tilde{\phi}_1(k_2) = \frac{\lambda}{2\pi}\frac{1}{\frac{k_1^2}{2m}-\frac{k_2^2}{2m}} = -\frac{\lambda}{2\pi}\frac{1}{\frac{k_2^2}{2m}-\frac{k_1^2}{2m}} = -\tilde{\phi}_2(k_1) \tag{2.88}$$

the second term disappears. Furthermore we treat the last one

$$\langle\tilde{\phi}_1|\tilde{\phi}_2\rangle = \frac{\lambda^2}{4\pi^2}\int_{-\infty}^{\infty}\frac{1}{(\frac{k_1^2}{2m}-\frac{p^2}{2m})(\frac{k_2^2}{2m}-\frac{p^2}{2m})}dp \ . \tag{2.89}$$

At first sight this expression seems to be complicated. However, if we use partial fraction decomposition, we will obtain

$$\langle\tilde{\phi}_1|\tilde{\phi}_2\rangle = \frac{\lambda^2}{4\pi^2}\left(\frac{1}{k_2^2-k_1^2}\int_{-\infty}^{\infty}\frac{1}{\frac{k_1^2}{2m}-\frac{p^2}{2m}}dp + \frac{1}{k_1^2-k_2^2}\int_{-\infty}^{\infty}\frac{1}{\frac{k_2^2}{2m}-\frac{p^2}{2m}}dp\right) \ . \tag{2.90}$$

With the relation (2.48) we can simplify this term without any further calculations to

$$\langle\tilde{\phi}_1|\tilde{\phi}_2\rangle = \frac{\lambda^2}{2\pi}\left(\frac{1}{k_2^2-k_1^2}\phi_1(0) + \frac{1}{k_1^2-k_2^2}\phi_2(0)\right) \ . \tag{2.91}$$

We already know the function $\phi(x)$ (see eq.(2.71)) and especially $\phi(0) = 0$. Hence we can write

$$\langle\tilde{\Psi}^e_{k_1}|\tilde{\Psi}^e_{k_2}\rangle = \frac{1}{4}\left(\delta(k_1-k_2)+\delta(k_2-k_1)\right) = \frac{1}{2}\delta(k_1-k_2) \tag{2.92}$$

There is a third case where we take an even and an odd function but it is easy to understand that the scalar product of such a combination of wave functions is zero due to the symmetric integration boundaries.

Chapter 3

Solution of the relativistic problem

This is the main part of this thesis. As already mentioned, we can solve this problem in the momentum space in a much easier way. Therefore the Schrödinger equation appears in the following form

$$\sqrt{m^2+p^2}\tilde{\Psi}(p)+\lambda\Psi(0)=E\tilde{\Psi}(p)\ . \tag{3.1}$$

We solve this equation for $\tilde{\Psi}(p)$ and obtain

$$\tilde{\Psi}(p)=\frac{\lambda\Psi(0)}{E-\sqrt{m^2+p^2}}\ . \tag{3.2}$$

In analogue to the non-relativistic treatment we rewrite

$$\Psi(0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\Psi}(p)dp=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\lambda\Psi(0)}{E-\sqrt{m^2+p^2}}dp\ , \tag{3.3}$$

with $\Psi(0)$ as a constant expression such that we cancel it on both sides

$$\frac{1}{\lambda}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{E-\sqrt{m^2+p^2}}dp\ . \tag{3.4}$$

This is the gap equation in the relativistic case. Here occurs the problem that demands renormalisation. For high momentum the mass can be neglected and the square root reduces to p. Hence in this region the integrand behaves like $\frac{1}{E-p}$ and the integral diverges logarithmically. This is a so-called ultraviolet divergence.
Below we concentrate on the case $\lambda<0$, where both bound and scattering states exist.

3.1 Solution for the bound state

Analogously to the non-relativistic considerations, we start with the bound state and its binding energy $E_B<m$. To solve this problem we use a trick: We subtract the diverging part of eq.(3.4) to obtain an expression which converges. Then we calculate this subtraction separately and finally add it again

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{E_B-\sqrt{m^2+p^2}}dp+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\sqrt{m^2+p^2}}dp=$$
$$\frac{E_B}{2\pi}\int_{-\infty}^{\infty}\frac{1}{E_B\sqrt{m^2+p^2}-(m^2+p^2)}dp\ . \tag{3.5}$$

We introduce the following definitions

$$\boxed{\alpha_B=\frac{E_B}{m}\quad,\quad\alpha=\frac{E}{m}}\ , \tag{3.6}$$

where E is the energy of an unbound particle and the index B refers to the bound state. Note that $\alpha_B<1$ and $\alpha>1$.
First of all, we calculate this integral in the bound case and we obtain

$$\frac{E_B}{2\pi}\int_{-\infty}^{\infty}\frac{1}{E_B\sqrt{m^2+p^2}-(m^2+p^2)}dp=\frac{-\alpha_B}{\pi\sqrt{1-\alpha_B^2}}\left(\frac{\pi}{2}+\arcsin(\alpha_B)\right)=\frac{1}{\lambda(E_B)}\ . \tag{3.7}$$

Now we consider the additional part by itself. As already discussed, the integral

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{m^2+p^2}} dp \tag{3.8}$$

diverges. To solve this problem, we use dimensional regularisation. The principle of this method is the following: An integral may diverge in a space of integer dimension (in our case in one dimension). However, we rewrite it in a form of dimension $D = 1+\epsilon$ and finally examine the behaviour for $\epsilon \to 0$, respectively $D \to 1$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{m^2+p^2}} dp \longrightarrow \frac{1}{(2\pi)^D} \int_{-\infty}^{\infty} \frac{1}{\sqrt{m^2+p^2}} d^D p \ . \tag{3.9}$$

We can parametrise the last expression in spherical coordinates such that the integral consists of an angular and a radial part. The angular part can be calculated directly and results in the surface of a $(D-1)$-dimensional unit sphere, S^{D-1}. We introduce a factor m^ϵin order to maintain the dimension of λ. The radial part remains to be calculated

$$\frac{m^\epsilon}{\lambda(\epsilon)} = \frac{S^{D-1}}{(2\pi)^D} \int_0^{\infty} \frac{p^{D-1}}{\sqrt{m^2+p^2}} dp = \frac{S^{D-1}}{(2\pi)^D} \frac{m^{D-1}\Gamma(\frac{1}{2}-\frac{D}{2})\Gamma(\frac{D}{2})}{2\sqrt{\pi}} \ , \tag{3.10}$$

where $\Gamma(t)$ is the Gamma function defined as

$$\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x} dx \quad , \quad \Gamma(t+1) = t\Gamma(t) \ . \tag{3.11}$$

We use [9]

$$S^{D-1} = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \ , \tag{3.12}$$

insert this in eq.(3.10) and obtain

$$\frac{m^\epsilon}{\lambda(\epsilon)} = \frac{S^{D-1}}{(2\pi)^D} \int_0^{\infty} \frac{p^{D-1}}{\sqrt{m^2+p^2}} dp = \frac{\pi^{\frac{-D-1}{2}}}{2^D} \Gamma(\frac{1}{2}-\frac{D}{2}) m^{D-1} \ . \tag{3.13}$$

We replace $D = 1+\epsilon$ and expand it analytically

$$\frac{m^\epsilon}{\lambda(\epsilon)} = \frac{\pi^{-\frac{1}{2}\epsilon-1}}{2^{1+\epsilon}} \Gamma(-\frac{\epsilon}{2}) m^\epsilon \cong \left(-\frac{1}{\pi\epsilon} - \frac{\gamma-\log(4\pi)}{2\pi} + \mathcal{O}(\epsilon)\right) m^\epsilon \ . \tag{3.14}$$

We terminate the expansion at the order ϵ^0 because we let ϵ go towards 0 after all and the first term then dominates completely. The second expression is a constant that we regard as well, correspondingly to the $\overline{MS}$ scheme. The γ in this term is the Euler-Mascheroni constant $\gamma \approx 0.577$.
So the final result for the gap equation looks as follows

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{E_B - \sqrt{m^2+p^2}} dp = \\ \frac{1}{\lambda(E_B,\epsilon)} &= \frac{-\alpha_B}{\pi\sqrt{1-\alpha_B^2}} \left(\frac{\pi}{2} + \arcsin(\alpha_B)\right) + \frac{1}{\pi\epsilon} + \frac{\gamma-\log(4\pi)}{2\pi} \ . \end{aligned} \tag{3.15}$$

We are now interested in the form of the wave function in position space. From eq.(3.2) we know that the wave function for the bound state in momentum space has the form

$$\tilde{\Psi}_B(p) = \frac{A\lambda\Psi(0)}{E_B - \sqrt{m^2+p^2}} = \frac{A'}{E_B - \sqrt{m^2+p^2}} \ , \tag{3.16}$$

where we introduced a constant A respectively A' (λ and $\Psi(0)$ are constants and can be put together) which we will calculate next.
The wave function has to be normalised

$$\begin{aligned} \langle\tilde{\Psi}_B(p)|\tilde{\Psi}_B(p)\rangle &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A'}{E_B - \sqrt{m^2+p^2}} \left(\frac{A'}{E_B - \sqrt{m^2+p^2}}\right)^* dp \\ &= \frac{A'^2}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(E_B - \sqrt{m^2+p^2})^2} dp \overset{!}{=} 1 \ . \end{aligned} \tag{3.17}$$

This is a rather long calculation. At the end we obtain

$$A' = \sqrt{\frac{2\pi m(1-\alpha_B^2)^{\frac{3}{2}}}{\pi + 2\arcsin(\alpha_B) + 2\alpha_B\sqrt{1-\alpha_B^2}}} \ . \tag{3.18}$$

In a final step we are interested in the form of the wave function in position space. Therefore we have to transform $\tilde{\Psi}(p)$ to position space (compare to eq.(2.47))

$$\Psi_B(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ipx}}{E_B - \sqrt{m^2+p^2}} dp \ . \tag{3.19}$$

This is a highly non-trivial integral and needs a long treatment, but can be solved with the help of complex analysis, a special parametrisation and line integration. This is done in appendix A.2. Here we present the result of the bound wave function in the relativistic case

$$\boxed{\Psi_B(x) = \frac{A' E_B}{\sqrt{m^2 - E_B^2}} e^{-\sqrt{m^2-E_B^2}|x|} + \frac{A'}{\pi}\int_m^{\infty} \frac{\sqrt{(\mu^2-1)}e^{-m\mu|x|}}{\alpha_B^2+\mu^2-1} d\mu} \ . \tag{3.20}$$

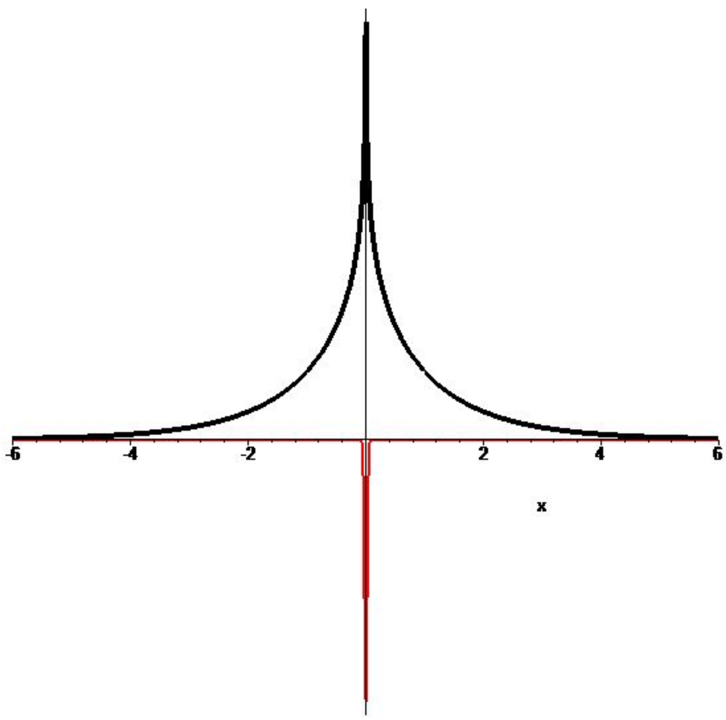

Figure 3.1: The wave function in position space ($m = 1$, $E_B = 0.6$). The fact that the wave function diverges at $x = 0$ is quite astonishing. Still, the probability to find the particle in an interval dx around $x = 0$ remains finite.

We want to know whether this result turns into the non-relativistic solution in the non-relativistic limit. This means that the binding energy of the particle is comparable to its mass ($E_B \lesssim m$).
We consider the condition

$$\frac{1}{\lambda(E_B)} = \frac{-\alpha_B}{\pi\sqrt{1-\alpha_B^2}}\left(\frac{\pi}{2} + \arcsin(\alpha_B)\right) \ . \tag{3.21}$$

We expand this expression as a series with $\alpha_B = 1$ and we obtain

$$\frac{-\alpha_B}{\pi\sqrt{1-\alpha_B^2}}\left(\frac{\pi}{2} + \arcsin(\alpha_B)\right) \cong \frac{i}{\sqrt{2}\sqrt{\alpha_B - 1}} + \frac{1}{\pi} + \mathcal{O}(\sqrt{\alpha_B - 1}) \ . \tag{3.22}$$

We see that the first term dominates for $\alpha_B \approx 1$. Therefore we can neglect all other terms and write

$$\frac{1}{\lambda(E_B)} = \frac{i}{\sqrt{2}\sqrt{\alpha_B - 1}} \; . \tag{3.23}$$

We solve for $\alpha_B = \frac{E_B}{m}$. Furthermore we write $E_B = E_B' + \Delta E_B$. The relativistic binding energy consists of a variable part E_B' and a constant part which remains fixed in the non-relativistic limit. Finally we take $\frac{E_B'}{m} \to 1$ and obtain

$$\frac{E_B}{m} - 1 = \frac{E_B' + \Delta E_B}{m} - 1 = \frac{E_B'}{m} - 1 + \frac{\Delta E_B}{m} = \frac{-\lambda^2}{2} \implies \Delta E_B = \frac{-m\lambda^2}{2} \; ,$$

which is nothing but the non-relativistic binding energy. Let us now see what happens to the solution for the bound state wave function in eq.(3.20) for $\frac{E_B}{m} \to 1$. In this case the first term in eq.(3.20) dominates. This can be easily seen. Hence the expression in the denominator tends to zero and therefore the pre-factor grows strongly. So we concentrate on the following expression

$$\sqrt{\frac{2\pi m(1-\alpha_B^2)^{\frac{3}{2}}}{\pi + 2\arcsin(\alpha_B) + 2\alpha_B\sqrt{1-\alpha_B^2}}\frac{\alpha_B}{\sqrt{1-\alpha_B^2}}}e^{-m\sqrt{1-\alpha_B^2}|x|} \; . \tag{3.24}$$

With $\alpha_B \to 1$ the last summand in the denominator of the normalisation factor can be neglected, with $2\arcsin(1) = \pi$ and fraction reduction we obtain

$$\sqrt{m\sqrt{1-\alpha_B^2}}e^{-m\sqrt{1-\alpha_B^2}|x|} \; . \tag{3.25}$$

After all we just have to examine the square root $\sqrt{1-\alpha_B^2}$. We rewrite $E_B = E_B' + \Delta E_B$ the same way as above

$$\sqrt{1-\alpha_B^2} = \sqrt{1 - \frac{(E_B' + \Delta E_B)^2}{m^2}} = \sqrt{1 - \left(\frac{E_B'}{m}\right)^2 - \frac{2E_B'\Delta E_B}{m^2} + \left(\frac{\Delta E_B}{m}\right)^2} \; , \tag{3.26}$$

for $\frac{E_B'}{m} \to 1$ and neglecting the quadratic terms, this results in

$$\sqrt{-\frac{2}{m}\Delta E_B} = \sqrt{-\frac{2}{m}\frac{-m\lambda^2}{2}} = \sqrt{\lambda^2} = |\lambda| \; . \tag{3.27}$$

If we insert this in eq.(3.25), we obtain exactly the wave function of the non-relativistic case.

3.2 Solution for the scattering states

3.2.1 The parity-odd solution

Just as in the non-relativistic case we take the following ansatz for an odd wave function in momentum space

$$\tilde{\Psi}_k(p) = \frac{1}{2i}\left(\delta(p-k) - \delta(p+k)\right) \; . \tag{3.28}$$

With

$$\Psi_k(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\Psi}_k(p)dp = \frac{1}{4\pi i}\left(\int_{-\infty}^{\infty}\delta(p-k)dp - \int_{-\infty}^{\infty}\delta(p+k)dp\right) = 0 \tag{3.29}$$

eq.(3.1) reduces to

$$\sqrt{m^2+p^2}\frac{1}{2i}\left(\delta(p-k) - \delta(p+k)\right) = E\frac{1}{2i}\left(\delta(p-k) - \delta(p+k)\right)$$
$$\implies \sqrt{m^2+k^2}\tilde{\Psi}(p) = E\tilde{\Psi}(p) \; . \tag{3.30}$$

$\tilde{\Psi}(p)$ is a solution under the condition

$$k = \pm\sqrt{E^2 - m^2} \; . \tag{3.31}$$

Remember that in the unbound case $E > m$ and therefore $k \in \mathbb{R}$.
To obtain the form of the wave function in position space we transform it

$$\Psi_k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2}\left(\delta(p-k) - \delta(p+k)\right) e^{ipx} dp =$$

$$\boxed{\Psi_k^o(x) = \frac{B(k)}{2\pi}\sin(kx)} \ . \tag{3.32}$$

This solution is the same as in the non-relativistic treatment except the value of the energy E. We know that $\Psi(0) = 0$. Hence the particle never stays at $x = 0$ and is not influenced by the potential.

3.2.2 The parity-even solution

As well we take an ansatz similar to the non-relativistic chapter

$$\tilde{\Psi}_k(p) = \frac{1}{2}\left(\delta(p-k) + \delta(p+k)\right) + \tilde{\phi}(p) \ . \tag{3.33}$$

Let us have a closer look at $\Psi(x)$

$$\Psi_k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{2}\left(\delta(p-k) + \delta(p+k)\right) + \tilde{\phi}(p)\right) e^{ipx} dp = \frac{1}{2\pi}\cos(kx) + \phi(x) \tag{3.34}$$

with $k = \pm\sqrt{E^2 - m^2}$ and especially

$$\Psi(0) = \frac{1}{2\pi} + \phi(0) \ . \tag{3.35}$$

What we still have to find out is $\tilde{\phi}(p)$ and finally $\phi(x)$. If we insert $\tilde{\phi}(p)$ in the Schrödinger equation, we obtain

$$\sqrt{m^2 + p^2}\tilde{\phi}(p) + \lambda\Psi_k(0) = E\tilde{\phi}(p) \ . \tag{3.36}$$

We solve it for $\tilde{\phi}(p)$, instert eq.(3.35) and integrate over all momenta (times $\frac{1}{2\pi}$)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\phi}(p)dp = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\lambda\left(\frac{1}{2\pi} + \phi(0)\right)}{E - \sqrt{m^2 + p^2}}dp = \phi(0) \ , \tag{3.37}$$

where we used eq.(2.48) for the last relation. All the expressions in the numerator of the integrand are independent of p and can be drawn out of the integral. With the definition

$$I(E) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{E - \sqrt{m^2 + p^2}}dp \tag{3.38}$$

we can rewrite

$$\lambda\left(\frac{1}{2\pi} + \phi(0)\right) I(E) = \phi(0) \tag{3.39}$$

and obtain

$$\phi(0) = \frac{1}{2\pi}\frac{\lambda I(E)}{1 - \lambda I(E)} \ . \tag{3.40}$$

We must calculate $I(E)$, whereupon we operate the same way as we did for the relativistic bound state. We subtract the diverging part and calculate it separately

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{E - \sqrt{m^2 + p^2}}dp + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\sqrt{m^2 + p^2}}dp$$

$$= \frac{E}{2\pi}\int_{-\infty}^{\infty}\frac{1}{E\sqrt{m^2 + p^2} - (m^2 + p^2)}dp = \frac{\alpha}{\pi\sqrt{\alpha^2 - 1}}\tanh^{-1}\left(\frac{\sqrt{\alpha^2 - 1}}{\alpha}\right) \ . \tag{3.41}$$

The subtracted integral is exactly the same as in eq.(3.10) and so we can write

$$I(E) = \frac{\alpha}{\pi\sqrt{\alpha^2 - 1}}\tanh^{-1}\left(\frac{\sqrt{\alpha^2 - 1}}{\alpha}\right) + \frac{1}{\pi\epsilon} + \frac{\gamma - \log(4\pi)}{2\pi} \ . \tag{3.42}$$

With eq.(3.35) and eq.(3.40) we rewrite $\tilde{\phi}(p)$ as follows

$$\tilde{\phi}(p) = \frac{\lambda\left(\frac{1}{2\pi} + \frac{1}{2\pi}\frac{\lambda I(E)}{1-\lambda I(E)}\right)}{E - \sqrt{m^2+p^2}} \ , \tag{3.43}$$

where $\lambda = \lambda(E_B, \epsilon)$ what we already calculated. Here comes the essential part of the renormalisation. Up to now we only mentioned that we will send ϵ towards zero, but we have not done anything so far. Let us see what will happen. From eq.(3.15) we see that $\lambda(E_B,\epsilon)$ tends to zero in the limit $\epsilon \to 0$. This implies that the first term in the numerator disappears. The second term needs some care (we simply consider the numerator)

$$\lambda\left(\frac{1}{2\pi} + \frac{1}{2\pi}\frac{\lambda I(E)}{1-\lambda I(E)}\right) \stackrel{\epsilon\to 0}{\longrightarrow} \frac{1}{2\pi}\frac{\lambda^2 I(E)}{1-\lambda I(E)} \ . \tag{3.44}$$

We expand the fraction in $\frac{1}{\lambda}$ and fill in our results for $\lambda = \lambda(E_B,\epsilon)$ and $I(E)$. We obtain

$$\frac{1}{2\pi}\frac{\lambda I(E)}{\frac{1}{\lambda} - I(E)} = \frac{1}{2\pi}\frac{\frac{\alpha}{\pi\sqrt{\alpha^2-1}}\tanh^{-1}\left(\frac{\sqrt{\alpha^2-1}}{\alpha}\right) + \frac{1}{\pi\epsilon} + \frac{\gamma-\log(4\pi)}{2\pi}}{\frac{-\alpha_B}{\pi\sqrt{1-\alpha_B^2}}\left(\frac{\pi}{2} + \arcsin(\alpha_B)\right) + \frac{1}{\pi\epsilon} + \frac{\gamma - \log(4\pi)}{2\pi}}$$

$$\times\left[\frac{-\alpha_B}{\pi\sqrt{1-\alpha_B^2}}\frac{\pi}{2} + \arcsin(\alpha_B) + \frac{1}{\pi\epsilon} + \frac{\gamma-\log(4\pi)}{2\pi} - \left(\frac{\alpha}{\pi\sqrt{\alpha^2-1}}\tanh^{-1}\left(\frac{\sqrt{\alpha^2-1}}{\alpha}\right) + \frac{1}{\pi\epsilon} + \frac{\gamma-\log(4\pi)}{2\pi}\right)\right]^{-1} \tag{3.45}$$

The numerator simplifies to 1 because in the limit $\epsilon \to 0$ the $\frac{1}{\pi\epsilon}$-terms dominate on both sides. In the denominator something very interesting occurs: The diverging parts cancel out each other. And this leads to the renormalised coupling constant of the δ-potential in relativistic quantum mechanics

$$\lambda(E_B, E) = \frac{-\pi}{\frac{E_B}{\sqrt{m^2-E_B^2}}\left(\frac{\pi}{2} + \arcsin\left(\frac{E_B}{m}\right)\right) + \frac{E}{\sqrt{E^2-m^2}}\tanh^{-1}\left(\frac{\sqrt{E^2-m^2}}{E}\right)} \ . \tag{3.46}$$

So we can write

$$\tilde{\Psi}_k(p) = \frac{1}{2}\left(\delta(p-x) + \delta(p+k)\right) + \frac{1}{2\pi}\frac{\lambda(E_B,E)}{E - \sqrt{m^2+p^2}} \tag{3.47}$$

and transform it back to position space. The term consisting of the δ-functions results in a cosine. Therefore we still have to calculate

$$\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2\pi}\frac{\lambda(E_B,E)e^{ipx}}{E-\sqrt{m^2+p^2}}dp = \frac{\lambda(E_B,E)}{4\pi^2}\int_{-\infty}^{\infty}\frac{e^{ipx}}{E-\sqrt{m^2+p^2}}dp \ . \tag{3.48}$$

This is a laborious integral which is handled in more detail in appendix A.3. Here we just give the result for the even wave function for the scattering state

$$\boxed{\Psi_k^e(x) = \frac{A(k)}{2\pi}\left[\cos(kx) + \lambda(E_B,E)\left(\frac{\sqrt{k^2+m^2}}{k}\sin(k|x|) - \frac{1}{\pi}\int_m^{\infty}\frac{\sqrt{(\mu^2-1)}e^{-m|x|\mu}}{\alpha^2+\mu^2-1}d\mu\right)\right]} \ . \tag{3.49}$$

3.2.3 Reflection and transmission coefficients

According to the non-relativistic procedure, the particle starts on the negative x-axis and moves toward the δ-peak. So the wave function can be parametrised the following form

$$\Psi(x) = \begin{cases} e^{ikx} + R(k)e^{-ikx} + \lambda(E_B,E)C(k)\chi(x) & \text{if } x<0 \\ T(k)e^{ikx} + \lambda(E_B,E)C(k)\chi(x) & \text{if } x>0 \end{cases} \ . \tag{3.50}$$

$R(k)$ and $T(k)$ are the reflection respectively transmission coefficients that we are looking for. The function $\chi(x)$ has no direct impact on the reflective or transmissive behaviour of the particle.
Let us start with the case $x > 0$. We are looking for a linear combination of the even and odd states (of the same energy) such that it can be written as the wave function above. We have

$$\Psi_k(x) = A(k)\left[\cos(kx) + \frac{\lambda(E_B,E)E}{k}\sin(kx) - \lambda(E_B,E)\chi(x)\right] + B(k)\sin(kx) \tag{3.51}$$

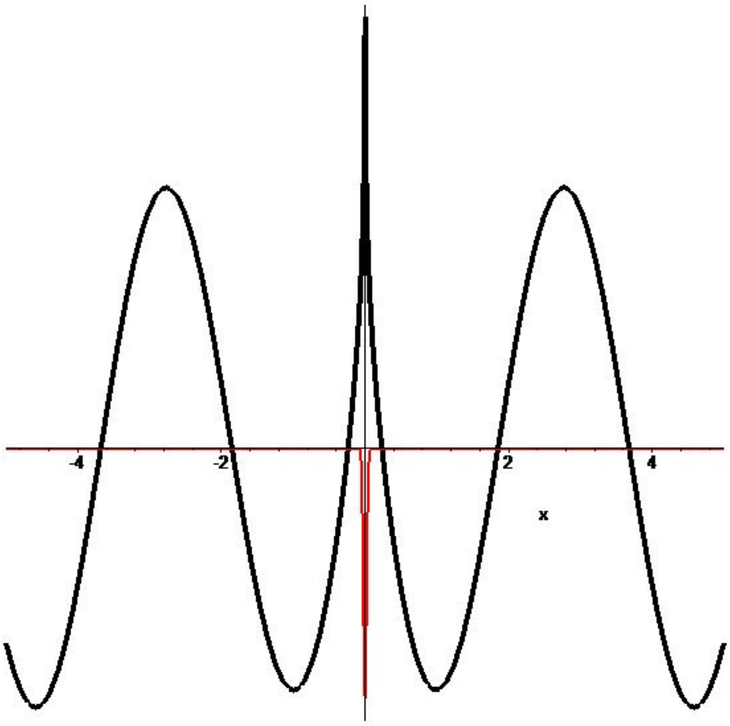

Figure 3.2: Example for the unbound, even wave function with $m = 1$, $E_B = 0.5$ and $E = 2$. In contrast to the non-relativistic results, the particle "feels" the potential not just at $x = 0$ but also from a distance. This can be explained by the form of the Schrödinger equation in position space: If we Taylor expand the square root, in the relativistic case the terms of high order do not disappear. Since high order derivatives can not be considered as point-like operations anymore, the particle is influenced also away from the origin.

with

$$\chi(x) = \frac{1}{\pi}\int_m^\infty \frac{\sqrt{(\mu^2-1)}e^{-m|x|\mu}}{\alpha^2+\mu^2-1}d\mu \ . \tag{3.52}$$

Now we rewrite the sine and cosine functions in terms of e^{ikx} and e^{-ikx} and we recapitulate

$$\Psi_k(x) = A(k)\left[\frac{1}{2}\left(1+\frac{\lambda(E_B,E)E}{ik}+\frac{B(k)}{iA(k)}\right)e^{ikx}+\frac{1}{2}\left(1-\frac{\lambda(E_B,E)E}{ik}-\frac{B(k)}{iA(k)}\right)e^{-ikx}-\lambda(E_B,E)\chi(x)\right] \tag{3.53}$$

where the e^{-ikx}-term has to vanish. This means

$$1-\frac{\lambda(E_B,E)E}{ik}-\frac{B(k)}{iA(k)}=0 \implies B(k)=A(k)\left(i-\frac{\lambda(E_B,E)E}{k}\right) \ . \tag{3.54}$$

So eq.(3.53) becomes

$$A(k)e^{ikx}-A(k)\lambda(E_B,E)\chi(x)=T(k)e^{ikx}+C(k)\lambda(E_B,E)\chi(x) \ . \tag{3.55}$$

When we compare the coefficients, we obtain

$$A(k)=T(k) \quad , \quad C(k)=-A(k) \ . \tag{3.56}$$

With this result we consider the case $x < 0$. The only difference is the expression $\sin(k|x|)$ which now is written as $\sin(k(-x))$ such that for negative x we still obtain the sine of a positive value of x. Again we replace the trigonometric functions by exponential terms and obtain

$$\Psi_k(x) = A(k)\left[\frac{1}{2}\left(1-\frac{\lambda(E_B,E)E}{ik}+\frac{B(k)}{iA(k)}\right)e^{ikx}+\frac{1}{2}\left(1+\frac{\lambda(E_B,E)E}{ik}-\frac{B(k)}{iA(k)}\right)e^{-ikx}-\lambda(E_B,E)\chi(x)\right] \ . \tag{3.57}$$

When we replace $B(k)$ in eq.(3.54) it results in

$$\Psi_k(x) = A(k)\left(1-\frac{\lambda(E_B,E)E}{ik}\right)e^{ikx}+A(k)\frac{\lambda(E_B,E)E}{ik}e^{-ikx}-A(k)\lambda(E_B,E)\chi(x) \ . \tag{3.58}$$

We compare the pre-factors with the actual wave function in (3.50) and obtain the following conditions

$$\begin{aligned} A(k)\left(1-\frac{\lambda(E_B,E)E}{ik}\right) = 1 \implies A(k) &= \frac{ik}{ik-\lambda(E_B,E)E} = T(k) \ , \\ R(k) = A(k)\frac{\lambda(E_B,E)E}{ik} &= \frac{\lambda(E_B,E)E}{ik-\lambda(E_B,E)E} \ , \\ B(k) = A(k)\left(i-\frac{\lambda(E_B,E)E}{ik}\right) &= i \ . \end{aligned} \tag{3.59}$$

In a final step we rewrite $E=\sqrt{m^2+k^2}$ and expand the fractions by $-i$

$$R(k) = \frac{-i\sqrt{k^2+m^2}\lambda(E_B,E)}{k+i\sqrt{k^2+m^2}\lambda(E_B,E)} \tag{3.60}$$

$$T(k) = \frac{k}{k+i\sqrt{k^2+m^2}\lambda(E_B,E)} \ . \tag{3.61}$$

We can check easily that

$$T(k) - R(k) = 1 \tag{3.62}$$

and

$$|R(k)|^2 + |T(k)|^2 = 1 \ . \tag{3.63}$$

In the non-relativistic limit we write $E=\sqrt{m^2+k^2}\approx m$ and $\lambda(E_B,E)\to\lambda(E_B)$. This gives us the same results for $T(k)$ and $R(k)$ as with the non-relativistic calculations in eq.(2.26) and eq.(2.27).

3.3 Orthogonality

Finally we check whether the obtained wave functions are orthogonal. Like in the non-relativistic case we consider the orthogonality in each case in the more favourable space.
One directly sees that the scalar product of the odd unbound state with the bound and the even unbound state is zero. Hence we are integrating an odd function (the product of an even and an odd function is again an odd function) from $-\infty$ to ∞ and this is zero. Therefore we must have a closer look only at

$$\langle \Psi_B(x)|\Psi_k^e(x)\rangle \ . \tag{3.64}$$

It is much easier to deal with this scalar product in momentum space. Then the wave functions have the following form

$$\tilde{\Psi}_B(p) = \frac{A}{E_B-\sqrt{m^2+p^2}} \tag{3.65}$$

and

$$\tilde{\Psi}_k^e(p) = \frac{1}{2}\left(\delta(p-k)+\delta(p+k)\right) + \frac{1}{2\pi}\frac{\lambda(E_B,E)}{E-\sqrt{m^2+p^2}} \ . \tag{3.66}$$

We determine

$$\langle \tilde{\Psi}_B(p)|\tilde{\Psi}_k^e(p)\rangle = \int_{-\infty}^{\infty} \frac{A\left(\frac{1}{2}\left(\delta(p-k)+\delta(p+k)\right)+\frac{1}{2\pi}\frac{\lambda(E_B,E)}{E-\sqrt{m^2+p^2}}\right)}{\left(E_B-\sqrt{m^2+p^2}\right)}dp \ , \tag{3.67}$$

where the integration can be split in the individual summands. The two integrals with the δ-function can be calculated easily and result in (with the definition $E=\sqrt{m^2+k^2}$)

$$\langle \tilde{\Psi}_B(p)|\tilde{\Psi}_k^e(p)\rangle = \frac{A}{E_B-E} + \frac{A\lambda(E_B,E)}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\left(E_B-\sqrt{m^2+p^2}\right)\left(E-\sqrt{m^2+p^2}\right)}dp \ . \tag{3.68}$$

With the help of partial fraction decomposition this is equal to

$$\langle \tilde{\Psi}_B(p)|\tilde{\Psi}_k^e(p)\rangle = \frac{A}{E_B-E} + \frac{A\lambda(E_B,E)}{E_B-E}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{E-\sqrt{m^2+p^2}}dp - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{E_B-\sqrt{m^2+p^2}}dp\right] \ , \tag{3.69}$$

but this can also be written as

$$\frac{A}{E_B - E} + \frac{A\lambda(E_B, E)}{E_B - E} \left[I(E) - I(E_B)\right] \ . \tag{3.70}$$

When we remember, that $\lambda(E_B, E) = \frac{1}{I(E_B) - I(E)}$, we obtain

$$\frac{A}{E_B - E} + \frac{A}{(E_B - E)} \frac{I(E) - I(E_B)}{(I(E_B) - I(E))} = \frac{A}{E_B - E} + \frac{-A}{E_B - E} = 0 \ . \tag{3.71}$$

So all obtained wave functions are orthogonal.

3.4 Asymptotic freedom

It is worth to pay some more attention to the renormalised coupling constant. In the non-relativistic case we obtained the relation

$$E_B = -\frac{\lambda^2 m}{2} \quad \Rightarrow \quad \lambda(E_B) = \pm\sqrt{\frac{-2E_B}{m}} \ . \tag{3.72}$$

In the relativistic case we obtain

$$\lambda(E_B, E) = \frac{-\pi}{\frac{E_B}{\sqrt{m^2 - E_B^2}} \left(\frac{\pi}{2} + \arcsin\left(\frac{E_B}{m}\right)\right) + \frac{E}{\sqrt{E^2 - m^2}} \tanh^{-1}\left(\frac{\sqrt{E^2 - m^2}}{E}\right)} \ . \tag{3.73}$$

We want to check if this will result in the expression in the non-relativistic limit by expanding $\lambda(E_B, E)$ for $E_B = m$ and $E = m$

$$\lambda(E_B, E) \longrightarrow \lambda(E_B) \approx \frac{\sqrt{-2m}}{m} \sqrt{E_B - m} - \frac{4}{m\pi}(E_B - m) + \mathcal{O}(\sqrt{(E_B - m)}^3) \ . \tag{3.74}$$

Only taking the first term and understanding the remaining difference between E_B and m as the non-relativistic binding energy, we obtain exactly the expected result.
The renormalised coupling constant in eq.(3.46), which occurs in the even wave function of an unbound particle, does not only depend on E_B but also on the particle's energy E. This is a very interesting and non-intuitive result. In the non-relativistic case, assuming that a binding energy is given or measured, we can calculate the non-relativistic coupling constant, which remains the same as well for unbound particles in the same potential. In the relativistic case this changes. λ does not have the same value for particles with different energies. It changes continuously with E, in this context we speak about a running coupling constant.
The gap eq.(3.4) for the bound case includes an integral which actually diverges. We worked with it in such a way that we were finally able to describe and "control" this divergence. In our further work, we had the situation that another integration occurred with the same diverging behaviour such that they cancelled out (fortunately). But this does not change the fact, that the gap equation diverges. Actually, what we observe is the difference of two integrals, which has physical relevance. As a consequence, an additional parameter E appears.

3.5 β-function

At the end of this bachelor thesis, we introduce the concept of the β-function. It is defined as follows

$$\boxed{\beta(\lambda(E_B, E), E) = E\frac{\partial|\lambda(E_B, E)|}{\partial E}} \ . \tag{3.75}$$

We multiply it with E to keep $\beta(\lambda(E_B, E))$ dimensionless. This function describes the behaviour of the renormalised coupling constant for a changing energy. The calculation is long but straightforward

$$E\frac{\partial}{\partial E}\left(\frac{-\pi}{\frac{E_B}{\sqrt{m^2 - E_B^2}} \left(\frac{\pi}{2} + \arcsin\left(\frac{E_B}{m}\right)\right) + \frac{E}{\sqrt{E^2 - m^2}} \tanh^{-1}\left(\frac{\sqrt{E^2 - m^2}}{E}\right)}\right)$$
$$= \frac{-\lambda(E_B, E)^2}{\pi}\left(\frac{m^2 \tanh^{-1}\left(\frac{\sqrt{E^2 - m^2}}{E}\right) + E\sqrt{E^2 - m^2}}{(E^2 - m^2)^{\frac{3}{2}}}\right) \ . \tag{3.76}$$

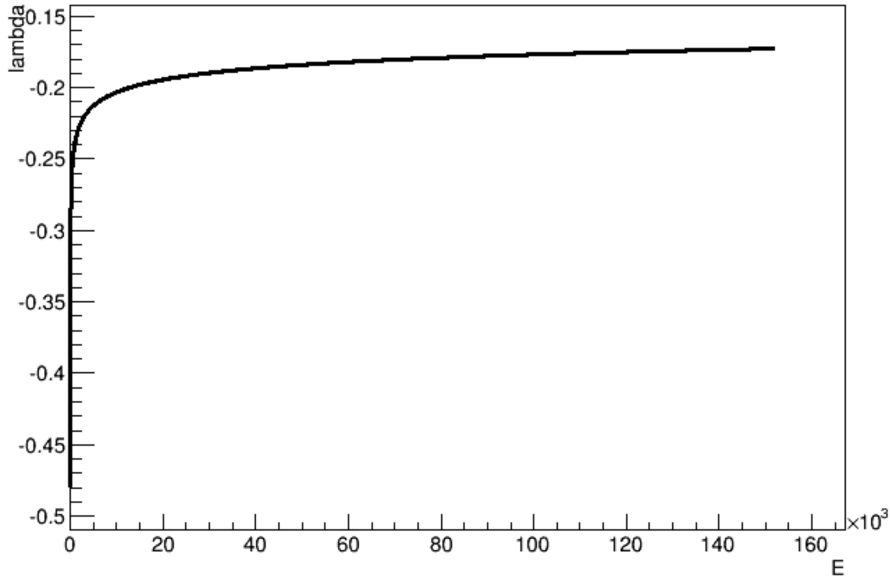

Figure 3.3: The running coupling constant $\lambda(E_B, E)$ with $m = 1$, $E_B = 0.9$ for an energy range $E = 1, ..., 10^5$. Especially for small E we can notice a fast changing $\lambda(E_B, E)$. This reduces for large energies but it is all the same still evident. For $E \to \infty$, $\lambda(E_B, E) \to 0$ because $\frac{\sqrt{E^2-m^2}}{E} \to 1$ and $\tanh^{-1}(1) = \infty$.

This is the final result but we can transform it in a more convenient form. We solve eq.(3.46) for $\tanh^{-1}$ and we obtain

$$\tanh^{-1}\left(\frac{\sqrt{E^2-m^2}}{E}\right) = \frac{-\sqrt{E^2-m^2}}{E}\left(\frac{\pi}{\lambda(E_B,E)} - \frac{\pi}{\lambda(E_B)}\right) \ . \tag{3.77}$$

Then the β-function can be written as follows

$$\beta(\lambda(E_B,E),E) = \frac{-\lambda(E_B,E)^2}{\pi} - \lambda(E_B,E)^2\frac{\zeta^2}{1-\zeta^2}\left(\frac{1}{\lambda(E_B,E)} - \frac{1}{\lambda(E_B)} + \frac{1}{\pi}\right) \ . \tag{3.78}$$

We used $\zeta = \frac{m}{E} = \alpha^{-1}$ to simplify the equation.

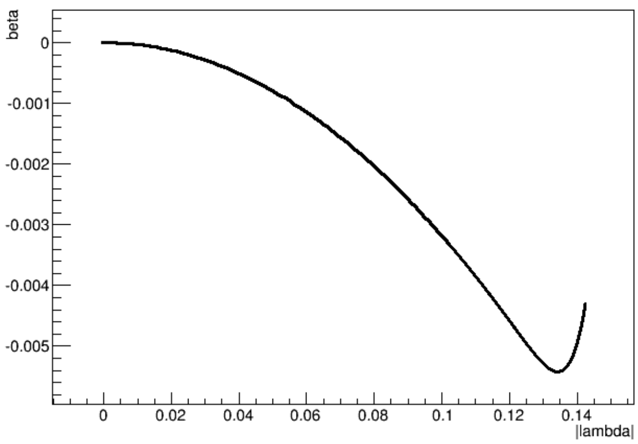

Figure 3.4: The β-function for $m = 1$ and $E_B = 0.99$. It is very interesting that in this case the function has a minimum for $|\lambda(E_B, E)| \approx 0.134$. It would be very interesting to examine this behaviour in more detail, but this would exceed this thesis. For small λ (this means for large energies E) the β-function behaves proportional to $-\lambda^2$. This is because for $E \to \infty$ $\zeta \to 0$. So only the first term of the β-function in eq.(3.78) remains.

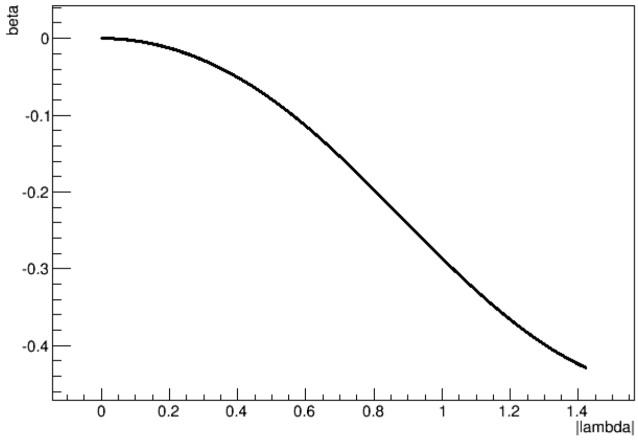

Figure 3.5: The β-function for $m = 1$ and $E_B = 0.5$.

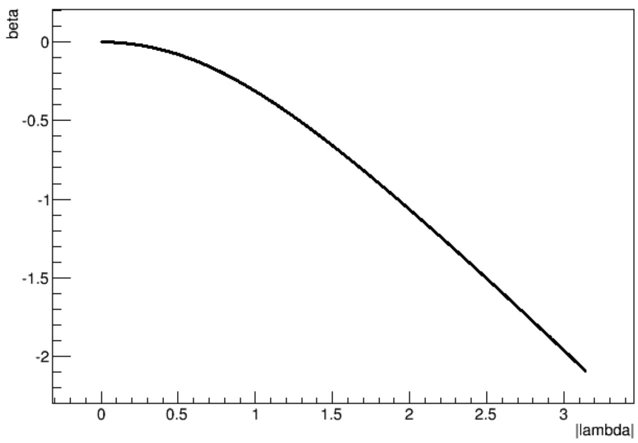

Figure 3.6: The β-function for $m = 1$ and $E_B = 0$. We can observe the same behaviour for small $|\lambda(E_B, E)|$ as above, but the lower energy effect (as in the case $E_B = 0.99$ and slightly for $E_B = 0.5$, too) disappears. One should also consider the change of the scales.

Chapter 4

Conclusion

The aim of this thesis was the solution of the Schrödinger equation

$$\sqrt{m^2 - \frac{\partial^2}{\partial x^2}}\Psi(x) + \lambda\delta(x)\Psi(x) = E\Psi(x) \ . \tag{4.1}$$

The physics behind this equation is very interesting and it can be called an in-between of non-relativistic quantum mechanics and quantum field theory. The Hamiltonian, its derivation and the wave function are well-known from quantum mechanics. However to handle this problem, I worked with a method which is commonly used in quantum field theory, the dimensional regularisation.
As a result, I obtained the wave function of the bound and unbound states in position space. Here something new occurred: A wave function which has a pole. This is no actual problem, because it is still square integrable. Furthermore, I examined the properties of the renormalised coupling constant $\lambda(E_B, E)$ which is asymptotically free, a characteristic that occurs mainly in quantum field theory. Finally, I calculated the β-function which has a special behaviour for $E_B \approx m$. However, due to lack of time, I did not examine it furthermore.
It is also interesting to consider the solution of this problem in the context of Leutwyler's classical no interaction theorem. It is possible that there is a quantum mechanical loop-hole: the δ-function potential.

If I had more time, I could expand this investigation to the strongly bound case (this means $E_B < 0$) and the ultra-strongly bound case ($E_B < -m$). Another situation, worth to consider, is the massless case $m = 0$. These questions have been studied in [2].

Acknowledgement

A special thank goes to Prof. Uwe-Jens Wiese for his support and guidance. The weekly meetings were very informative and helpful.

I would also like to thank my fellow students Igor Jurosevic and Hernando Sobrino who worked on the same topic and with whom I had some very interesting and enlightening discussions.

Appendices

Appendix A

Mathematical remarks

Here we take a more detailed look at certain integrals which are too long to be presented in the main part of this thesis. We assume the residue theorem as proven [10]

$$\boxed{\oint_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} \eta(\gamma, z_k) Res(f, z_k)}\ , \tag{A.1}$$

where z_k are the poles of $f(z)$ and $\eta(\gamma, z_k)$ is the number of turns in each case.

A.1 $\int_{-\infty}^{\infty} \frac{e^{ipx}}{E - \frac{p^2}{2m}} dp$

When we analyse the function, we realize that there are two poles, namely on the real axis at $p = \pm\sqrt{2Em}$. We exclude E and obtain

$$\int_{-\infty}^{\infty} \frac{e^{ipx}}{E - \frac{p^2}{2m}} dp = \frac{1}{E} \int_{-\infty}^{\infty} \frac{e^{ipx}}{1 - \frac{p^2}{2mE}} dp \qquad \begin{cases} p = \sqrt{2mE}\ q \\ dp = \sqrt{2mE}\ dq \end{cases}$$

$$\Rightarrow \frac{\sqrt{2mE}}{E} \int_{-\infty}^{\infty} \frac{e^{ix\sqrt{2mE}q}}{1 - q^2} dq \overset{\alpha = \sqrt{2mE}}{=} \frac{\alpha}{E} \int_{-\infty}^{\infty} \frac{e^{ix\alpha q}}{1 - q^2} dq\ . \tag{A.2}$$

We neglect the pre-factor. As a further step expand the function in the complex plane $q \to z$. This is allowed, because we deal with "nice" functions such as the exponential function and polynomials. The integrand has now two poles on the real axis at $z = \pm 1$. We use a concrete integration path which excludes them (see fig.(A.1)).
With the residual theorem we then obtain

$$\int_\gamma \frac{e^{i\alpha x z}}{1 - z^2} dz = 0\ . \tag{A.3}$$

We examine each of the six parts individually. We begin with the semi-circle of radius R and show that it disappears for $R \to \infty$

$$\int_6 \frac{e^{i\alpha x z}}{1 - z^2} dz \le \left| \int_6 \frac{e^{i\alpha x z}}{1 - z^2} dz \right| \le \int_6 \left| \frac{e^{i\alpha x z}}{1 - z^2} \right| |dz| = \int_6 \frac{\left|e^{i\alpha x z}\right|}{|1 - z^2|} |dz|$$

$$= \int_6 \frac{1}{|1 - z^2|} |dz| \le \sup_{|z| = R} \left(\frac{1}{1 - z^2} \right) \pi R = \frac{\pi R}{1 - R^2} \overset{R \to \infty}{\longrightarrow} 0\ . \tag{A.4}$$

The other integration path is on the real axis, where we exclude the two poles. We parametrise these exclusions as semi-circles with radius ϵ and finally take the limit $\epsilon \to 0$. What remains is the integration over the real axis (except for the two divergences)

$$\int_2 \frac{e^{i\alpha x z}}{1 - z^2} dz \qquad \begin{cases} z \to -1 - \epsilon e^{-it} \\ dz = \frac{dz}{dt} dt = i\epsilon e^{it} \end{cases}$$

$$\Rightarrow \int_0^\pi \frac{e^{i\alpha x\left(-1 - \epsilon e^{-it}\right)}\left(i\epsilon e^{-it}\right)}{1 - \left(-1 - \epsilon e^{-it}\right)^2} dt = e^{-i\alpha x} \int_0^\pi \frac{e^{-i\alpha x \epsilon e^{-it}} i\epsilon e^{-it}}{-2\epsilon e^{-it} - \left(\epsilon e^{-it}\right)^2} dt\ . \tag{A.5}$$

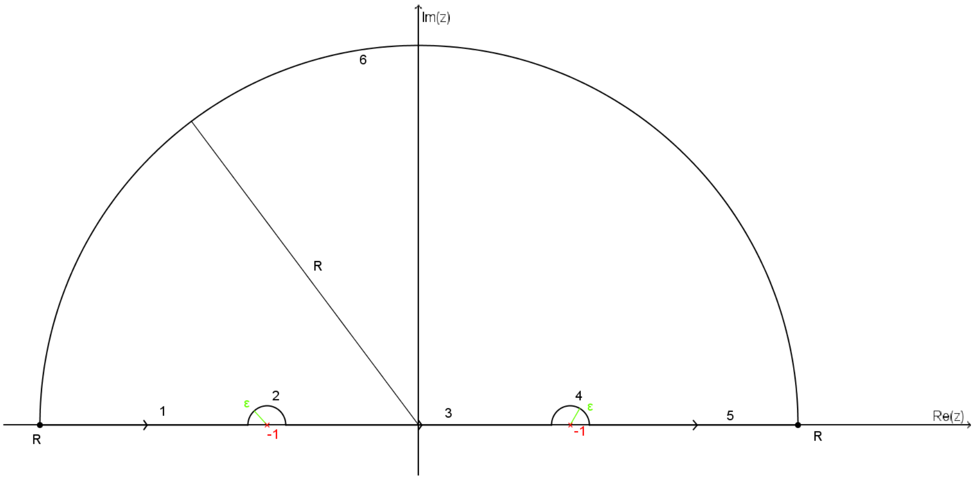

Figure A.1: The path in te complex plane.

Now we take the limit $\epsilon \to 0$. Without a strictly mathematical treatment, we can assume that the limit and the integration can be exchanged

$$e^{-i\alpha x}\lim_{\epsilon\to 0}\int_0^\pi \frac{i\epsilon e^{-it}e^{-i\alpha x\epsilon e^{-it}}}{-2\epsilon e^{-it}-\left(\epsilon e^{-it}\right)^2}dt = e^{-i\alpha x}\int_0^\pi \lim_{\epsilon\to 0}\frac{i\epsilon e^{-it}e^{-i\alpha x\epsilon e^{-it}}}{-2\epsilon e^{-it}-\left(\epsilon e^{-it}\right)^2}dt$$
$$= e^{-i\alpha x}\int_0^\pi \lim_{\epsilon\to 0}\frac{ie^{-it}e^{-i\alpha z\epsilon e^{-it}}}{-2e^{-it}-\epsilon e^{-2it}}dt = e^{-i\alpha x}\int_0^\pi \frac{ie^{-it}}{-2e^{-it}}dt = \pi\frac{e^{-i\alpha x}}{2i} \ . \tag{A.6}$$

We proceed in exactly the same way for the integration path 4 with a slightly different parametrisation

$$\int_4 \frac{e^{i\alpha xz}}{1-z^2}dz \qquad \begin{cases} z \to 1-\epsilon e^{-it} \\ dz = \frac{dz}{dt}dt = i\epsilon e^{it} \end{cases}$$
$$\Rightarrow \lim_{\epsilon\to 0}\int_0^\pi \frac{e^{i\alpha x\left(1-\epsilon e^{-it}\right)}\left(i\epsilon e^{-it}\right)}{1-\left(1-\epsilon e^{-it}\right)^2}dt = e^{i\alpha x}\int_0^\pi \lim_{\epsilon\to 0}\frac{ie^{i\alpha x\epsilon e^{-it}}e^{-it}}{2e^{-it}-\epsilon e^{-2it}}dt$$
$$= e^{i\alpha x}\int_0^\pi \frac{ie^{-it}}{2e^{-it}}dt = -\pi\frac{e^{i\alpha x}}{2i} \ . \tag{A.7}$$

Using eq.(A.1), we obtain

$$\int_\gamma \frac{e^{i\alpha xz}}{1-z^2}dz = \int_1 ...dz + \int_2 ...dz + \int_3 ...dz + \int_4 ...dz + \int_5 ...dz + \int_6 ...dz = 0$$
$$\Rightarrow \int_1 + \int_3 + \int_5 = \int_{-\infty}^{\infty} \frac{e^{i\alpha xq}}{1-q^2}dq = \pi\frac{e^{i\alpha x}}{2i} - \pi\frac{e^{-i\alpha x}}{2i} = \pi\sin(\alpha x) \ . \tag{A.8}$$

Finally we have

$$\int_{-\infty}^{\infty} \frac{e^{ipx}}{E-\frac{p^2}{2m}}dp = \frac{\sqrt{2mE}}{E}\pi\sin(\sqrt{2mE}|x|) \ . \tag{A.9}$$

The reason why we can also write $|x|$ instead of x is the following: If we take a negative value for x, the integral that we want to calculate is not actually the reverse Fourier transform because the exponent is negative for $p > 0$. So we have to parametrise $p \to -p$ what gives us a factor -1. So this is $-\pi\sin(\alpha x)$ for $x < 0$. We will use the same argument in the other calculations as well.

A.2 $\frac{A'}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ipx}}{E_B-\sqrt{m^2+p^2}}dp$

Again we use the residue theorem. We have a pole at $z = \sqrt{\alpha_B^2-1}$. Because $\alpha_B < 1$, the pole is on the imaginary axis

$$Res\left(\frac{e^{imxz}}{\alpha_B-\sqrt{1+z^2}}, z = \sqrt{\alpha_B^2-1}\right) = \frac{-\alpha_B}{\sqrt{\alpha_B^2-1}}e^{-m\sqrt{1-\alpha_B^2}|x|} \ . \tag{A.10}$$

With the integration path described in fig.(A.2) we write

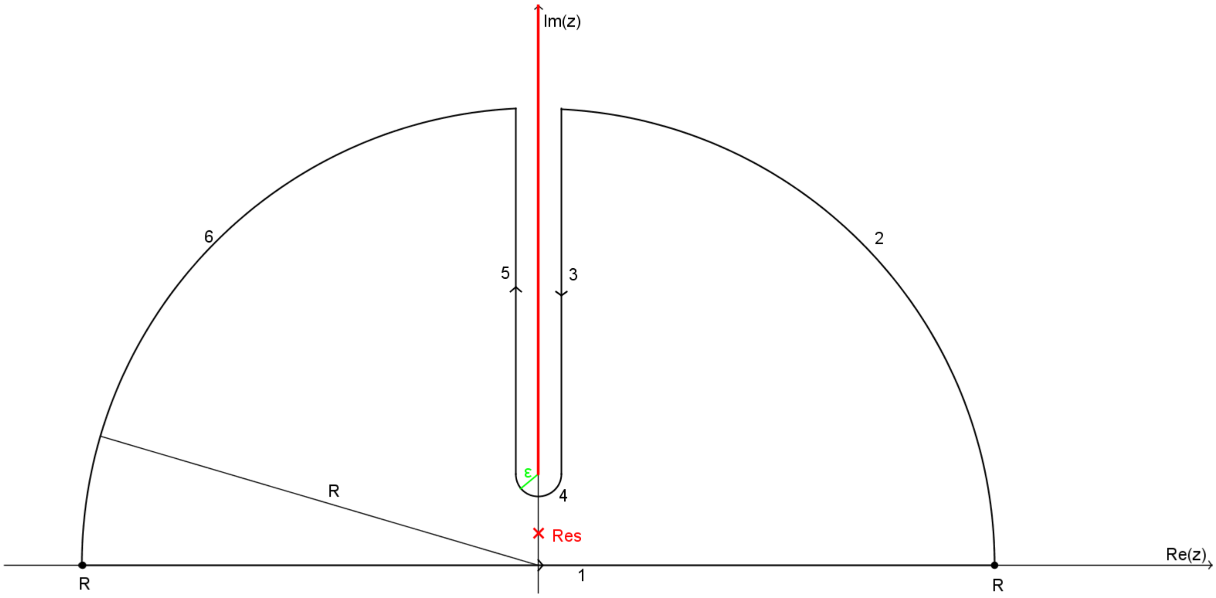

Figure A.2: Integration path to determine the relativistic bound state.

$$\frac{A'}{2\pi}\int_\gamma \frac{e^{imxz}}{\alpha_B - \sqrt{1+z^2}}dz = \frac{-iA\alpha_B}{\sqrt{\alpha_B^2-1}}e^{-m\sqrt{1-\alpha_B^2}|x|} \ . \tag{A.11}$$

The reason why we can not simply cross the imaginary axis is the discontinuous behaviour of the square root of an imaginary number. The quarter-circle paths disappear for $R \to \infty$ with the same argument as before. We have a closer look at the integrals 3 to 5. We must consider the cut carefully. Therefore we go along the axis at the distance ϵ and finally take the limit $\epsilon \to 0$,

$$\int_3 \frac{e^{imxz}}{\alpha_B - \sqrt{1+z^2}}dz \qquad \begin{cases} z = i\mu + \epsilon \\ dz = \frac{dz}{d\mu}d\mu = id\mu \end{cases}$$
$$\implies \int_R^m \frac{e^{imx(i\mu+\epsilon)}i}{\alpha_B - \sqrt{1+(i\mu+\epsilon)^2}}d\mu \overset{\epsilon\to 0}{=} \int_\infty^m \frac{e^{-mx\mu}i}{\alpha_B-\sqrt{1-\mu^2}}d\mu \ . \tag{A.12}$$

Now consider the integral path of a semi-circle with central point at $z = im$ and radius ϵ

$$\int_4 \frac{e^{imxz}}{\alpha_B - \sqrt{1+z^2}}dz \qquad \begin{cases} z = im + \epsilon e^{-it} \\ dz = -i\epsilon e^{-it}dt \end{cases}$$
$$\implies \int_0^\pi \frac{-i\epsilon e^{-it}e^{imx\left(im+\epsilon e^{-it}\right)}}{\alpha_B - \sqrt{1+(im+\epsilon e^{-it})^2}}dt \leq |\epsilon| \int_0^\pi \frac{|e^{-it}e^{imx\left(im+\epsilon e^{-it}\right)}|}{|\alpha_B - \sqrt{1+(im+\epsilon e^{-it})^2}|}dt$$
$$\leq |\epsilon| \int_0^\pi \frac{e^{-mx(m-\sin(t))}}{|\alpha_B - \sqrt{1+(im+\epsilon e^{-it})^2}|}dt \quad . \tag{A.13}$$

If we take the limit $\epsilon \to 0$, this integral disappears.
Finally we go upwards on the other side of the imaginary axis. We use another parametrisation and obtain (be aware of the opposite sign before the square root)

$$\int_5 \frac{e^{imxz}}{\alpha_B + \sqrt{1+z^2}}dz \qquad \begin{cases} z = i\mu - \epsilon \\ dz = id\mu \end{cases}$$
$$\implies \int_m^R \frac{e^{imx(i\mu+\epsilon)}i}{\alpha_B + \sqrt{1+(i\mu-\epsilon)^2}}d\mu = \int_m^\infty \frac{e^{-mx\mu}i}{\alpha_B+\sqrt{1-\mu^2}}d\mu \ . \tag{A.14}$$

We solve eq.(A.11) for the integral path 1 and we obtain (with the use of the rule for exchanging the integration limits that causes a change of sign)

$$
\begin{aligned}
&\frac{A}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ipx}}{E_B-\sqrt{m^2+p^2}}dp = \\
&\frac{-iA\alpha_B}{\sqrt{\alpha_B^2-1}}e^{-m\sqrt{1-\alpha_B^2}|x|}-\frac{A}{2\pi}\left(\int_{\infty}^{m}\frac{e^{-mx\mu}i}{\alpha_B-\sqrt{1-\mu^2}}d\mu+\int_{m}^{\infty}\frac{e^{-mx\mu}i}{\alpha_B+\sqrt{1-\mu^2}}d\mu\right) \\
&=\frac{-A\alpha_B}{\sqrt{1-\alpha_B^2}}e^{-m\sqrt{1-\alpha_B^2}|x|}-\frac{A}{2\pi}\int_{m}^{\infty}\frac{-2i\sqrt{(1-\mu^2)}e^{-m|x|\mu}}{\alpha_B^2-(1-\mu^2)}d\mu \\
&=\frac{-A\alpha_B}{\sqrt{1-\alpha_B^2}}e^{-m\sqrt{1-\alpha_B^2}|x|}-\frac{A}{\pi}\int_{m}^{\infty}\frac{\sqrt{(\mu^2-1)}e^{-m|x|\mu}}{\alpha_B^2+\mu^2-1}d\mu \quad . \qquad (A.15)
\end{aligned}
$$

One might complain that there is the wrong sign. However we know that the physical properties of a quantum mechanical system are invariant under a phase shift, so we can comprehend this as a shifting with the angle π with no further consequences.

$$
-\Psi(x)=e^{\pm i\pi}\Psi(x) \qquad (A.16)
$$

A.3 $\int_{-\infty}^{\infty}\frac{e^{ipx}}{E-\sqrt{m^2+p^2}}dp$

Compared to the bound state, we have two poles on the real axis at $x=\pm\sqrt{E^2-m^2}$. We exclude them by integrating around them with a semi-circle of radius ϵ. So we have no residues inside and we can write

$$
\int_{\gamma}\frac{e^{ipx}}{E-\sqrt{m^2+p^2}}dp=0\ . \qquad (A.17)
$$

Therefore we must only calculate two other integrals, namely 7 and 9

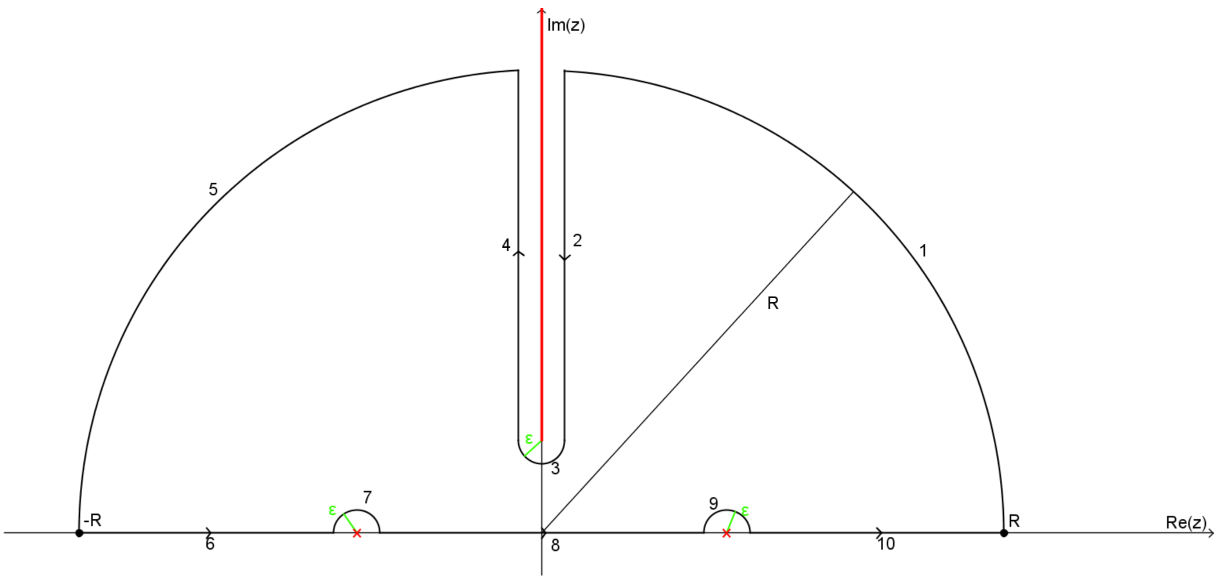

Figure A.3: The integration path we take to solve this integral.

$$
\begin{aligned}
&\int_{7}\frac{e^{ipx}}{E-\sqrt{m^2+p^2}}dp \qquad \begin{cases} p=-\sqrt{E^2-m^2}-\epsilon e^{-it} \\ dp=i\epsilon e^{-it}\end{cases} \\
&\Rightarrow e^{-i\sqrt{E^2-m^2}x}\int_{0}^{\pi}\frac{i\epsilon e^{ix\epsilon e^{-it}}e^{-it}}{E-\sqrt{E^2+2\epsilon\sqrt{E^2-m^2}e^{-it}+(\epsilon e^{-it})}}dt\ . \qquad (A.18)
\end{aligned}
$$

We insert the Taylor series for the square root

$$\sqrt{E^2 + 2\epsilon\sqrt{E^2 - m^2}e^{-it} + (\epsilon e^{-it})} \overset{\epsilon=0}{\cong} E + \frac{\sqrt{E^2 - m^2}}{E}e^{-it}\epsilon + \mathcal{O}(\epsilon^2) \tag{A.19}$$

and obtain for $\epsilon \to 0$

$$\Rightarrow e^{-i\sqrt{E^2-m^2}x} \int_0^{\pi} \frac{i\epsilon e^{-it}}{E - E - \frac{\sqrt{E^2-m^2}}{E}\epsilon e^{-it}} dt = \frac{-i\pi E}{\sqrt{E^2 - m^2}} e^{-i\sqrt{E^2-m^2}x} \ . \tag{A.20}$$

A fully analogous approach (with a slightly different parametrisation) leads to

$$\int_9 \frac{e^{ipx}}{E - \sqrt{m^2 + p^2}} dp = \frac{i\pi E}{\sqrt{E^2 - m^2}} e^{i\sqrt{E^2-m^2}x} \ . \tag{A.21}$$

Together with the integral along the cut, which gives exactly the same result as in the bound state case right before (simply with α instead of α_B), and with $k = \sqrt{E^2 - m^2}$ we obtain

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ipx}}{E - \sqrt{m^2 + p^2}} dp = \frac{E}{k} \sin(k|x|) - \frac{1}{\pi} \int_m^{\infty} \frac{\sqrt{(\mu^2 - 1)}e^{-m|x|\mu}}{\alpha^2 + \mu^2 - 1} d\mu \ . \tag{A.22}$$

Bibliography

[1] http://en.wikipedia.org/wiki/Modern_physics, 11.9.2014.

[2] M.H. Al-Hashimi, A.M. Shalaby, U.-J. Wiese, Asymptotic freedom, dimensional transmutation, and an infrared conformal fixed point for the δ-function potential in one-dimensional relativistic quantum mechanics, Physical Rewiew D 89, 125023, 2014.

[3] D.J. Griffiths, Quantenmechanik - Lehr- und Übungsbuch, 2. Auflage, Pearson Verlag, 2012.

[4] D.G. Currie, T.F. Jordan, E.C.G. Sudarshan, Relativistic Invariance and Hamiltonian Theories of Interacting Particles, Reviews of Modern Physics 35, 350, 1963.

[5] H. Leutwyler, Il Nuovo Cimento 37, N.2, 1965.

[6] D. Klauser, Relativistic Quantum Mechanics of Interacting Particles, Universität Bern, 2004.

[7] W. Bietenholz and L. Prado, Revolutionary physics in reactionary Argentina, Physics Today, 2014.

[8] Formeln und Tafeln, Mathematik - Physik, 11. Auflage, Orell Füssli Verlag AG Zürich, 2006.

[9] Duden - Rechnen und Mathematik, 6. Auflage, S.358, Mannheim, 2000.

[10] Wiler, Skriptum Vorlesung Komplexe Analysis, Universität Bern, 2013.