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# Supersymmetric Quantum Mechanics

**Bachelor Thesis**

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## **Abstract**

This bachelor thesis is an introduction to supersymmetry in one dimensional quantum mechanics. Beginning with the factorization of Hamiltonian we will develop tools to solve energy spectra for many Hamiltonians in a very simple way. At the end we will use all the different aspects we looked at to solve the radial equation of the hydrogen atom.



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# Chapter 1

## Introduction

Supersymmetry (often abbreviated SUSY) is a mathematical concept which arose from theoretical arguments and led to an extension of the Standard Model (SM) as an attempt to unify the forces of nature. It is a symmetry which relates fermions (half integer spin) and bosons (integer spin) by transforming fundamental particles into *superpartners* with the same mass and a difference of  $\frac{1}{2}$  spin. This symmetry, however, has never been observed in nature which means that it needs to be broken, if it exists. This would then allow for the superpartners to be heavier than the corresponding original particles. It was out of the search for spontaneous SUSY breaking that SUSY for quantum mechanics (SUSY QM) was born. The idea is to study symmetry breaking in quantum mechanics to get a better understanding of this process and then draw conclusions for quantum field theory (QFT). After SUSY was introduced into quantum mechanics people started to realize that this field was interesting by itself and not only as a testing ground for QFT. It became clear that SUSY QM gives deeper insight into the factorization method introduced by Infeld and Hull [1] and the solvability of potentials. It even led to the discovery of new solvable potentials. For potentials which are not exactly solvable SUSY allows us to develop very powerful approximation methods such as SUSY WKB which is more precise than the classical WKB approximation. In 1983 the new concept of shape invariant potentials (SIP) was introduced by Gendenshtein [2]. It is a relation between two partner potentials, which if it is satisfied, tells us that the two potentials have the same dependence in the variable and may only differ in other parameters. For potentials which satisfy this condition we can solve the energy spectrum as well as the eigenfunctions analytically. We may also calculate reflection and transmission coefficients algebraically. The problem of classification of the SIPs has not yet been solved, but there are already different classes of shape invariant potentials. They are grouped by the transformation of the parameters such as scaling or translation. The aim of this thesis is a brief introduction to the SUSY theory for quan-

tum mechanics. This will be illustrated by simple examples and we will end with the calculation of the energy spectrum of the hydrogen atom. The spectrum of the hydrogen atom will be calculated using SUSY and the SIP condition and will serve as a conclusion of the SUSY theory learned in the first part of the thesis. The information for this Thesis was taken from the books [3],[4] and papers [5], [6], [7], [8], [9], [10], [11] and [12] as well as from wikipedia.org and www.sunclipse.org.

## Chapter 2

# SUSY QM in 1D

### 2.1 Factorization and partner Hamiltonian

We start with the general Hamiltonian in one dimensional quantum mechanics

$$\mathbb{H}_1 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x), \quad (2.1)$$

and try to factorize it. We want to do this so we can reduce the second order derivative to a derivative of first order to simplify the Schrödinger equation. To do so, we take a different approach on solving the SCHRÖDINGER equation. Instead of assuming the potential to be given, we define the ground state wavefunction  $\psi_0$  of the Hamiltonian to be nodeless and to vanish at  $x = \pm\infty$ . We set the ground state energy to zero, which can be done without loss of generality since we can just shift the potential by a constant value to obtain this.

The Schrödinger equation for  $\psi_0$  reads

$$0 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_0 + V_1(x) \psi_0. \quad (2.2)$$

Since the ground state is nodeless, i.e. it is never zero except at  $x = \pm\infty$ , one can solve the SCHRÖDINGER equation for the potential

$$V_1(x) = \frac{\hbar^2}{2m} \frac{\psi_0''(x)}{\psi_0(x)}. \quad (2.3)$$

In fact, this means that once we know the ground state we also know the potential up to a constant.

We now start by factorizing the Hamiltonian as follows:

$$\mathbb{H} = \mathbb{A}^\dagger \mathbb{A}, \quad (2.4)$$

where

$$\mathbb{A} = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x), \quad \mathbb{A}^\dagger = \frac{-\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x). \quad (2.5)$$

$W(x)$  is generally referred to as the *superpotential*.

The relation between the potential and the superpotential can be found by inserting the two factors eq.(2.5) in the Hamiltonian

$$\begin{aligned}
\mathbb{H}\psi(x) &= \left( \frac{-\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x) \right) \left( \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x) \right) \psi(x) \\
&= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) - \frac{\hbar}{\sqrt{2m}} [W'(x)\psi(x) + \psi'(x)W(x)] \\
&\quad + W(x) \frac{\hbar}{\sqrt{2m}} \psi'(x) + W(x)^2 \psi(x) \\
&= \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \frac{\hbar}{\sqrt{2m}} W'(x) + W(x)^2 \right] \psi(x) \\
&\Rightarrow V(x) = W^2 - \frac{\hbar}{\sqrt{2m}} W'(x)
\end{aligned} \tag{2.6}$$

which is the well-known RICCATI equation. With the potential obtained in eq.(2.3) we now solve the RICCATI equation for the superpotential  $W(x)$  and get

$$W(x) = -\frac{\hbar}{\sqrt{2m}} \frac{\psi_0'(x)}{\psi_0(x)}. \tag{2.7}$$

The solution to the RICCATI equation was obtained by recognizing that, once we satisfy  $\mathbb{A}\psi_0(x) = 0$ , we have a solution  $H_1\psi_0(x) = 0$  as we proposed at the beginning. From these arguments we can directly draw the conclusion

$$\mathbb{A}\psi_0 = 0 \Rightarrow \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} \psi_0 + W(x)\psi_0 = 0, \tag{2.8}$$

$$\Rightarrow \psi_0 = N \exp \left( -\frac{\sqrt{2m}}{\hbar} \int^x W(y) dy \right). \tag{2.9}$$

This means that we only need to solve a first order differential equation to obtain the ground state eigenfunction if the superpotential is known. This is a great simplification compared to the Schrödinger equation which is a second order differential equation.

By reversing the order in the factorization we create a new Hamiltonian, which we refer to as the *partner* Hamiltonian. The new Hamiltonian is of the form

$$\mathbb{H}_2 = \mathbb{A}\mathbb{A}^\dagger, \tag{2.10}$$

where  $\mathbb{A}$  and  $\mathbb{A}^\dagger$  are defined in eq.(2.5). With the same calculations as above we get the new potential in dependence of the superpotential to be

$$V_2(x) = W^2 + \frac{\hbar}{\sqrt{2m}} W'(x). \tag{2.11}$$

This potential is the so called *supersymmetric partner potential*. Let us now take a closer look at the relations between the two Hamiltonians connected

through SUSY.

We begin with the relation between the two energy spectra. For the moment we assume  $E_n^{(1)} \neq 0$  and look at the case  $E_n^{(1)} = 0$  later. The energy levels of the first Hamiltonian can be extracted from the SCHRÖDINGER equation

$$\mathbb{H}_1 \psi_n^{(1)}(x) = \mathbb{A}^\dagger \mathbb{A} \psi_n^{(1)}(x) = E_n^{(1)} \psi_n^{(1)}(x). \quad (2.12)$$

For  $\mathbb{H}_2 = \mathbb{A} \mathbb{A}^\dagger$  we take the ansatz  $\psi_m^{(2)}(x) = \mathbb{A} \psi_n^{(1)}(x)$ :

$$\mathbb{H}_2 (\mathbb{A} \psi_n^{(1)}(x)) = \mathbb{A} \mathbb{A}^\dagger \mathbb{A} \psi_n^{(1)}(x) = E_n^{(1)} (\mathbb{A} \psi_n^{(1)}(x)). \quad (2.13)$$

Of course these calculations can also be made for the second Hamiltonian  $\mathbb{H}_2$ :

$$\mathbb{H}_2 \psi_m^{(2)}(x) = \mathbb{A} \mathbb{A}^\dagger \psi_m^{(2)}(x) = E_m^{(2)} \psi_m^{(2)}(x), \quad (2.14)$$

which implies

$$\mathbb{H}_1 (\mathbb{A}^\dagger \psi_m^{(2)}(x)) = \mathbb{A}^\dagger \mathbb{A} \mathbb{A}^\dagger \psi_m^{(2)}(x) = E_m^{(2)} (\mathbb{A}^\dagger \psi_m^{(2)}(x)). \quad (2.15)$$

It follows from these calculations that  $\mathbb{A} \psi_n^{(1)}(x)$  is, in fact an eigenfunction of  $\mathbb{H}_2$  and  $\mathbb{A}^\dagger \psi_m^{(2)}(x)$  respectively an Eigenfunction of  $\mathbb{H}_1$ . And the corresponding energy level is that of the partner Hamiltonian. Therefore we can conclude, that the partner Hamiltonians have the same spectrum up to possible zero-modes.

We now compare the Eigenfunctions of the two Hamiltonians. As extracted from eq.(2.13)  $\psi_n^{(1)}$  is an Eigenfunction of  $\mathbb{H}_1$  and  $\mathbb{A} \psi_n^{(1)}$  is an Eigenfunction of  $\mathbb{H}_2$  with the same Eigenvalue  $E_n^{(1)}$ . Thus we can identify  $\mathbb{A} \psi_n^{(1)}$  with an Eigenfunction  $\psi_m^{(2)} = \mathbb{A} \psi_n^{(1)}$  of  $\mathbb{H}_2$ . By normalizing the eigenfunctions we get:

$$1 = \int \psi_m^{*(2)} \psi_m^{(2)} = \int \psi_n^{*(1)} \mathbb{A}^\dagger \mathbb{A} \psi_n^{(1)} = E_n^{(1)} \int \psi_n^{*(1)} \psi_n^{(1)}, \quad (2.16)$$

and therefore

$$\psi_m^{(2)} = \left(E_n^{(1)}\right)^{-\frac{1}{2}} \mathbb{A} \psi_n^{(1)} \quad (2.17)$$

is the normalized eigenfunction of  $\mathbb{H}_2$ .

Since  $\mathbb{A} \psi_0^{(1)} = 0$  eq.(2.17) tells us that there is no zero energy ground state for  $\mathbb{H}_2$ . This fact in combination with the eq.(2.13) and eq.(2.15) gives us the possibility to identify  $m$  with  $n$  through  $m = n - 1$ . This means that the lowest energy level of the second Hamiltonian  $\mathbb{H}_2$  is the same as the first excited energy level of  $\mathbb{H}_1$ . Concluding our calculations we already have a few useful relations between the two partners:

$$E_n^{(2)} = E_{n+1}^{(1)}, \quad (2.18)$$

$$\psi_n^{(2)} = \left(E_{n+1}^{(1)}\right)^{-\frac{1}{2}} \mathbb{A} \psi_{n+1}^{(1)}, \quad (2.19)$$

$$\psi_{n+1}^{(1)} = \left(E_n^{(2)}\right)^{-\frac{1}{2}} \mathbb{A}^\dagger \psi_n^{(2)} \quad (2.20)$$

These formulas also lead to the conclusion that

$$\mathbb{A}^\dagger \psi_0^{(2)} = 0 \Rightarrow -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} \psi_0^{(2)} + W(x) \psi_0^{(2)} = 0, \quad (2.21)$$

$$\Rightarrow \psi_0^{(2)} = N \exp \left( \frac{\sqrt{2m}}{\hbar} \int^x W(y) dy \right). \quad (2.22)$$

The relations eq.(2.18) to eq.(2.20) are easily understood by looking at the following figure:

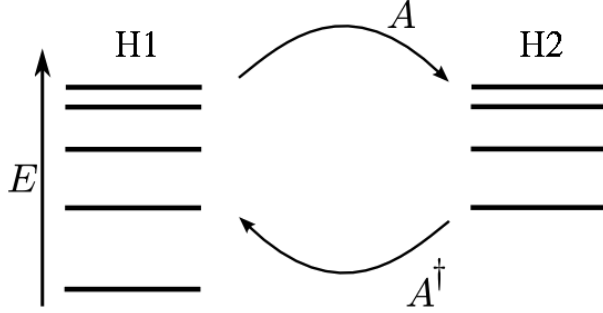


Figure 2.1: Energy spectrum relations of two partner Hamiltonians

As we see, the operators  $\mathbb{A}$  and  $\mathbb{A}^\dagger$  relate the energy states of the two Hamiltonians, we also see that the spectra are degenerate except for the missing zero energy level of the second Hamiltonian. We immediately see a similarity to the raising and lowering operators for the harmonic oscillator, except that in this case we have two different potentials. From the above formulas we conclude that as soon as we have an exactly solvable potential with at least one bound state, supersymmetry allows us to construct a partner potential with the same spectrum except for the zero energy ground state. Furthermore the formulas eq.(2.18) to eq.(2.20) give us all important relations to calculate the eigenfunctions and the energy levels.

Later on we will look at an example for partner Hamiltonians and their energy spectrum, but let us first take a brief look at two partner potentials and their shape. It is very interesting to realize that the shape of two partners does not need to be similar and they still have the same spectrum as explained above. As a simple example we start with the superpotential

$$W(x) = ax^3, \quad (2.23)$$

which then leads to the two partners

$$V_1(x) = a^2 x^6 - 3ax^2, \quad (2.24)$$

$$V_2(x) = a^2 x^6 + 3ax^2. \quad (2.25)$$

by looking at their graphs we see in fig.(2.2) that they differ a lot in their shape, since one potential is a double well and the other just a single well.

However, SUSY tells us that they both have the same energy spectrum which we will show explicitly in an example later on.

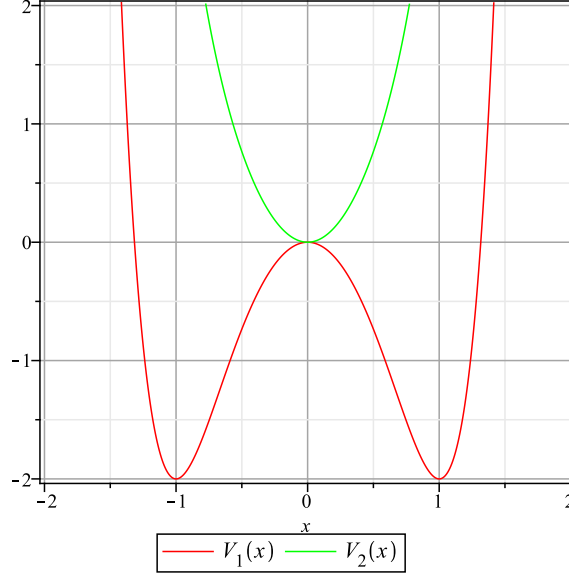


Figure 2.2: Two partner potentials with different shapes

## 2.2 $N=2$ SUSY QM algebra

To get a better understanding of the degeneracy of the partner Hamiltonian spectra let us take a closer look at the underlying algebra. In Quantum Field Theory (QFT) the SUSY algebra is an extension of the Poincaré algebra. In our case it just extends the usual symmetries in Quantum Mechanics, this means that it has two generators ( $\mathbb{Q}, \mathbb{Q}^\dagger \Rightarrow N = 2$ ), which relate the two partner Hamiltonians. This relation between the two partners can also be interpreted in the sense that the two generators exchange the fermionic and bosonic degrees of freedom. This is discussed in *Supersymmetry in Quantum Mechanics*[3] and will not be discussed further in this thesis since it does not play an important role in the calculation of the energy spectrum.

In this thesis we only look at one dimensional quantum mechanics with the Hamiltonian as generator of translations in time.

The SUSY algebra in 1D QM is defined by the following (anti)commutation relations:

$$[\mathbb{H}, \mathbb{Q}] = [\mathbb{H}, \mathbb{Q}^\dagger] = 0, \quad (2.26)$$

$$\{\mathbb{Q}, \mathbb{Q}^\dagger\} = \mathbb{H}, \quad (2.27)$$

$$\{\mathbb{Q}, \mathbb{Q}\} = \{\mathbb{Q}^\dagger, \mathbb{Q}^\dagger\} = 0, \quad (2.28)$$

where the SUSY Hamiltonian contains both partner Hamiltonians

$$\mathbb{H} = \begin{pmatrix} \mathbb{H}_1 & 0 \\ 0 & \mathbb{H}_2 \end{pmatrix}.$$

From the factorization eq.(2.4) and eq.(2.10) of the two Hamiltonians we get for the generators

$$\mathbb{Q} = \begin{pmatrix} 0 & 0 \\ \mathbb{A} & 0 \end{pmatrix},$$

$$\mathbb{Q}^\dagger = \begin{pmatrix} 0 & \mathbb{A}^\dagger \\ 0 & 0 \end{pmatrix},$$

which are often referred to as *Supercharges*.

As we see, both  $\mathbb{Q}$  and  $\mathbb{Q}^\dagger$  commute with the SUSY Hamiltonian, this fact is responsible for the degeneracy of the two spectra.

Let us now take a look at the ground state of  $\mathbb{H}$ . We start by defining the ground state

$$|0\rangle = |\psi_0\rangle = \begin{pmatrix} |\psi_0^{(1)}\rangle \\ |\psi_0^{(2)}\rangle \end{pmatrix}, \quad (2.29)$$

where the vector entries follow from eq.(2.8)

$$\mathbb{A} |\psi_0^{(1)}\rangle = 0 \Rightarrow |\psi_0^{(1)}\rangle(x) = N \exp\left(-\int^x W(y)dy\right) \quad (2.30)$$

$$\mathbb{A}^\dagger |\psi_0^{(2)}\rangle = 0 \Rightarrow |\psi_0^{(2)}\rangle(x) = N \exp\left(\int^x W(y)dy\right), \quad (2.31)$$

which in the algebraic notation takes the following form

$$\mathbb{Q} |0\rangle = \mathbb{Q}^\dagger |0\rangle = 0. \quad (2.32)$$

With SUSY being unbroken only one of the two ground states is normalizable. By convention we choose the normalizable eigenfunction to correspond to the first Hamiltonian, therefore we get

$$|\psi_0\rangle = \begin{pmatrix} |\psi_0^{(1)}\rangle \\ 0 \end{pmatrix}. \quad (2.33)$$

As already discussed earlier, we see that there is no zero energy ground state for the partner Hamiltonian. Actually supersymmetry only tells us that one of the two partners has a zero energy ground state as long as SUSY is unbroken. It's just a convention to define  $\mathbb{H}_1$  to be the one with the zero energy level. Often in supersymmetry we identify the first Hamiltonian with a bosonic system and the second Hamiltonian with a fermionic. In this sense, the supercharges are operators which change bosonic degrees of freedom into fermionic one and vice versa. This idea is mostly studied in Quantum Field Theory and is not of bigger importance for the one dimensional quantum mechanics.



## 2.3 SUSY Example

As a simple illustration of supersymmetric partner potentials, let us look at the infinite square well potential and its SUSY partner.

$$V_{isw}(x) = \begin{cases} 0 & : 0 \leq x \leq a \\ \infty & : else \end{cases}$$

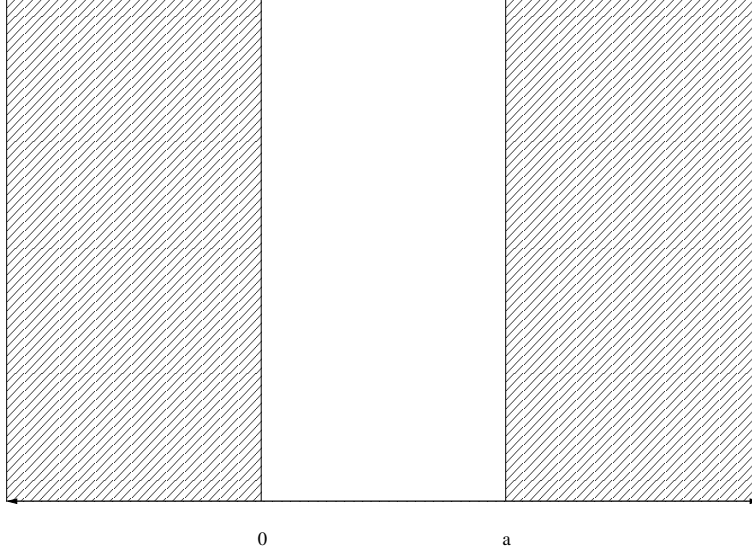


Figure 2.3: The infinite square well

The corresponding Hamiltonian

$$\mathbb{H}_{isw} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_{isw}(x), \quad (2.34)$$

is solved by the ansatz:

$$\psi(x) = A \cdot \sin(kx), \quad (2.35)$$

with the restrictions

$$\psi(0) = \psi(a) = 0. \quad (2.36)$$

From eq.(2.35) and eq.(2.36) we extract the following eigenfunction

$$\psi_{isw}(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right), \quad (2.37)$$

with its derivate

$$\psi'_{isw}(x) = \sqrt{\frac{2}{a}} \frac{\pi}{a} \cos\left(\frac{\pi}{a}x\right). \quad (2.38)$$

To be able to factorize the Hamiltonian we need the lowest energy of  $\mathbb{H}_1$  to be zero. Therefore we shift the potential down by the known ground state energy

$$E_0 = \frac{\pi^2 \hbar^2}{2ma^2}, \quad (2.39)$$

leading to the shifted potential

$$\tilde{V}_1(x) = V_1(x) - E_0. \quad (2.40)$$

With all conditions satisfied for SUSY and the given ground state eigenfunction and its derivative we use eq.(2.7) to obtain the superpotential

$$W(x) = -\frac{\hbar}{\sqrt{2m}} \frac{\pi \cos\left(\frac{\pi}{a}x\right)}{a \sin\left(\frac{\pi}{a}x\right)} = -\frac{\hbar}{\sqrt{2m}} \frac{\pi}{a} \cot\left(\frac{\pi}{a}x\right). \quad (2.41)$$

using eq.(2.11) this leads to the shifted partner potential

$$\tilde{V}_2(x) = \frac{\hbar^2 \pi^2}{2ma^2} \cdot \frac{\cos\left(\frac{\pi}{a}x\right)^2 + 1}{\cos\left(\frac{\pi}{a}x\right)^2 - 1} = \frac{\hbar^2 \pi^2}{2ma^2} \left[ 2 \cdot \operatorname{cosec}^2\left(\frac{\pi}{a}x\right) - 1 \right]. \quad (2.42)$$

The shifted Hamiltonian  $\tilde{\mathbb{H}}_1 = \mathbb{H}_{isw} - E_0$  has the energy spectrum

$$\tilde{E}_n^{(1)} = \frac{n^2 \pi^2 \hbar^2}{2ma^2} \quad n = 0, 1, 2, \dots \quad (2.43)$$

Here we recognize that the shifted Hamiltonian possesses a zero energy state. SUSY allows us to immediately calculate the spectrum of the partner potential without even looking at the SCHRÖDINGER equation of the partner Hamiltonian. This is astonishing since the two potentials do not have the same shape, and do not represent the same quantum mechanical problem. The only difference in the spectrum is the missing zero energy ground state. Therefore the shifted second Hamiltonian has the energy spectrum

$$\tilde{E}_n^{(2)} = \frac{(n+1)^2 \pi^2 \hbar^2}{2ma^2} \quad n = 0, 1, 2, \dots \quad (2.44)$$

Shifting back to the initial problem leads to the following energy spectra for the partner Hamiltonians

$$E_n^{(1)} = \frac{(n+1)^2 \pi^2 \hbar^2}{2ma^2} \quad (2.45)$$

$$E_n^{(2)} = \frac{(n+2)^2 \pi^2 \hbar^2}{2ma^2} \quad n = 0, 1, 2, \dots \quad (2.46)$$

The eigenfunctions of the infinite square well are easily obtained as explained in chapter 2.2 in Introduction to Quantum Mechanics [4] and read as follows

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right), \quad (2.47)$$

which using eq.(2.17), then lead to the eigenfunctions of the new Hamiltonian. We see that the lowest energy eigenstate is annihilated by  $\mathbb{A}$  and all higher eigenstates lose one node, but the general shape of the eigenfunctions remains similar.

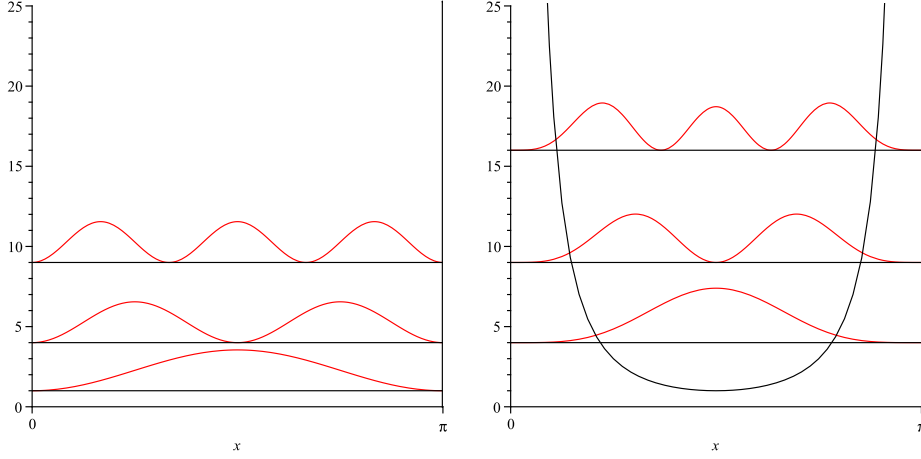


Figure 2.4: Infinite square well and the partner potential  $V_2(x)$

On the left we have the infinite square well and its first three energy eigenvalues  $E_{0,1,2}$  with the squared norm of their relating eigenfunctions  $|\psi_{0,1,2}(x)|^2$ . On the right we have the same information but for the SUSY partner  $V_2(x)$ . For simplicity and clarity we set  $\hbar = 2m = 1$  and multiplied the eigenfunctions by a factor to make them more visible.

This is a very simple example which shows how easy energy spectra of different Hamiltonians can be calculated through supersymmetry. Even though the second Hamiltonian looks complicated, its spectrum and eigenfunctions are very easily obtained.

## 2.4 Broken Supersymmetry

If we have a known potential with at least one bound state, we showed that it is possible to create a partner potential with the same spectrum. Now let us consider a different situation where the superpotential is known and from there we construct two partner potentials. As long as  $\psi_0^{(1)}(x)$  is normalizable, supersymmetry is unbroken. For this to be true  $W(x)$  needs to be positive (negative) for large positive (negative)  $x$ . This follows directly from eq.(2.9). If this is not the case, there will be no zero energy ground state for  $\mathbb{H}_1$  and we say that supersymmetry is broken. For broken supersymmetry the two spectra are degenerate and the operators  $\mathbb{A}, \mathbb{A}^\dagger$  do not change the number

of nodes in the eigenfunctions. Therefore we get for the energies

$$E_n^{(2)} = E_n^{(1)} > 0, \quad (2.48)$$

and the eigenfunctions

$$\psi_n^{(2)} = \left(E_n^{(1)}\right)^{-\frac{1}{2}} \mathbb{A} \psi_n^{(1)}, \quad (2.49)$$

$$\psi_n^{(1)} = \left(E_n^{(2)}\right)^{-\frac{1}{2}} \mathbb{A}^\dagger \psi_n^{(2)}. \quad (2.50)$$

A simple way of determining if supersymmetry is broken was introduced by Edward Witten. He defined the so called *Witten index*

$$\Delta = \text{Tr}(-1)^{N_F}, \quad (2.51)$$

where  $N_F$  is the *fermion number*. The fermion number has either the value 0 or 1 and represents the number of fermions in this state. Because of the Pauli principle there can never be more than one fermion in one state.

Let us look at an example where supersymmetry is broken. Then there are no zero energy ground states and all the bosonic ( $N_f = 0$ ) and fermionic ( $N_f = 1$ ) states come in pairs with the same energy. These pairs cancel each other in the contribution to  $\Delta$  and therefore we obtain  $\Delta = 0$ .

On the contrary, if we assume that there are ground states with zero energy, then these states contribute to  $\Delta$  by

$$\Delta = n_B - n_F, \quad (2.52)$$

where  $n_B$  and  $n_F$  are the respective number of zero energy ground states. Therefore it follows that

$$\Delta \neq 0 \Rightarrow \text{supersymmetry is unbroken.} \quad (2.53)$$

In one dimensional quantum mechanics it's even an equality

$$\Delta \neq 0 \Leftrightarrow \text{supersymmetry is unbroken.} \quad (2.54)$$

Because examples for broken supersymmetry are easily constructed and understood, I will only mention one class of superpotentials which break supersymmetry

$$W(x) = gx^n. \quad (2.55)$$

For  $n$  even, supersymmetry will always be broken and for  $n$  odd we will find a normalizable groundstate and hence SUSY is not broken. The proof for this is very simple and goes as follows.

As discussed above for unbroken SUSY, we need the groundstate to be

normalizable. However if this is not possible, SUSY is broken. For the ground state

$$\psi_0(x) = N \exp \left( -\frac{\sqrt{2m}}{\hbar} \int^x W(y) dy \right), \quad (2.56)$$

to be normalizable it needs to vanish at  $\pm\infty$ , which will be the case if the exponent converges to  $-\infty$  for  $x \rightarrow \pm\infty$ . From these arguments we extract

$$\int_{-\infty}^0 W(y) dy = \infty, \quad (2.57)$$

$$\int_0^{\infty} W(y) dy = \infty. \quad (2.58)$$

Since  $W(y)$  is a polynomial potential and the exponent  $n$  is even we know that the potential will be symmetric and the two equations above will never be satisfied at the same time. This means that SUSY will always be broken for such potentials.

For  $n$  being odd and  $g$  being positive the equations eq.(2.57) and eq.(2.58) will always be satisfied and SUSY unbroken. If  $g$  is negative there will also be a normalizable groundstate because

$$\psi_0^{(2)}(x) = N \exp \left( \frac{\sqrt{2m}}{\hbar} \int^x W(y) dy \right), \quad (2.59)$$

has the opposite sign in the exponent. In this case we just switch the Hamiltonians ( $\mathbb{H}_1 \rightleftharpoons \mathbb{H}_2$ ) because it is a convention to assign the normalizable ground state to the first Hamiltonian  $\mathbb{H}_1$ .

## 2.5 Hierarchy of Hamiltonians

As we learned in section 2.1 SUSY allows us to create partner Hamiltonians with the same energy spectrum. We did this by shifting the first Hamiltonian by a constant to get a zero energy ground state, so we could factorize the Hamiltonian. The idea now is to also shift the new found Hamiltonian by a constant to get a zero energy ground state for  $\mathbb{H}_2$  and forget about  $\mathbb{H}_1$  for the moment. This then allows us to construct a third partner potential. We then restart the procedure of shifting the potential by a constant, factorizing the Hamiltonian and constructing a partner potential. By repeating these steps we get a chain of partner Hamiltonians with almost the same spectrum. Because we always lose the ground state when constructing the partner Hamiltonian the spectrum gets reduced by one energy level in each step.

As we have seen in the last section, we know the relations between two partner Hamiltonians, these will also be valid for each Hamiltonian in the chain, meaning that if we have an exactly solvable Hamiltonian we will be able to extract the energy spectrum for the whole chain as well as all their eigenfunctions. Furthermore we can reconstruct all eigenfunctions of the first Hamiltonian from the knowledge of the ground states of all the Hamiltonians in the chain. These properties will prove very useful in combination with the **Shape Invariant Potential** integrability condition.

Let us begin by repeating the factorization and construction of the first partner Hamiltonian. Since the first Hamiltonian does not naturally need to have a zero energy ground state we rewrite the Hamiltonian as follows

$$\mathbb{H}_1 = \mathbb{A}_1^\dagger \mathbb{A}_1 + E_0^{(1)} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x) \quad (2.60)$$

and the corresponding potential

$$V_1(x) = W_1(x)^2 - \frac{\hbar}{\sqrt{2m}} W_1'(x) + E_0^{(1)} \quad (2.61)$$

with  $\mathbb{A}$  and  $\mathbb{A}^\dagger$  as in (2.5) and  $E_0^{(1)}$  the ground state energy of  $\mathbb{H}_1$ , which basically is just a shift of the Hamiltonian. As it follows from section 2.1, we can immediately write down the partner Hamiltonian

$$\mathbb{H}_2 = \mathbb{A}_1 \mathbb{A}_1^\dagger + E_0^{(1)} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_2(x). \quad (2.62)$$

with the potential

$$\begin{aligned} V_2(x) &= W_1(x)^2 + \frac{\hbar}{\sqrt{2m}} W_1'(x) + E_0^{(1)}, \\ \Rightarrow V_2(x) &= V_1(x) + \frac{2\hbar}{\sqrt{2m}} W_1'(x) = V_1(x) - \frac{2\hbar}{\sqrt{2m}} \frac{d^2}{dx^2} \ln(\psi_0^{(1)}). \end{aligned} \quad (2.63)$$

As seen before this yields

$$\begin{aligned} E_n^{(2)} &= E_{n+1}^{(1)}, \\ \psi_n^{(2)} &= \left(E_{n+1}^{(1)} - E_0^{(1)}\right)^{-\frac{1}{2}} \mathbb{A} \psi_{n+1}^{(1)}. \end{aligned} \quad (2.64)$$

So far we just repeated what was done in section (sec. 2.1), but now we will continue the chain of Hamiltonians and construct a new partner Hamiltonian to the second Hamiltonian. The ground state energy of the second Hamiltonian is  $E_0^{(2)} = E_1^{(1)}$ , which allows us to write

$$\mathbb{H}_2 = \mathbb{A}_1 \mathbb{A}_1^\dagger + E_0^{(1)} = \mathbb{A}_2^\dagger \mathbb{A}_2 + E_0^{(2)} = \mathbb{A}_2^\dagger \mathbb{A}_2 + E_1^{(1)}. \quad (2.65)$$

The factorization was done exactly the same way as in section (2.1)

$$\mathbb{A}_2 = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W_2(x), \quad \mathbb{A}_2^\dagger = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W_2(x). \quad (2.66)$$

where

$$W_2(x) = -\frac{\hbar}{\sqrt{2m}} \frac{\left(\psi_0^{(2)}(x)\right)'}{\psi_0^{(2)}(x)} = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} \ln \left(\psi_0^{(2)}\right). \quad (2.67)$$

Now we construct a new partner Hamiltonian by reversing the order of the factors

$$\mathbb{H}_3 = \mathbb{A}_2 \mathbb{A}_2^\dagger + E_1^{(1)} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_3(x), \quad (2.68)$$

and its corresponding potential thus takes the form

$$\begin{aligned} V_3(x) &= W_2(x)^2 + \frac{\hbar}{\sqrt{2m}} W_2'(x) + E_1^{(1)} = V_2(x) - \frac{2\hbar}{\sqrt{2m}} \frac{d^2}{dx^2} \ln \left(\psi_0^{(2)}\right) \\ &= V_1(x) - \frac{2\hbar}{\sqrt{2m}} \frac{d^2}{dx^2} \ln \left(\psi_0^{(1)}\right) - \frac{2\hbar}{\sqrt{2m}} \frac{d^2}{dx^2} \ln \left(\psi_0^{(2)}\right) \\ &= V_1(x) - \frac{2\hbar}{\sqrt{2m}} \frac{d^2}{dx^2} \ln \left(\psi_0^{(1)} \psi_0^{(2)}\right). \end{aligned} \quad (2.69)$$

Similar to the first partner Hamiltonian we can draw some useful conclusion for the third partner from the above calculations

$$\begin{aligned} E_n^{(3)} &= E_{n+1}^{(2)} = E_{n+2}^{(1)}, \\ \psi_n^{(3)} &= \left(E_{n+1}^{(2)} - E_0^{(2)}\right)^{-\frac{1}{2}} \mathbb{A}_2 \psi_{n+1}^{(2)} \\ &= \left(E_{n+2}^{(1)} - E_1^{(1)}\right)^{-\frac{1}{2}} \left(E_{n+2}^{(1)} - E_0^{(1)}\right)^{-\frac{1}{2}} \mathbb{A}_2 \mathbb{A}_1 \psi_{n+1}^{(1)}. \end{aligned} \quad (2.70)$$

As we see, the solutions of the third Hamiltonian can be expressed through the solutions of the first Hamiltonian. This means that the number of Hamiltonians in the chain is restricted by the number of bound states of the initial

Hamiltonian. For example if there are  $p$  bound states for the initial Hamiltonian we are able to construct a chain of  $p - 1$  Hamiltonians, with the  $m$ 'th Hamiltonian having the same spectrum as  $\mathbb{H}_1$  except for the  $(m - 1)$  lowest energy eigenvalues. This means that we can write a general formula for a chain of Hamiltonians connected through supersymmetry. I will not list them here, since it is more interesting to look at these chains for **Shape Invariant Potentials** (SIP). What one needs to keep in mind is the initial shift performed to obtain a zero energy ground state. To get the actual spectrum of the potential we need to shift back the energy for the first Hamiltonian and this then automatically shifts all the other energy levels. The following figure shows the general principle of a Hamiltonian chain.

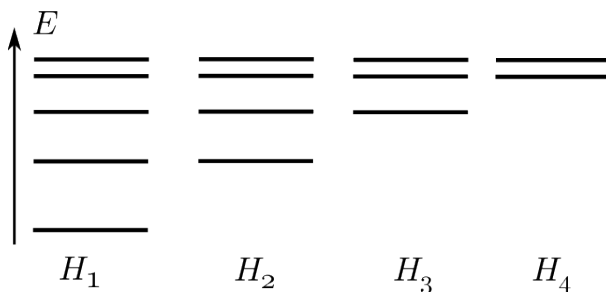


Figure 2.5: Energy spectrum of a Hamiltonian chain

As seen in figure 2.1 the energy levels of the different Hamiltonians are related through  $\mathbb{A}$  and  $\mathbb{A}^\dagger$  which also applies to the chain of Hamiltonians. The graph also shows the reduction of energy levels in every step in the chain.



The figure 2.6 shows the chain of HAMILTONIANS produced from the initial Hamiltonian

$$\mathbb{H} = \frac{\hbar}{2m} \frac{d^2}{dx^2} + V(x), \quad (2.71)$$

with

$$V_{isw}(x) = \begin{cases} 0 & : 0 \leq x \leq a \\ \infty & : else \end{cases}. \quad (2.72)$$

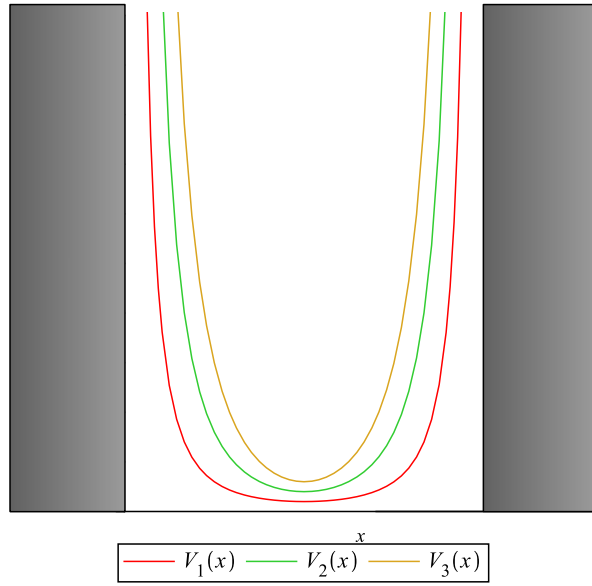


Figure 2.6: The infinite square well and the first three partner potentials of the Hamiltonian chain

In this figure it is clear that every new partner potential will have a higher ground state energy level. In the next section we will use these principles on a special group of potentials which will allow us to solve for the energy spectrum in a very elegant way.



## Chapter 3

# Shape Invariant Potentials

Shape invariant potentials are a great success of supersymmetry in quantum mechanics. As we saw in chapter 2, SUSY allows us to build chains of Hamiltonians with very simple relations between the Hamiltonians in the chain. In this chapter we want to take this idea a step further and look at the benefits of the shape invariance condition. We will see that every well known exactly solvable potential can be solved using SUSY and SIP in a very elegant way. We will also lay the foundation for our calculation of the radial energy spectrum of the hydrogen atom.

We start by looking at two partner potentials  $V_1(x)$  and  $V_2(x)$  related through SUSY. We call a potential **shape invariant** if the following condition is satisfied

$$V_2(x; a_1) = V_1(x; a_2) + R(a_1), \quad (3.1)$$

where  $a_1$  and  $a_2$  are two different sets of parameters related through some function  $a_2 = f(a_1)$ . A very important aspect of SIP is that the remainder  $R(a_1)$  is independent of  $x$ . This simple condition in combination with the hierarchy of Hamiltonians results in a very powerful tool for calculating the energy spectrum if supersymmetry is unbroken. It is important to note that the condition is an equation where the second potential is a function with the set of parameters  $a_1$  and the first potential is written with the new parameters  $a_2$  and the remainder is a function of the old parameters  $a_1$ . One should always pay attention when checking this condition because it can be very confusing with the two sets of parameters.

### 3.1 Energy Spectrum

We start by taking two partner Hamiltonians related through supersymmetry. Then we know that they have the same energy spectrum and their eigenfunctions are related through the relations in eq.(2.19). Since we as-

sume supersymmetry to be unbroken we also know that

$$E_0^{(1)}(a_1) = 0, \quad (3.2)$$

$$\psi_0^{(1)}(x; a_1) = N \exp \left( - \int^x W_1(y; a_1) dy \right). \quad (3.3)$$

Let us now construct a chain of Hamiltonians which satisfies the condition eq.(3.1). We start with the first Hamiltonian

$$\mathbb{H}_1 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x; a_1), \quad (3.4)$$

and its partner Hamiltonian

$$\begin{aligned} \mathbb{H}_2 &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_2(x; a_1) \\ &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x; a_2) + R(a_1), \end{aligned} \quad (3.5)$$

where we used eq.(3.1) to obtain this.

Continuing the Hamiltonian chain we construct the third partner by shifting the second Hamiltonian and applying the same steps as before. The shifted second Hamiltonian  $\tilde{\mathbb{H}}_2$  has the form

$$\tilde{\mathbb{H}}_2 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x; a_2), \quad (3.6)$$

which leads to the third shifted Hamiltonian

$$\begin{aligned} \tilde{\mathbb{H}}_3 &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_2(x; a_2) \\ &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x; a_3) + R(a_2). \end{aligned} \quad (3.7)$$

The reason for shifting the second Hamiltonian lies in the conditions necessary for the factorization of the Hamiltonian, namely that the ground state energy of one of the partners needs to be zero. As we shifted the second Hamiltonian down by  $R(a_1)$  we need to shift back to get the real energy spectrum. By doing so we get the following first three Hamiltonians

$$\begin{aligned} \mathbb{H}_1 &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x; a_1), \\ \mathbb{H}_2 &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x; a_2) + R(a_1), \\ \mathbb{H}_3 &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x; a_3) + R(a_2) + R(a_1). \end{aligned} \quad (3.8)$$

In some cases, the first Hamiltonian is already shifted to obtain a zero energy ground state. In this case this shift needs to be added to each Hamiltonian

to obtain the actual energy spectrum. The relations between the parameters are expressed by the following functions

$$a_2 = f(a_1), \quad (3.9)$$

$$a_3 = f(a_2) = f(f(a_1)). \quad (3.10)$$

We now turn to the energy spectrum of the three Hamiltonians and their relations. Since we assume supersymmetry to be unbroken we know that  $\mathbb{H}_1$  has a zero energy ground state and we also know that  $\mathbb{H}_2$  has the same spectrum except for that zero energy ground state. Because we are interested in the energy difference of the two ground states we take a closer look at the second Hamiltonian and immediately realize that

$$\begin{aligned} \mathbb{H}_2 \psi_0^{(1)}(x; a_2) &= \underbrace{\left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x; a_2) \right] \psi_0^{(1)}(x; a_2)}_{\mathbb{H}_1(x; a_2) \psi_0^{(1)}(x; a_2) = 0} + R(a_1) \psi_0^{(1)}(x; a_2) \\ &= R(a_1) \psi_0^{(1)}(x; a_2). \end{aligned} \quad (3.11)$$

$$= R(a_1) \psi_0^{(1)}(x; a_2). \quad (3.12)$$

We see that the ground state eigenfunction  $\psi_0^{(1)}$  of the first Hamiltonian is also an eigenfunction of the second Hamiltonian just with a different set of parameters  $a_2 = f(a_1)$ . This lets us conclude that the ground state energy of the second Hamiltonian is just the remainder,

$$E_0^{(2)} = R(a_1). \quad (3.13)$$

Further we know from chapter 2 that this energy level is equal to the first excited energy level of  $\mathbb{H}_1$ . Let us now look at the  $k$ 'th partner Hamiltonian

$$\mathbb{H}_k = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x; a_k) + \sum_{i=1}^{k-1} R(a_i). \quad (3.14)$$

As already discussed for the second Hamiltonian we again look for the ground state eigenvalue

$$\begin{aligned} \mathbb{H}_k \psi_0^{(1)}(x; a_k) &= \underbrace{\left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x; a_k) \right] \psi_0^{(1)}(x; a_k)}_{\mathbb{H}_1(x; a_k) \psi_0^{(1)}(x; a_k) = 0} + \sum_{i=1}^{k-1} R(a_i) \psi_0^{(1)}(x; a_k). \end{aligned} \quad (3.15)$$

Once again this means that the ground state energy of the  $k$ 'th Hamiltonian is

$$E_0^{(k)} = \sum_{i=1}^{k-1} R(a_i), \quad (3.16)$$

which, as we know from the last chapter, is equivalent to the  $(k - 1)$ 'th energy level of the initial Hamiltonian  $\mathbb{H}_1$ . Hence to obtain the energy spectrum of the first Hamiltonian we just need to know all ground states of the Hamiltonian chain. Therefore from eq.(3.16) it follows for the spectrum of  $\mathbb{H}_1$  that

$$E_n^{(1)}(a_1) = \sum_{i=1}^n R(a_i); \quad E_0^{(1)} = 0. \quad (3.17)$$

It is important to keep in mind that sometimes  $E_0^{(1)} = 0$  is not always the case and a shift is necessary to obtain this situation. In such a case we need to add the shift at the end of our calculations to obtain the actual energy spectrum.

From the above thoughts we can also draw conclusions for the eigenfunctions of the first Hamiltonian. To do so we recall equation eq.(2.20) and look at a Hamiltonian in the chain. We know from equation eq.(3.14) that the Hamiltonian  $\mathbb{H}_k$  has the ground state  $\psi_0^{(1)}(x; a_k)$  from which we now go back one step and get

$$\psi_1^{(1)}(x; a_{k-1}) \propto \mathbb{A}^\dagger(x; a_{k-1})\psi_0^{(1)}(x; a_k), \quad (3.18)$$

which is the first excited state of  $H_{k-1}$ . By repeating this step we get the unnormalized  $k$ 'th excited state of the first Hamiltonian

$$\psi_k^{(1)}(x; a_1) \propto \mathbb{A}^\dagger(x; a_1)\mathbb{A}^\dagger(x; a_2)\dots\mathbb{A}^\dagger(x; a_k)\psi_0^{(1)}(x; a_{k+1}) \quad (3.19)$$

As we already saw in figure 2.1 there is an explicit relation between two partner Hamiltonians and their eigenfunctions which leads to the following relation between different eigenfunctions of the initial Hamiltonian

$$\psi_k^{(1)}(x; a_1) = \frac{1}{\sqrt{E_{k-1}^{(1)}}}\mathbb{A}^\dagger(x; a_1)\psi_{k-1}^{(1)}(x; a_2). \quad (3.20)$$

This means that for potentials which satisfy the SIP condition, we only need to find the first ground state eigenfunction and the function  $f(a)$  which determines the change of the parameters as well as the remainder and from there we can calculate all eigenfunctions and energy levels.

### 3.2 Example

Let us illustrate the idea of SIP by a simple example. For simplicity, in this example we set  $\hbar = 2m = 1$ . As a comprehensive example let us take a look at the radial equation of the 3D oscillator, this is also a good preparation for the upcoming solution of the radial radial part of the hydrogen atom.

The superpotential of the oscillator in 3D has the following form (as seen in chapter 4 in Supersymmetry in Quantum Mechanics [3])

$$W(r) = \frac{1}{2}\omega r - \frac{(l+1)}{r}, \quad (3.21)$$

where  $l$  is the *azimuthal quantum number*. We know a simple relation between the superpotential and the partner potentials from eq.(2.6). From there we get

$$V_1(r) = \frac{1}{4}\omega^2 r^2 - \omega \left( l + \frac{3}{2} \right) + \frac{l(l+1)}{r^2} \quad (3.22)$$

and

$$V_2(r) = \frac{1}{4}\omega^2 r^2 - \omega \left( l + \frac{1}{2} \right) + \frac{(l+1)(l+2)}{r^2} \quad (3.23)$$

with their corresponding Hamiltonians

$$\mathbb{H}_1 = -\frac{d^2}{dr^2} + V_1(r; l) \quad (3.24)$$

$$\mathbb{H}_2 = -\frac{d^2}{dr^2} + V_2(r; l) \quad (3.25)$$

Figure 3.1 shows the superpotential with the two partner potentials.

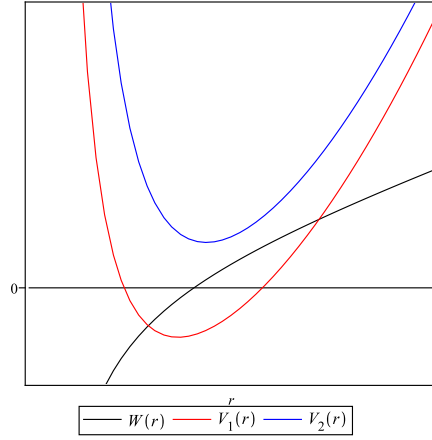


Figure 3.1: The superpotential  $W(r)$  with the two corresponding partner potentials  $V_1(r)$  and  $V_2(r)$

Let us now check whether SUSY is unbroken so we can use SIP and the Hamiltonian chain to extract the energy spectrum. For SUSY to be unbroken we need one of the ground states to be normalizable and have a zero energy. From eq.(2.9) we can calculate the ground state eigenfunction

$$\psi_0^{(1)} = N \exp \left( -\frac{\sqrt{2m}}{\hbar} \int^r W(y) dy \right) = N \exp \left( -\frac{1}{4}\omega r^2 + \ln(r)l + \ln(r) \right), \quad (3.26)$$

which when inserted in the Hamiltonian  $\mathbb{H}_1$  yields a zero energy ground state. Therefore SUSY is unbroken and we are allowed to use the formulas extracted with SIP and the Hamiltonian chain. Already a short look at the two potentials lets us guess that they will satisfy the SIP condition eq.(3.1). To check the SIP condition we first need to rename the parameter  $l$  in the first potential to something new, let's say  $l_2$ , and insert them in the equation

$$\frac{1}{4}\omega^2 r^2 - \omega \left( l + \frac{1}{2} \right) + \frac{(l+1)(l+2)}{r^2} = \frac{1}{4}\omega^2 r^2 - \omega \left( l_2 + \frac{3}{2} \right) + \frac{l_2(l_2+1)}{r^2} + R(l).$$

Since we are interested in the relation  $f(l) = l_2$  we only look at the terms with an  $r$ -dependency. That means we only have to solve a simple quadratic equation to obtain

$$l_2 = f(l) = l + 1. \quad (3.27)$$

Now we plug this into the above SIP condition and solve for the remainder

$$R(l) = 2\omega. \quad (3.28)$$

We are now interested in expanding the chain of Hamiltonians to get the whole energy spectrum of the 3D oscillator. To do so, we forget about the first Hamiltonian and shift the second Hamiltonian by a constant to receive a zero energy ground state. As discussed in the section 3.1 this down shift is identical to the calculated remainder  $R(l)$ . This leads to the new Hamiltonian

$$\tilde{\mathbb{H}}_2 = -\frac{d^2}{dr^2} + V_1(r; l_2) \quad (3.29)$$

with a zero energy ground state.

It is clear that the new partner Hamiltonian (potential), will also satisfy the SIP condition with the same relations as before. Therefore we do not need to calculate all the Hamiltonians in the chain. We only need our result from the first partner Hamiltonians. From there, using

$$E_n^{(1)}(a_1) = \sum_{i=1}^n R(a_i), \quad (3.30)$$

since we know that  $R(l)$  is actually not depended on  $l$  we can extract,

$$E_n^{(1)} = \sum_{i=1}^n R(l_i) = 2n\omega, \quad (3.31)$$

which is the same result as obtained in Supersymmetry in Quantum Mechanics [3].

SIP also allows us to obtain the eigenfunctions for the first Hamiltonian as



shown in the preceding section. After finding the ground state eigenfunction for the first Hamiltonian it is a simple exercise to extract the ground state eigenfunctions for the other Hamiltonians since they only differ in the parameter but not in the  $r$ -dependence. We are then ready to calculate all eigenfunctions of the first Hamiltonian. Let us begin by finding the ground state eigenfunction for  $\mathbb{H}_1$ . This is done with help of eq.(2.30) which yields

$$\begin{aligned}\psi_0^{(1)}(r) &= N^{-1} \exp \left( - \int^r \left[ \frac{1}{2} \omega r' - \frac{(l+1)}{r'} dr' \right] \right) \\ &= N^{-1} \exp \left( -\frac{1}{4} \omega r^2 + \ln(r)l + \ln(r) \right),\end{aligned}\quad (3.32)$$

with the normalization

$$N = \int_0^\infty \left[ \exp \left( -\frac{1}{4} \omega r^2 + \ln(r)l + \ln(r) \right) \right]^2 dr. \quad (3.33)$$

Now we just need to construct the operator

$$\mathbb{A}^\dagger = -\frac{d}{dr} + W(r; a_1) = -\frac{d}{dr} + \frac{1}{2} \omega r - \frac{(l+1)}{r}, \quad (3.34)$$

and we are set to construct all eigenfunctions of the 3D oscillator. For the first excited state we do this explicitly as a guide, the other eigenfunctions are then easily obtained by repeating the same steps.

To obtain the first excited eigenfunction we apply the operator in eq.(3.34) to the ground state,

$$\begin{aligned}\psi_1^{(1)}(r; a_1) &= \mathbb{A}^\dagger(r; a_1) \psi_0^{(1)}(r; a_2) \\ &= -\frac{d}{dr} \psi_0^{(1)}(r; a_2) + W(r; a_1) \psi_0^{(1)}(r; a_2) \\ &= -\frac{d}{dr} \psi_0^{(1)}(r; l+1) + W(r; l) \psi_0^{(1)}(r; l+1),\end{aligned}\quad (3.35)$$

then we normalize the eigenfunction by

$$N_1 = \int_0^\infty \left| \psi_1^{(1)}(r; a_1) \right|^2 dr \quad (3.36)$$

and have already found the first excited eigenstate.

Figure 3.2 shows the first two eigenfunctions and the related eigenvalue implanted in the potential.

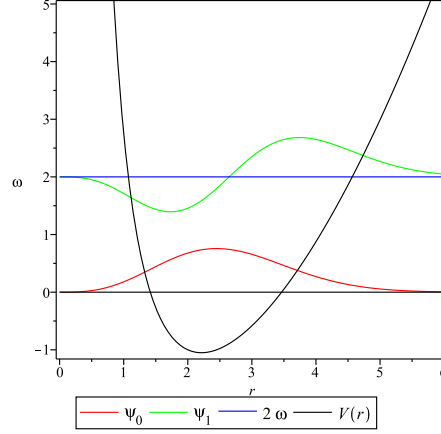


Figure 3.2: First two radial eigenfunctions of the 3D oscillator

In figure 3.3 we see the ground and first excited state energy level with the squared norm of their corresponding eigenfunctions  $|\psi_{0,1}(r)|^2$ . To conclude this example we see that using SUSY, the hierarchy of Hamiltonians and the SIP condition, we can solve analytically solvable problems in a very elegant way. There are only first order differential equations as well as simple quadratic equations that need to be solved to obtain the desired results. In the book *Supersymmetry in Quantum Mechanics* [3] there are many more examples of problems which are easily solved using SIP.

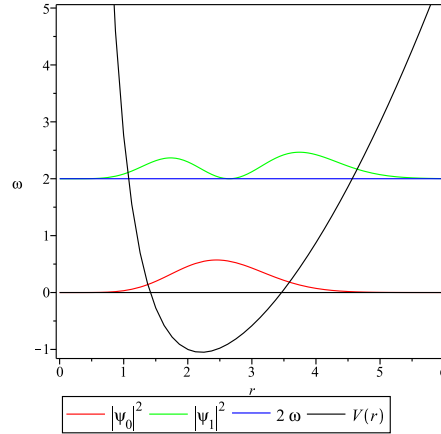


Figure 3.3: Energy levels of the 3-D oscillator

## Chapter 4

# Hydrogen Atom

### 4.1 Introduction

The hydrogen atom is a textbook problem in quantum mechanics. Because of its characteristics to have a heavy essentially motionless proton and a light electron orbiting around it, makes it more easy solvable for quantum mechanics. That's why this example is often looked at when calculating energy levels. In this chapter we want to look at a different way of calculating this spectrum than the classical way by using SUSY (and factorizing the Hamiltonian) and the SIP condition. We will only look at the radial equation since this part satisfies the SIP condition. For a classical way of calculating the spectrum one may consult chapter 4.2 in *Introduction to Quantum Mechanics* [4].

### 4.2 Radial Equation

The motion of the electron is governed by the Coulomb force with the potential

$$V(r) = -\frac{e^2}{4\pi\epsilon_0} \frac{1}{r}. \quad (4.1)$$

By plugging this into the radial Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \left[ V(r) + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \right] u(r) = E_0 u(r) \quad (4.2)$$

for the shifted radial potential we get

$$\tilde{V}(r) = \left[ -\frac{1}{4} \frac{e^2}{\pi\epsilon_0} \right] \frac{1}{r} + \left[ \frac{\hbar^2 l(l+1)}{2m} \right] \frac{1}{r^2} - E_0. \quad (4.3)$$



Figure 4.1: Radial Coulomb Potential

Our next step is to find the corresponding partner potential and verify the SIP condition. Let us begin by finding the superpotential through the differential equation

$$\tilde{V}(r) = W(r)^2 - \frac{\hbar}{\sqrt{2m}} W'(r), \quad (4.4)$$

knowing what  $\tilde{V}(r)$  looks like, we make the ansatz

$$W(r) = C - \frac{D}{r}, \quad (4.5)$$

which leads to

$$\begin{aligned} \tilde{V}(r) &= C^2 - \frac{2CD}{r} + \frac{D^2}{r^2} - \frac{D}{r^2} \frac{\hbar}{\sqrt{2m}} \\ &= C^2 - \frac{1}{r} 2CD + \frac{1}{r^2} \left( D^2 - \frac{\hbar}{\sqrt{2m}} D \right). \end{aligned} \quad (4.6)$$

Since  $C^2$  does not depend on  $r$ , we can directly identify this constant with the ground state energy  $E_0 = -C^2$ . By comparing the coefficients we further get

$$-2CD = -\frac{e^2}{4\pi\epsilon_0}, \quad (4.7)$$

$$D^2 - \frac{\hbar}{\sqrt{2m}} D = \frac{\hbar^2}{2m} l(l+1), \quad (4.8)$$

from where we extract

$$C = \frac{\sqrt{2m}}{\hbar} \frac{e^2}{2 \cdot 4\pi\epsilon_0(l+1)}, \quad (4.9)$$

$$D = \frac{\hbar}{\sqrt{2m}}(l+1). \quad (4.10)$$

It is interesting to see that from this short calculation we already found the ground state energy of the radial HAMILTONIAN (for  $l = 0$ ).

$$E_0 = -C^2 = -\frac{e^4}{4 \cdot 16\pi^2\epsilon_0^2(l+1)^2} \frac{2m}{\hbar^2} = -2.18 \cdot 10^{-18} J \approx -13.6 \text{ eV}. \quad (4.11)$$

Plugging eq.(4.9) and eq.(4.10) in the ansatz eq.(4.5) gives us the superpotential

$$W(r) = \frac{\sqrt{2m}}{\hbar} \frac{e^2}{2 \cdot 4\pi\epsilon_0(l+1)} - \frac{\left(\frac{\hbar}{\sqrt{2m}}(l+1)\right)}{r} \quad (4.12)$$

from which we will continue to the partner potential through eq.( 2.11)

$$V_2(r) = \left[-\frac{1}{4} \frac{e^2}{\pi\epsilon_0}\right] \frac{1}{r} + \left[\frac{\hbar^2(l+1)(l+2)}{2m}\right] \frac{1}{r^2} + \left[\frac{e^4 m}{32\pi^2 \hbar^2 \epsilon_0^2 (l+1)^2}\right]. \quad (4.13)$$

Comparing the partner potential with the first potential eq.(4.3) we may already assume that the SIP condition eq.(3.1) will be satisfied.

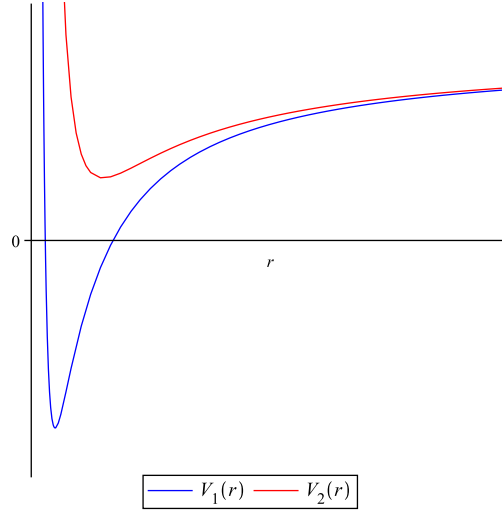


Figure 4.2: The two first hydrogen partner potentials ( $l = 1$ )

In fact, it is easy to see that the relation between the parameters

$$a_2 = f(a_1) \Rightarrow f(l) = l + 1, \quad (4.14)$$

leads to the remainder

$$R(l) = \frac{e^4 m (2l + 3)}{32\pi^2 \hbar^2 \epsilon_0^2 (l + 1)^2 (l + 2)^2}. \quad (4.15)$$

As discussed in chapter 3 this remainder is equal to the energy gap between the zero energy and the first excited state therefore we get

$$E_{\Delta_{0 \rightarrow 1}} = \frac{e^4 m (2l + 3)}{32\pi^2 \hbar^2 \epsilon_0^2 (l + 1)^2 (l + 2)^2}, \quad (4.16)$$

and since we know the ground state energy is not zero, we shift the energy down by the previous shift  $E_0$  and get

$$E_1 = -\frac{e^4 m}{32\pi^2 \hbar^2 \epsilon_0^2 (l + 1)^2} + \frac{e^4 m (2l + 3)}{32\pi^2 \hbar^2 \epsilon_0^2 (l + 1)^2 (l + 2)^2}. \quad (4.17)$$

Since we are interested in the whole energy spectrum we will try to extract a formula for the  $n$ 'th energy level  $E_n$ . We have all the knowledge necessary to do so, because we know the remainder  $R(l)$  between two partner potentials and the relation for the parameter  $f(l) = l + 1$ . Together with the two calculated energy levels this leads to

$$E_n = E_0 + \sum_{i=1}^n \frac{e^4 m (2(l + n - 1) + 3)}{32\pi^2 \hbar^2 \epsilon_0^2 (l + n)^2 (l + n + 1)^2}. \quad (4.18)$$

For  $l = 0$ , we can rewrite this sum into a direct formula which reads

$$E_n = \frac{e^4 m}{32\pi^2 \hbar^2 \epsilon_0^2 (n + 1)^2}, \quad (4.19)$$

which is the known formula for the energy levels and yields the following energies.

Energy Level	[J]	[eV]
$E_0$	$-2.179 \cdot 10^{-18}$	-13.605
$E_1$	$-5.450 \cdot 10^{-19}$	-3.401
$E_2$	$-2.422 \cdot 10^{-19}$	-1.512
$E_3$	$-1.362 \cdot 10^{-19}$	-0.851
$E_4$	$-8.722 \cdot 10^{-20}$	-0.544
$E_5$	$-6.058 \cdot 10^{-20}$	-0.378

Table 4.1: First 6 energy levels of the hydrogen atom.

which is in good agreement with the known energies of the hydrogen atom. The example of the radial energy spectrum of the hydrogen atom shows how powerful these tools can be in quantum mechanics.

# Conclusion

The main aim of the thesis was to give a short introduction to SUSY in quantum mechanics, starting out from the factorization of Hamiltonians and leading to some powerful tools for investigating energy spectra of Hamiltonians. Concerning energy spectra of Hamiltonians, the shape invariance condition is one of the most important achievements of SUSY in quantum mechanics. With it we can calculate the energy spectrum of a Hamiltonian with just a few steps without having to go through the trouble of solving the Schrödinger equation. This shows that even though SUSY in quantum mechanics started as a testing ground for quantum field theory, it turned out to be interesting in its own right. The methods developed with the help of SUSY do not require any knowledge of quantum field theory, therefore they can well be used by undergraduate students to see a different approach to known text book problems in quantum mechanics and thereby lead to a deeper understanding of the solvability of potentials. Furthermore, the reduction of the order of derivatives due to the factorization makes it significantly easier to find eigenfunctions to given Hamiltonians. Even though the factorization method is known since a long time, it is usually only used to solve the harmonic oscillator problem, whereas we saw in this thesis that it can be very useful for a wide range of potentials.

The SUSY algebra was mentioned in this thesis for completeness reasons, but was not further investigated, because it was not of big importance to our calculations. Nevertheless, this would be an interesting topic to look into, specially when considering more than two generators. Moreover, the investigation of SUSY breaking (which plays an important role in quantum field theory) in quantum mechanics could lead to interesting results or new methods which could be applied in quantum field theory. I would like to mention that the possibilities of SUSY in quantum mechanics are still much broader than those looked at in this thesis. For example the relations of transmission and reflection coefficients of partner potentials are an interesting problem to examine, especially potentials without reflection could lead to deeper insight where this non reflection comes from. Many of these topics could be investigated by undergraduate students to make them more accessible to other students, since most papers on these topics are written for more sophisticated readers.





# Appendix A

## Calculations

### A.1 Proof for N=2 SUSY algebra

Here is the short proof of the (anti)commutation relations which define the SUSY algebra. Equation eq.(2.26) follows from

$$[\mathbb{H}, \mathbb{Q}] = \begin{pmatrix} \mathbb{H}_1 & 0 \\ 0 & \mathbb{H}_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \mathbb{A} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \mathbb{A} & 0 \end{pmatrix} \begin{pmatrix} \mathbb{H}_1 & 0 \\ 0 & \mathbb{H}_2 \end{pmatrix} \quad (\text{A.1})$$

$$\Rightarrow [\mathbb{H}, \mathbb{Q}] = \begin{pmatrix} 0 & 0 \\ \mathbb{H}_2 \mathbb{A} - \mathbb{A} \mathbb{H}_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \mathbb{A} \mathbb{A}^\dagger \mathbb{A} - \mathbb{A} \mathbb{A}^\dagger \mathbb{A} & 0 \end{pmatrix} = 0, \quad (\text{A.2})$$

and eq.(2.27) comes from

$$\{\mathbb{Q}, \mathbb{Q}^\dagger\} = \begin{pmatrix} 0 & 0 \\ \mathbb{A} & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbb{A}^\dagger \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathbb{A}^\dagger \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \mathbb{A} & 0 \end{pmatrix} \quad (\text{A.3})$$

$$\Rightarrow \{\mathbb{Q}, \mathbb{Q}^\dagger\} = \begin{pmatrix} \mathbb{A}^\dagger \mathbb{A} & 0 \\ 0 & \mathbb{A} \mathbb{A}^\dagger \end{pmatrix} = \begin{pmatrix} \mathbb{H}_1 & 0 \\ 0 & \mathbb{H}_2 \end{pmatrix}. \quad (\text{A.4})$$

At last eq.(2.28) is proven by

$$\{\mathbb{Q}, \mathbb{Q}\} = \begin{pmatrix} 0 & 0 \\ \mathbb{A} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \mathbb{A} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \mathbb{A} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \mathbb{A} & 0 \end{pmatrix} \quad (\text{A.5})$$

$$\Rightarrow \{\mathbb{Q}, \mathbb{Q}\} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (\text{A.6})$$



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