

# Quantum Link Models with Many Rishon Flavours and with Many Colours

by

Boris Schlittgen

BSc Mathematics with Applied Mathematics / Mathematical Physics  
University of London

Imperial College of Science, Technology and Medicine  
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Author .....

Department of Mathematics

25 April, 2001

Certified by .....

Uwe-Jens Wiese

Associate Professor

Thesis Supervisor

Accepted by .....

Daniel J. Kleitman

Chairman, Committee on Applied Mathematics

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## Abstract

Quantum spin and quantum link models provide an unconventional regularisation of field theory in which classical fields arise via dimensional reduction of discrete variables. This D-theory regularisation leads to the same continuum theories as the conventional approach. We show this by deriving the low-energy effective Lagrangians of D-theory models using coherent state path integral techniques. We illustrate our method for the  $(2 + 1)$ -d Heisenberg quantum spin model which is the D-theory regularisation of the 2-d  $O(3)$  model. We then construct the D-theory formulation of the principal chiral model using symmetry considerations. We show that in the continuum limit the resulting  $(2+1)$ -d quantum spin model with  $SU(N)_L \times SU(N)_R \times U(1)_{L=R}$  symmetry is indeed equivalent to the 2-d principal chiral model. In the quantum link formulation of  $U(N)$  and  $SU(N)$  lattice Yang-Mills theory the gluons are collective excitations of coloured rishon fermions propagating in an additional Euclidean dimension which ultimately disappears via dimensional reduction. We show that  $(4 + 1)$ -d  $SU(N)$  quantum link models reduce to ordinary 4-d Yang-Mills theory in the continuum limit. Quantum link models with  $U(N)$  gauge symmetry can be formulated in terms of colour-neutral rishon pairs. These pairs are described by fields in the coset space  $U(2dM)/[U(dM) \times U(dM)]$  where  $d$  is the dimension of space-time and  $M$  is the number of rishon flavours. The coset field theory has a  $U(M)$  gauge symmetry with the gauge transformations residing on the links of the lattice. We demonstrate confinement in the strong coupling limit of this theory.

Thesis Supervisor: Uwe-Jens Wiese

Title: Associate Professor

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# Contents

<b>1</b>	<b>Introduction</b>	<b>10</b>
<b>2</b>	<b>From the (2+1)-d Heisenberg Quantum Spin Model to the 2-d Classical <math>O(3)</math> Model</b>	<b>19</b>
2.1	The 2-d Classical $O(3)$ Model . . . . .	20
2.2	The Heisenberg Quantum Spin Model . . . . .	21
2.3	The Low-Energy Effective Theory of the Heisenberg Model . . . . .	22
2.3.1	The Antiferromagnetic Case . . . . .	24
2.3.2	The Ferromagnetic Case . . . . .	29
<b>3</b>	<b><math>U(N)</math> Quantum Spins and Quantum Links</b>	<b>32</b>
<b>4</b>	<b>The Principal Chiral Model</b>	<b>38</b>
4.1	The Standard Formulation . . . . .	39
4.2	The Quantum Spin Formulation . . . . .	40
4.3	The Low-Energy Effective Theory of the Quantum Spin Model . . . . .	41
<b>5</b>	<b>Gauge Theory</b>	<b>44</b>
5.1	The Standard Formulation . . . . .	44
5.2	The Quantum Link Formulation . . . . .	52
5.3	The Low-Energy Effective Theory of the Quantum Link Model . . . . .	59
<b>6</b>	<b>Large <math>N</math> Yang-Mills Theory</b>	<b>64</b>
6.1	The Large $N$ Limit of Standard QCD . . . . .	64

6.2	Standard Yang-Mills Theory in the Strong Coupling Limit . . . . .	68
6.3	The $\Phi$ -Model . . . . .	72
6.4	The Strong Coupling Expansion of the $\Phi$ -Model . . . . .	81
6.5	The $\Phi$ -Model Formulation of the Principal Chiral Model . . . . .	87
<b>7</b>	<b>Conclusions</b>	<b>93</b>
<b>A</b>	<b>Some Facts About Compact Semi-Simple Lie Groups</b>	<b>96</b>
A.1	The Group Algebra . . . . .	97
A.2	Representations . . . . .	99
A.3	$SU(N)$ . . . . .	101
<b>B</b>	<b>Coherent States</b>	<b>105</b>
B.1	Generalised Coherent States . . . . .	105
B.2	Coherent States for $SU(2N)$ . . . . .	110
B.3	Coset Decomposition of a $GL(N, \mathbb{C})$ matrix . . . . .	116
<b>C</b>	<b>Integrals over <math>U(dM)</math> and <math>U(2dM)/[U(dM) \times U(dM)]</math> manifolds</b>	<b>118</b>

# List of Figures

2-1	Dimensional reduction of a D-theory . . . . .	29
3-1	Choosing a representation for $SU(2N)$ by which states at each lattice site/link transform. . . . .	35
6-1	Double line representation of Feynman diagrams . . . . .	67
6-2	Integrating over a common link of adjacent plaquettes . . . . .	70
6-3	Wilson's gauge theory: Lowest order contribution to the Wilson loop in the strong coupling expansion . . . . .	71
6-4	Choosing a representation for the $U(2dM)$ algebra of the $\hat{\Phi}$ -model. . . . .	77
6-5	Connected and disconnected graphs in $\Phi$ -theory and Wilson's theory . . . . .	85
6-6	$\Phi$ -model: Lowest order contribution to the Wilson loop in the strong coupling expansion . . . . .	85
6-7	Choosing a representation for the $U(MV)$ algebra of $\hat{\Phi}$ -operators. . . . .	89
6-8	Lowest order contribution to the spin-spin correlation function in the strong coupling expansion. . . . .	92
A-1	Young graphs and permissible Young tableaux . . . . .	104

# List of Tables

6.1	Integrals of the Strong Coupling Expansion . . . . .	84
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*The means by which we live are derived from the four dimensions of time and space. Humanness is created in a fifth dimension. It is the gap which is filled by imagination. Nothing may appear in it which is not derived from the other four. Humanness is material, not transcendental. But the fifth dimension has its own imperatives. It impels us to turn the other four dimensions into ideology and art. It is the source of value, of existential meaning. In it we create our ends. (...) The fifth dimension's structure has no independent reality. Its ideologies and gods are fictitious. But self-consciousness is synonymous with the fifth dimension: so the fictions have the force of reality.*

Edward Bond [1]

# Chapter 1

## Introduction

In the context of particle physics, the conventional approach to field theories is to define a field Lagrangian whose fundamental degrees of freedom are continuous field variables. Such a theory is quantised by introducing a partition function, which is a functional integral over all possible field configurations. In perturbative calculations of correlation functions and other observables, ultraviolet divergences of the resulting integrals are ubiquitous and need to be dealt with at each order of the perturbation expansion. To regularize the ultraviolet divergences beyond perturbation theory it is natural to introduce a space-time lattice. For example, in Wilson's lattice gauge theory gluons are represented by classical  $SU(N)$  parallel transporter matrices. Similarly, the fundamental degrees of freedom of a lattice  $O(3)$  model are classical unit vectors. The continuum limit of lattice field theories is attained at a second-order phase transition, at which the correlation length diverges. At such a critical point, lattice artifacts due to finite lattice spacing disappear. Reaching such a phase transition usually requires fine-tuning of the parameters of the theory. For example, both in Wilson's lattice gauge theory, and in the lattice  $O(3)$  model, the continuum limit is reached by letting the coupling  $g \rightarrow 0$ , because both models are asymptotically free. However, the simulation of lattice models near a phase transition is notoriously difficult due to a phenomenon called critical slowing down. Since at a second order phase transition fluctuations at all length-scales are present, thermal averaging over an ensemble of configurations requires larger and larger statistics, the closer one gets

to the phase transition. Only a few algorithms, called cluster algorithms, do not suffer from critical slowing down and allow simulations close, and in some cases even at a phase transition. Unfortunately such algorithms exist only for a limited number of models, and those models which are of primary interest in particle physics have so far resisted simulation by cluster algorithms.

In this thesis, we follow a different approach to the quantisation of field theories. Inspired by examples from statistical mechanics, where models whose fundamental degrees of freedom are described by quantum operators that act in a Hilbert space play a prominent role, we develop such models in the context of particle physics. In this approach to field theory — D-theory — classical fields are replaced by discrete variables (quantum spins or quantum links) that undergo dimensional reduction. In D-theory, the partition function is a trace in the Hilbert space in which the quantum operators act. However, the physics described by D-theory models should be the same as that described by the corresponding models formulated in the standard way. This is achieved by the mechanism of dimensional reduction.

The D-theory approach to field theories is motivated by the quest for better simulation algorithms. Since the configuration space of D-theory models is entirely discrete, in fact even finite on a finite lattice, such models are much more appropriate for computer simulations than standard models with a continuous configuration space. Furthermore, the continuum limit of D-theory models is attained as the extent of the fifth dimension is taken large, thus requiring no fine-tuning of coupling constants. Most importantly, however, the operators in D-theory Hamiltonians allow a representation in terms of constituent fermion operators, called rishons. Recently, meron cluster algorithms have been developed, which allow the efficient simulation of fermion systems, and other systems, which suffer from complex action or minus sign problems [2, 3]. One such algorithm is based on the D-theory formulation of the  $O(3)$  model and for the first time allowed the simulation of the  $O(3)$  model at non-zero chemical potential [4]. Work is also being done on the development of a cluster algorithm for  $U(1)$  gauge theory, based on the D-theory formulation.

To illustrate the ideas involved in the D-theory formulation of field theories, con-

sider the Heisenberg quantum spin model. At zero temperature the  $O(3)$  symmetries of both, the  $(2+1)$ -d ferromagnetic and antiferromagnetic Heisenberg quantum spin models break spontaneously, giving rise to massless Goldstone bosons — the so-called magnons or spin waves. These magnons are collective excitations of many quantum spins and are effectively described by a continuous classical field. It is remarkable that these continuous degrees of freedom emerge from a microscopic theory of purely discrete quantum spins. The low-energy effective theory of magnons is an  $O(3)$  model in  $(2+1)$  dimensions. At small, non-zero temperature, and hence at finite extent  $\beta$  of the Euclidean time dimension, the correlation length  $\xi$  of the Goldstone bosons is large compared to  $\beta$  and hence the system undergoes dimensional reduction to the 2-d  $O(3)$  model. In this case, the Coleman-Hohenberg-Mermin-Wagner theorem [5, 6, 7] implies that  $\xi$  must become finite and that the magnons pick up a nonperturbatively generated mass gap  $m = 1/\xi$ . As a consequence of asymptotic freedom of the 2-d  $O(3)$  model,  $\xi$  is exponentially large in  $\beta$ ,  $\xi \sim \exp(2\pi\rho_s\beta)$ , where  $\rho_s$  is the spin stiffness of the underlying quantum spin system [8, 9, 10, 11]. Hence,  $\rho_s\beta = 1/g^2$  plays the role of the coupling constant of the dimensionally reduced theory. The continuum limit of that theory is reached by varying the extent  $\beta$  of the extra dimension, not by adjusting a bare coupling constant.

Dimensional reduction of discrete variables is not limited to the quantum Heisenberg model. In fact, it is a generic phenomenon that gives rise to the D-theory formulation of field theory. For example, as we show in this thesis, the 2-d principal chiral model that is traditionally formulated in terms of continuous classical  $U(N)$  matrix fields can also be expressed as a system of generalized quantum spins in  $(2+1)$  dimensions (cf. [12]). The components of these quantum spins are generators of an  $SU(2N)$  algebra. Again, the discrete variables undergo dimensional reduction to 2-d if the  $(2+1)$ -d system has massless Goldstone bosons. We identify appropriate representations of the  $SU(2N)$  algebra for which this is indeed the case.

Gauge theories can be formulated in terms of discrete quantum links which are gauge covariant generalizations of quantum spins. A quantum link is an  $N \times N$  parallel transporter matrix whose elements are generators of  $SU(2N)$ . The dimensional

reduction of quantum link models works differently from the case of quantum spins. While in the spin case an infinite correlation length arises as a result of the spontaneous breakdown of a global symmetry, for gauge theories spontaneous symmetry breaking leads to a massive Higgs phase with a finite correlation length. Moreover, confined phases in non-Abelian gauge theories also have finite correlation lengths and therefore do not lead to dimensional reduction. However, gauge theories in five dimensions can exist in a non-Abelian Coulomb phase with massless unconfined gluons [13, 14]. The massless gluons of a  $(4 + 1)$ -d quantum link model are collective excitations of many quantum links, just as magnons are collective excitations of quantum spins. If a  $(4 + 1)$ -d quantum link model exists in a Coulomb phase for an infinite extent  $\beta$  of the fifth dimension, it will undergo dimensional reduction to an ordinary 4-d gauge theory once  $\beta$  becomes finite [15]. This is a consequence of the confinement hypothesis, which is the gauge analogue of the Coleman-Hohenberg-Mermin-Wagner theorem of the spin case. In particular, a gluon cannot remain massless when  $\beta$  becomes finite because it then effectively lives in a 4-d world and hence should be confined. The corresponding finite correlation length  $\xi$  is related to the glueball mass  $m = 1/\xi$ . As before, asymptotic freedom of 4-d non-Abelian gauge theories implies that  $\xi$  is exponentially large in  $\beta$ ,  $\xi \sim \exp(8\pi^2\beta/11e^2N)$ , where  $e$  is the dimensionful gauge coupling of the underlying  $(4 + 1)$ -d quantum link model. Hence, the role of the coupling constant of the dimensionally reduced theory is played by  $\beta/e^2 = 1/g^2$ . So again, the continuum limit of the theory is reached by varying the extent  $\beta$  of the extra dimension, not by adjusting a bare coupling constant.

As just explained, taking the continuum limit of a quantum link model requires a fifth dimension in order to obtain a large gluonic correlation length. It is then very natural to make use of the fifth dimension to include quarks as domain wall fermions. In particular, Shamir's variant [16, 17] of Kaplan's original proposal [18] provides a suitable realization of full quantum link QCD. Again, the correlation length of the quarks is controlled by the extent of the fifth dimension and is exponentially large in  $\beta$ . Consequently, one reaches both the chiral and the continuum limit by sending  $\beta$  to infinity. This requires no fine-tuning of bare coupling constants, which makes

D-theory an attractive alternative to the traditional approach to field theory [19].

Models with discrete degrees of freedom have been studied before. Generalizations of antiferromagnetic quantum spin models were discussed by Read and Sachdev in [20] for the case of an  $SU(N)$  symmetry group. In [21], Randjbar-Daemi, Salam and Strathdee considered discrete spin systems with a general symmetry group and showed how the continuum limit of such theories corresponds to sigma-model-type field theories. They considered both ferro- and antiferromagnetic cases. In a follow up to that paper, they also investigated the renormalization group flow of a particular continuum theory on the manifold  $SU(3)/[U(1) \times U(1)]$  [22]. A quantum link model with a  $U(1)$  gauge symmetry was first constructed by Horn in [23]. Orland and Rohrlich introduced an  $SU(2)$  quantum link model [24]. In the present context, it was realized in [15] that models with discrete variables can give rise to ordinary field theories, including QCD [19], via dimensional reduction. A detailed analysis of how the physics of conventional lattice gauge theory with  $U(1)$  gauge group is reproduced by the  $U(1)$  quantum link model is given in [25].

A key issue in the previous discussion is the existence of massless Goldstone bosons in quantum spin models, and of massless Coulombic gauge bosons in quantum link models. In the  $(2+1)$ -d spin  $1/2$  antiferromagnetic quantum Heisenberg model it was unclear for some time if the  $O(3)$  symmetry is spontaneously broken. For larger spin representations, however, one can prove analytically that this is indeed the case. By now, detailed numerical simulations have shown that spontaneous symmetry breaking also occurs for spin  $1/2$  [26, 27]. For the  $(2+1)$ -d  $SU(N)_L \times SU(N)_R \times U(1)_{L=R}$  symmetric quantum spin model constructed in this thesis it is a priori not clear if spontaneous symmetry breaking occurs, and thus if massless Goldstone bosons exist. As mentioned above, the degrees of freedom of quantum link models are described by quantum operators acting in a Hilbert space. The symmetries of the model determine the algebraic structure of these operators and in particular, we find that they are generators of the algebra of  $SU(2N)$ . However, the representation in which these generators live is not determined. In this thesis we consider representations for  $SU(2N)$  corresponding to rectangular Young tableaux, of  $N$  rows and  $M$  columns. We thus

show that for a sufficiently large representation of the embedding algebra  $SU(2N)$ , i.e. for large enough  $M$ , this model becomes a Wilson-type lattice principal chiral model in 3-d. Such a model is known to exist in a phase of spontaneously broken symmetry [28, 29] and hence we conclude that at finite extent  $\beta$  of the third dimension it undergoes dimensional reduction to the usual 2-d  $SU(N)_L \times SU(N)_R \times U(1)_{L=R}$  principal chiral model. Similarly, for quantum link models we consider representations of  $SU(2N)$  with rectangular Young tableaux as described above. We show that for sufficiently large  $M$ , the low-energy behavior of a  $(4+1)$ -d quantum link model is that of a Wilson-type lattice gauge theory in 5 dimensions. From numerical simulations, we know that such a model can exist in a non-Abelian Coulomb phase [13, 14] and hence undergoes dimensional reduction to a 4-d  $SU(N)$  Yang-Mills theory once the extent of the fifth dimension becomes finite. This shows that D-theory is indeed a valid regularization of these models. It would be interesting and of practical importance to investigate if the massless phases arise also for small representations of  $SU(2N)$ . This requires detailed future numerical studies.

It has been known for a long time that in the limit as the number of colours gets large, a simplification of ordinary  $SU(N)$  gauge theory takes place. This is rather surprising in view of the fact that the theory then includes an infinite number of fields. Nevertheless, one can show that a systematic expansion in  $1/N$  exists. However, nobody has succeeded in calculating even the first term of this expansion, which would amount to finding the so-called master field. The first simplification of the large  $N$  limit is achieved by the fact that the  $SU(N)$  gauge theory becomes indistinguishable from a  $U(N)$  gauge theory, up to corrections of order  $1/N^2$ . This allows for a representation of Feynman graphs, in which the colour indices of the gauge field propagator follow separate lines, corresponding to a fundamental and an anti-fundamental representation. This double line representation can be used to keep track of the scaling of Feynman diagrams with  $N$  [30]. So far nobody has managed to sum all the planar diagrams, or equivalently, to construct the master field. Still, even without actually solving  $N = \infty$  QCD, it was shown that its spectrum contains weakly coupled glueballs and mesons [31, 32]. Furthermore, using large  $N$  counting

rules, interesting hadron properties have been successfully described (see [33] for a comprehensive review).

Quantum link models offer a completely new approach towards investigating the large  $N$  limit of gauge theories. The choice of a representation with rectangular Young tableaux for the embedding algebra  $SU(2N)$  allows us to write the quantum link operators as products of fermionic creation and annihilation operators. These constituent fermion operators are called *rishons*, which is Hebrew and means “the first”. Rishons carry one colour index and one rishon flavour index. We denote the number of rishon flavours by  $M$ , and it determines the size of the representation of the  $SU(2N)$  embedding algebra. This implies that by using rishons, the two colour indices of quantum link operators are separated completely. Amazingly, the quantum link Hamilton operators for a  $U(N)$  gauge theory can thus be reformulated entirely in terms of colour singlet operators. These operators, which we denote by  $\widehat{\Phi}$ , are the generators of  $U(2dM)$ , where  $d$  is the number of space-time dimensions. They live on the lattice sites, unlike quantum link operators, which live on links. We show that in order to obtain a large  $N$  limit which is consistent with the 't Hooft limit of standard  $SU(N)$  gauge theories, we need  $M \propto N \rightarrow \infty$ . Unfortunately, this implies that the large  $N$  limit of  $\widehat{\Phi}$ -models is not necessarily easier to solve than that of standard gauge theories. Still, our hope is that the  $\Phi$ -field formulation of gauge theories will provide new insight into gauge theories, and may perhaps even lead to the construction of the master field. The  $\widehat{\Phi}$ -model Hamiltonian has a  $U(M)$  gauge symmetry, whose generators live on the links of the lattice, and not on the sites, like the  $U(N)$  colour gauge transformations of the quantum link model. The rishon Hamiltonian thus commutes with two sets of generators, those of  $U(N)$  colour gauge transformations, and those of  $U(M)$  rishon flavour gauge transformations. It turns out that imposing Gauss' law for the  $U(M)$  gauge symmetry of the  $\widehat{\Phi}$ -model corresponds to choosing a representation with rectangular Young tableau in the quantum link model. Analogously, imposing Gauss' law for the  $U(N)$  gauge symmetry of the quantum link model fixes the representation of the  $\widehat{\Phi}$  operators.

Using coherent state techniques, we map the  $\widehat{\Phi}$ -model to a Euclidean field theory,



whose degrees of freedom are elements of the coset space  $U(2dM)/[U(dM) \times U(dM)]$ . In the first part of this thesis we show that in the limit as the number of rishon flavours gets large, ordinary Young Mills theory is recovered from the quantum link model. If we now increase the number of rishon flavours with  $N$ , as we take  $N$  large, it is not clear if the resulting theory still has the correct properties of large  $N$  QCD. In order to investigate this question, we calculate the expectation value of a Wilson loop in strong coupling and find that it obeys an area law. Thus, this new theory is indeed confining. In the future, other physical properties of this theory should be calculated in order to confirm or refute if this is in fact the theory of QCD in the large  $N$  limit. In particular, a very interesting question is whether the theory is asymptotically free in the weak coupling limit. This is part of ongoing research, and we comment on this and other future directions, some complications that arise, as well as ideas of how to resolve them.

This thesis is organised as follows. In chapter 2, we introduce the 2-d classical  $O(3)$  model and the Heisenberg quantum spin model. We illustrate the basic ideas of D-theory by showing how these two models are related. In chapter 3, we introduce quantum spins and quantum links for models with  $U(N)$  symmetries. As a new contribution of this thesis, the operators constructed in this chapter are used in chapter 4 to formulate the principal chiral model as a quantum spin model. This quantum spin model is then related to the principal chiral model in the usual formulation of continuous variables. In chapter 5, we review the quantum link formulation of Yang-Mills theory. A new contribution is the third section of this chapter, in which we show how ordinary Yang-Mills theory emerges from quantum link models after dimensional reduction. In chapter 6 we give a review of large  $N$  Yang-Mills theory and strong coupling methods in Wilson's formulation of the theory. We then present new research regarding the large  $N$  limit of quantum link models. Chapter 7 contains our conclusions. In the appendices we give some mathematical background to the techniques used in this thesis. In Appendix A, we explain some simple facts about Lie groups, Lie algebras and their representations. In Appendix B, we explain how to set up systems of coherent states for  $SU(N)$  groups. Finally, in Appendix C we show how to

compute integrals over  $U(N)$  manifolds and over coset spaces  $U(2N)/[U(N) \times U(N)]$ .

## Chapter 2

# From the (2+1)-d Heisenberg Quantum Spin Model to the 2-d Classical $O(3)$ Model

In this chapter we illustrate the basic ideas of D-theory by explaining the relationship between the (2 + 1)-d Heisenberg quantum spin model and the 2-d  $O(3)$  model of classical spins. Depending on the sign of the coupling constant, the ground state of the Heisenberg model is either antiferromagnetic or ferromagnetic. We consider both possibilities here, and show how both give rise to the 2-d  $O(3)$  model after dimensional reduction. The antiferromagnetic Heisenberg model has been used to describe the physics of undoped precursors of high-temperature superconductors, materials such as  $\text{La}_2\text{CuO}_4$  and  $\text{Sr}_2\text{CuO}_2\text{Cl}_2$ . The ferromagnetic Heisenberg model has been used to describe ferromagnetic behaviour in metals. In both these systems, the ground state spontaneously breaks the  $SU(2)$  symmetry, giving rise to massless Goldstone excitations. We show how these excitations lead to dimensional reduction of the model, once the extent of the third dimension is made finite. The resulting effective theory is the 2-d  $O(3)$  model. The 2-d  $O(3)$  model has also been studied in a particle theory context. It is asymptotically free and has a nonperturbatively generated mass gap, as well as instantons and  $\theta$ -vacua. It has therefore been used as a simple toy model for QCD in four dimensions. In the D-theory approach to field theory, we want

to learn about the 2-d  $O(3)$  model by studying the (2+1)-d Heisenberg model, for example by numerical techniques. This approach is the opposite of the one taken in condensed matter physics, where, as explained above, the 2-d  $O(3)$  model is treated as an effective model, which describes the physics of Heisenberg spin models.

## 2.1 The 2-d Classical $O(3)$ Model

The continuum action of the 2-d  $O(3)$  model is given by

$$S[\mathbf{s}] = \frac{1}{2g^2} \int d^2x \partial_\mu \mathbf{s} \cdot \partial_\mu \mathbf{s}, \quad (2.1)$$

where  $\mathbf{s}$  is a classical, 3-component unit vector that assumes continuous values. Clearly, this action is invariant under global  $O(3)$  transformations  $\mathbf{s}' = R\mathbf{s}$ , where  $R^T R = \mathbb{1}$ . The standard procedure to regularise this theory beyond perturbation theory is to follow Wilson and introduce a lattice as an ultraviolet cut-off. Partial derivatives are then replaced by finite differences and (after dropping an irrelevant constant), the lattice action takes the form,

$$S[\mathbf{s}] = -\frac{1}{g^2} \sum_{x,\mu} \mathbf{s}_x \cdot \mathbf{s}_{x+\hat{\mu}}. \quad (2.2)$$

Here,  $\hat{\mu}$  is the unit-vector in the  $\mu$ -direction. The theory is quantised by writing down the partition function, which is a path integral over classical field configurations,

$$Z = \int \mathcal{D}\mathbf{s} \exp(-S[\mathbf{s}]). \quad (2.3)$$

Due to asymptotic freedom, the continuum limit of the lattice-regularized theory is attained by taking the bare coupling constant  $g$  to zero. In this limit, the correlation length  $\xi \sim \exp(2\pi/g^2)$  diverges exponentially, thus eclipsing any short-distance lattice artifacts.

## 2.2 The Heisenberg Quantum Spin Model

D-theory follows a radically different approach to field quantisation. Instead of performing a path integral over continuous classical fields, those fields are replaced by discrete quantum variables. For example, the above 2-d  $O(3)$  field theory is formulated in terms of quantum spins with a Heisenberg model Hamiltonian,

$$\hat{H} = J \sum_{x,\hat{\mu}} \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_{x+\hat{\mu}}. \quad (2.4)$$

The components of the spin vectors  $\hat{\mathbf{S}}_x$  are the generators of  $SO(3)$  and they satisfy the usual commutation relations

$$[\hat{S}_x^i, \hat{S}_y^j] = i\delta_{xy}\epsilon_{ijk}\hat{S}_x^k. \quad (2.5)$$

Notice that we are free to choose any representation of  $SO(3)$  for the generators  $\hat{\mathbf{S}}_x$ , not just spin 1/2. The  $SO(3)$  symmetry of the quantum Hamiltonian is expressed as  $[\hat{H}, \hat{\mathbf{S}}] = 0$ , where  $\hat{\mathbf{S}} = \sum_x \hat{\mathbf{S}}_x$ .

The case  $J < 0$  corresponds to a ferromagnet and  $J > 0$  to an antiferromagnet with a Néel-ordered ground state. The partition function for the Heisenberg model is given by

$$Z = \text{Tr} \exp(-\beta\hat{H}), \quad (2.6)$$

where the trace is taken in a large Hilbert space, the direct product of the Hilbert spaces corresponding to individual lattice sites. The Hamiltonian evolves the system in an extra dimension, giving rise to a (2+1)-dimensional field theory. For a condensed matter quantum spin system the additional dimension is Euclidean time. In D-theory, however, Euclidean time is part of the 2-d lattice and the additional Euclidean dimension will ultimately disappear via dimensional reduction.

## 2.3 The Low-Energy Effective Theory of the Heisenberg Model

As discussed in the introduction, dimensional reduction requires an infinite correlation length, which in this case is due to the existence of massless Goldstone bosons. One way of addressing the question of symmetry breaking and hence dimensional reduction, is to investigate the limit of large spin  $S$ . To do this, we set up a spin coherent state representation of the path integral as discussed in [34]. The mathematical theory underlying the construction and properties of coherent states is explained in [35] and reviewed in appendix B. The highest weight vector of the representation with spin  $S$  is denoted by  $|S, S\rangle$ . To generate a system of coherent states, we must therefore act with all group elements on this state. A general  $SO(3)$  group element can be parametrised using the three Euler angles as

$$R(\chi, \theta, \phi) = \exp(i\phi\widehat{S}^3) \exp(i\theta\widehat{S}^2) \exp(i\chi\widehat{S}^3). \quad (2.7)$$

Hence, we obtain the system of coherent states, with the following parametrisation,

$$|\mathbf{s}\rangle = R(\chi, \theta, \phi) |S, S\rangle = \exp(i\phi\widehat{S}^3) \exp(i\theta\widehat{S}^2) \exp(i\chi\widehat{S}^3) |S, S\rangle. \quad (2.8)$$

Notice that  $\exp(i\chi\widehat{S}^3) |S, S\rangle = \exp(i\chi S) |S, S\rangle$ , generating the isotropy subgroup for the highest weight vector,  $|S, S\rangle$ . Choosing  $\chi \equiv 0$  then corresponds to taking a cross section in the fiber bundle with base  $X = SO(3)/SO(2)$  and fiber  $SO(2)$ .

The coherent states are now parametrised only by  $\theta$  and  $\phi$ , which fall in the ranges  $\theta \in [0, \pi]$  and  $\phi \in [-\pi, \pi)$ , so that we can think of  $\mathbf{s}$  as a vector on the unit sphere,  $\mathbf{s} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ . To obtain an expression for the coherent states as a superposition of  $\widehat{S}^3$ -eigenstates, we introduce Schwinger bosons —  $\hat{a}, \hat{a}^\dagger$  and  $\hat{b}, \hat{b}^\dagger$  — which are two sets of boson creation and annihilation operators, satisfying the usual commutation relations. These are the analogue of the rishon operators, which we introduce in the next chapter to express operators of an  $SU(2N)$  algebra. Note

that the properties of a D-theory Hamiltonian are completely determined, once a representation for its quantum operators has been chosen. In particular, the physics is independent of whether this representation is expressed in terms of bosonic or fermionic creation and annihilation operators. For convenience we choose rishon operators of later chapters to be fermions. In terms of the Schwinger bosons just introduced we can express the spin operators as,

$$\widehat{S}^1 + i\widehat{S}^2 = \widehat{a}^\dagger \widehat{b}, \quad \widehat{S}^1 - i\widehat{S}^2 = \widehat{b}^\dagger \widehat{a}, \quad \widehat{S}^3 = \frac{1}{2}(\widehat{a}^\dagger \widehat{a} - \widehat{b}^\dagger \widehat{b}). \quad (2.9)$$

The additional constraint  $\widehat{n}_a + \widehat{n}_b = 2S$  fixes the representation of spin  $S$ . We can use the raising and lowering operators to generate the other weight vectors,

$$|S, m\rangle = \frac{(\widehat{a}^\dagger)^{S+m}}{\sqrt{(S+m)!}} \frac{(\widehat{b}^\dagger)^{S-m}}{\sqrt{(S-m)!}} |0\rangle. \quad (2.10)$$

To rotate such a state by  $R$  as above, we note that

$$\begin{aligned} \begin{pmatrix} \widehat{a}^\dagger \\ \widehat{b}^\dagger \end{pmatrix}' &= \begin{pmatrix} R \widehat{a}^\dagger R^{-1} \\ R \widehat{b}^\dagger R^{-1} \end{pmatrix} = \exp\left(i\frac{\chi}{2} \sigma^z\right) \exp\left(i\frac{\theta}{2} \sigma^y\right) \exp\left(i\frac{\phi}{2} \sigma^z\right) \begin{pmatrix} \widehat{a}^\dagger \\ \widehat{b}^\dagger \end{pmatrix} \\ &= \begin{pmatrix} u \exp\left(i\frac{\chi}{2}\right) & v \exp\left(i\frac{\chi}{2}\right) \\ -v^* \exp\left(-i\frac{\chi}{2}\right) & u^* \exp\left(-i\frac{\chi}{2}\right) \end{pmatrix} \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix}. \end{aligned} \quad (2.11)$$

Here,  $u(\theta, \phi) = \cos(\theta/2) \exp(i\phi/2)$  and  $v(\theta, \phi) = \sin(\theta/2) \exp(-i\phi/2)$ . This leads to the following expression for the coherent states (recall that  $\chi \equiv 0$ ),

$$|s\rangle = \frac{(\widehat{a}^\dagger)^{2S}}{\sqrt{(2S)!}} |0\rangle = \sqrt{(2S)!} \sum_m \frac{u^{S+m} v^{S-m}}{\sqrt{(S+m)!(S-m)!}} |S, m\rangle. \quad (2.12)$$

These coherent states are not a basis for the Hilbert space, because they are not all linearly independent. They do, however, form an over-complete set of states and we can express the identity operator as a superposition of coherent state projection operators. The measure of integration we use here is  $[(2S+1)/4\pi] ds = [(2S+$

1)/4\pi] \sin \theta d\theta d\phi,

$$\frac{2S+1}{4\pi} \int d\mathbf{s} |\mathbf{s}\rangle \langle \mathbf{s}| = \sum_m |S, m\rangle \langle S, m| = \mathbf{1}. \quad (2.13)$$

Another important property is the following eigenvalue relation,

$$\mathbf{s} \cdot \widehat{\mathbf{S}} |\mathbf{s}\rangle = S |\mathbf{s}\rangle. \quad (2.14)$$

These two properties, (2.13) and (2.14) will be used in the next two subsections to set up coherent state path integrals for both, the Heisenberg antiferromagnet and the ferromagnet.

### 2.3.1 The Antiferromagnetic Case

The ground state of the antiferromagnetic Heisenberg model is Néel-ordered. Thus, we divide the lattice into two sublattices,  $A$  and  $B$ . Lattice sites belong to sublattice  $A$  or  $B$ , depending on whether  $x_1 + x_2$  is even or odd, respectively. For sites of sublattice  $A$ , systems of coherent states are constructed in the way just explained. For sublattice  $B$ , we construct a system of coherent states starting out with the lowest state vector  $|S, -S\rangle$  instead. In the notation of (2.12), an expression for the coherent states in terms of  $\widehat{S}^3$  eigenstates is

$$|\mathbf{s}\rangle = \sqrt{(2S)!} \sum_m \frac{(-v^*)^{S+m} (u^*)^{S-m}}{\sqrt{(S+m)!(S-m)!}} |S, m\rangle. \quad (2.15)$$

The resolution of the identity in these coherent states takes exactly the same form as in (2.13), while the equivalent of (2.14) is now

$$\mathbf{s} \cdot \widehat{\mathbf{S}} |\mathbf{s}\rangle = -S |\mathbf{s}\rangle. \quad (2.16)$$

The big Hilbert space on which the entire Hamiltonian acts is just the direct product of all the Hilbert spaces associated with individual lattice sites. A system of coherent states in this big Hilbert space is therefore given by a direct product of the



coherent state systems we have derived for lattice sites belonging to sublattices  $A$  and  $B$ . Using properties (2.14) and (2.16) it is straightforward to obtain an expression for the expectation value of the Heisenberg Hamiltonian in a coherent state,

$$\mathcal{H}[\mathbf{s}] = \langle \mathbf{s} | H | \mathbf{s} \rangle = -\frac{S^2}{2} J \sum_{x, \hat{\mu}} \mathbf{s}_x \cdot \mathbf{s}_{x+\hat{\mu}}. \quad (2.17)$$

Using these ingredients, we can express the partition function (2.6) as a path integral over coherent states. The standard procedure is to divide up the “time” interval  $\beta$  into  $N_\varepsilon$  small intervals of width  $\varepsilon = \beta/N_\varepsilon$ , and to insert a resolution of the identity (2.13) in between each time slice. Eventually, we take  $N_\varepsilon \rightarrow \infty$ . We can manipulate the expression for the path integral using (2.17). Assuming field configurations that are smooth in the third direction, we write

$$\frac{\mathbf{s}_x(\tau + \varepsilon) - \mathbf{s}_x(\tau)}{\varepsilon} \rightarrow \dot{\mathbf{s}}_x + \mathcal{O}(\varepsilon). \quad (2.18)$$

In our parametrisation of the coherent states, the overlap between neighbouring states is given by

$$\langle \mathbf{s}(\tau + \varepsilon) | \mathbf{s}(\tau) \rangle = \exp \left( -iS\varepsilon \sum_x \eta_x \dot{\phi}_x \cos(\theta_x) \right). \quad (2.19)$$

Here,  $\eta_x = 1$  for  $x$  in sublattice  $A$  and  $\eta_x = -1$  for  $x$  in sublattice  $B$ . We thus get the path integral

$$Z = \int \mathcal{D}\mathbf{s} \exp(-S[\mathbf{s}]), \quad (2.20)$$

where

$$S[\mathbf{s}] = -iS \sum_x \eta_x \omega[\mathbf{s}_x] + \int_0^\beta d\tau \mathcal{H}[\mathbf{s}], \quad (2.21)$$

and  $\omega[\mathbf{s}] = -\int_0^\beta d\tau \dot{\phi} \cos \theta = -\oint_{\phi_0}^{\phi_0} d\phi \cos(\theta_\phi)$  is a Berry phase term. The geometric nature of the Berry phase term is evident, as it depends only on the path on the unit sphere traced out by the spin, and not on the explicit dependence of this path on  $\tau$ . In fact, this term measures the area enclosed by the path  $\mathbf{s}(\tau)$  on  $S^2$ . It may be

written in gauge invariant form as

$$\omega[\mathbf{s}] = \int_0^\beta d\tau \int_0^1 dv \mathbf{s} \cdot (\partial_\tau \mathbf{s} \times \partial_v \mathbf{s}). \quad (2.22)$$

We have introduced the interpolating field  $\mathbf{s}(\tau, v)$ , which depends on an additional (fourth) dimension and obeys the boundary conditions  $\mathbf{s}(\tau, 1) = \mathbf{s}(\tau)$ ,  $\mathbf{s}(\tau, 0) = \mathbf{s}(\tau', 0)$  and  $\mathbf{s}(0, v) = \mathbf{s}(\beta, v)$ . This field is therefore parametrised on a disc with  $v = 1$  as the boundary. The only requirement on the interpolation  $\mathbf{s}(t, v)$  from the boundary of the disc to the interior is that it be a smooth function of  $v$ . In particular, the value of the Berry phase term differs from one chosen interpolation to another only by an integer multiple of  $4\pi$ . This is a direct consequence of the fact that the second homotopy group of  $S^2$  is  $\Pi_2(S^2) = \mathbb{Z}$ . Hence, for the value of the path integral to be independent of the chosen interpolation, the spin  $S$  needs to be quantised in half-integer units.

To determine the low-energy behavior of the model, we expand about the Néel ordered ground state. Haldane proposed separating the long and short wavelength fluctuations by expanding the field  $\mathbf{s}$  as follows [36, 37],

$$\mathbf{s}_x \approx \mathbf{s}(x) \sqrt{1 - a^2 |\mathbf{L}(x)|^2} + \eta_x a \mathbf{L}(x). \quad (2.23)$$

We have introduced two new fields, the unimodular Néel-field  $\mathbf{s}(x)$ , with  $|\mathbf{s}(x)|^2 = 1$ , and the transverse canting field  $\mathbf{L}(x)$ . They satisfy  $\mathbf{s}(x) \cdot \mathbf{L}(x) = 0$ . The motivation for this change of variables is the underlying assumption that the quantum ground state is sufficiently close to the classical Néel state, so that the Néel field  $\mathbf{s}$  captures the long-distance dynamics. The canting field  $\mathbf{L}$  should be small, and its fluctuations should not destroy the approximate Néel order of the ground state. Equivalently, we assume the existence of an intermediate cut-off scale  $\Lambda$ , where  $\Lambda$  is much larger than the inverse correlation length, and therefore encloses the physically relevant region. Furthermore,  $\Lambda$  needs to be smaller than the inverse lattice spacing, so that edge effects do not destroy the dynamics at the centre of the Brillouin zone. Then a Gaussian approximation to the path integral is valid. The existence of such an intermediate cut-off scale  $\Lambda$  will be verified a posteriori. Assuming its existence for

now, we can expand the action in powers of the lattice spacing  $a$ , up to order  $a^2$  and take the continuum limit  $\sum_x a^2 \rightarrow \int d^2x$ . Upon dropping an irrelevant constant, the  $\mathcal{H}$ -term leads to

$$\mathcal{H}[\mathbf{s}] = \frac{S^2 J}{4} \int d^2x (|\partial_\mu \mathbf{s}|^2 + 8|\mathbf{L}|^2). \quad (2.24)$$

The Berry phase term can be shown to become

$$-iS \sum_x \eta_x \omega[\mathbf{s}_x] = -i\Gamma - iaS \int_0^\beta d\tau \sum_x (\mathbf{s} \times \partial_\tau \mathbf{s} \cdot \mathbf{L}). \quad (2.25)$$

Here,  $\Gamma = S \sum_x \eta_x \omega[\mathbf{s}(x)]$  is the topological Berry phase associated with the Néel field  $\mathbf{s}(x)$ . It was argued in [36, 37] that while in one dimension this term leads to very different behavior for integer and half-integer spins, in higher dimensions it vanishes due to cancelling contributions from neighbouring sites. We can therefore disregard it in the low-energy effective action. In the full action, we can complete the square and perform the Gaussian integral over the shifted field

$$\mathbf{L}' = \mathbf{L} - \frac{i}{4SJ a} (\mathbf{s} \times \partial_\tau \mathbf{s}) \quad (2.26)$$

to obtain an action

$$S[\mathbf{s}] = \int_0^\beta d\tau \int d^2x \frac{\rho_s}{2} \left( \partial_\mu \mathbf{s} \cdot \partial_\mu \mathbf{s} + \frac{1}{c^2} \partial_\tau \mathbf{s} \cdot \partial_\tau \mathbf{s} \right). \quad (2.27)$$

Here,  $\rho_s = S^2 J/2$  is the spin stiffness, and  $c = \sqrt{2} J S a$  is the spin wave velocity. We have also used that  $|\mathbf{s} \times \partial_\tau \mathbf{s}|^2 = |\partial_\tau \mathbf{s}(x)|^2$ . Notice that the Néel field of (2.27),  $\mathbf{s}(x)$ , is a unit vector field. It therefore lives in  $S^2$ , which is the coset space corresponding to a symmetry breaking pattern  $SO(3) \rightarrow SO(2)$ . Thus, the low-energy effective theory in (2.27) is a theory of Goldstone bosons associated with this symmetry breaking. The inverse correlation length of this model is

$$\xi^{-1} \propto \exp(-2\pi\beta\rho_s), \quad (2.28)$$

and since  $\rho_s \propto S^2$ , we see that for large enough  $S$ ,  $\xi^{-1}$  is small enough to guarantee

the existence of the intermediate cut-off scale  $\Lambda$ .

The mechanism of dimensional reduction was explained in [10] and we reproduce the argument here. Consider a system of dimensions  $L \times L \times \beta$ . For  $L = \beta = \infty$ , the ground state of the Heisenberg antiferromagnet is Néel-ordered, which breaks the  $SO(3)$  symmetry and as discussed above, we obtain the low-energy theory of Goldstone bosons of (2.27). If we now consider the case in which the extent of the additional dimension is taken to be finite, then the Coleman-Mermin-Wagner theorem tells us that there cannot be massless excitations in a slab [5, 6, 7]. The Goldstone bosons must therefore pick up a small, non-perturbatively generated mass. In [10], Hasenfratz and Niedermayer proposed using a block spin transformation to map the 3-d  $O(3)$ -model in a slab of finite extent  $\beta$  to a 2-d lattice  $O(3)$  model. One averages the fields over blocks of size  $\beta$  in the third direction and size  $\beta c$  in the two spatial directions, decreasing the original cutoff  $1/a$  down to  $1/a' \sim 1/\beta$ . One thus obtains a two-dimensional lattice field theory, which has as its degrees of freedom the block-averaged fields living at the block centers. The lattice spacing of the new theory is therefore equal to  $\beta c$ , which is different from the lattice spacing of the original quantum Heisenberg model (see figure 2-1). Using chiral perturbation theory, Hasenfratz and Niedermayer expressed the coupling of the new Wilsonian lattice field theory as

$$1/g^2 = \beta\rho_s + \mathcal{O}(1/\beta\rho_s). \quad (2.29)$$

In the same paper, the authors also extended an earlier result by Chakravarty, Halperin and Nelson [8, 9] for the dependence of the correlation length on the inverse temperature. Using the 3-loop  $\beta$ -function for the 2-d  $O(3)$  model together with its exact mass gap, they found

$$\xi = \frac{ec}{16\pi\rho_s} \exp(2\pi\beta\rho_s) \left[ 1 - \frac{1}{4\pi\beta\rho_s} + \mathcal{O}\left(\frac{1}{\beta^2\rho^2}\right) \right]. \quad (2.30)$$

The constant  $e$  in this formula is just the base of the natural logarithm.

The continuum limit is reached by taking the extent of the additional dimension to infinity, since the correlation length diverges exponentially as  $\beta \rightarrow \infty$ . However,

Figure 2-1: Dimensional Reduction of a D-theory: Averaging the  $(d + 1)$ -dimensional effective field of the D-theory over blocks of size  $\beta$  in the extra dimension and  $a'$  in the physical directions results in an effective  $d$ -dimensional Wilsonian lattice field theory with lattice spacing  $a'$ . For the Heisenberg antiferromagnet,  $a' = \beta c$ , and for the ferromagnet,  $a' = \sqrt{\rho_s \beta / S}$ .

in this limit the extent of the third dimension is much smaller than the correlation length, i.e.  $\xi \gg \beta c$ . Thus, the fields will effectively be constant in the  $\tau$ -direction, and the theory undergoes dimensional reduction.

### 2.3.2 The Ferromagnetic Case

In the ferromagnetic case,  $J < 0$ , we proceed along the same lines as in the antiferromagnetic case. Since the ground state consists of all parallel spins, there is no need to introduce sublattices  $A$  and  $B$ . The system of coherent states with properties (2.13) and (2.14) is sufficient for generators on all lattice sites. As before, we set up a coherent state path integral,

$$Z = \int \mathcal{D}\mathbf{s} \exp(-S[\mathbf{s}]), \quad (2.31)$$

where

$$S[\mathbf{s}] = -iS \sum_x \omega[\mathbf{s}_x] + \int_0^\beta d\tau \mathcal{H}[\mathbf{s}]. \quad (2.32)$$

The  $\mathcal{H}$  term in the action is now

$$\mathcal{H}[\mathbf{s}] = \langle \mathbf{s} | \hat{H} | \mathbf{s} \rangle = \frac{S^2}{2} J \sum_{x,\mu} \mathbf{s}_x \cdot \mathbf{s}_{x+\hat{\mu}}. \quad (2.33)$$

The nature of the Berry phase term  $\omega[\mathbf{s}]$  was explained in detail in section 2.3.1. With the simple parametrisation  $s^i(\tau, v) = v s^i(\tau)$  for  $i = 1, 2$ , we can perform the integral over  $v$  to obtain an expression for the Berry phase that is equivalent to the one given in (2.22), and hence arrive at the following continuum action,

$$S[\mathbf{s}] = \int_0^\beta d\tau \int d^2x \left[ iS(1 + s^3)^{-1} (\partial_\tau s^1 s^2 - \partial_\tau s^2 s^1) + \frac{\rho_s}{2} \partial_\mu \mathbf{s} \cdot \partial_\mu \mathbf{s} \right]. \quad (2.34)$$

Here,  $\rho_s = S^2|J|/2$  is the spin stiffness. This result was also obtained by Leutwyler in [38] using chiral perturbation theory. He showed that this theory has a non-relativistic dispersion relation, attributed to the existence of a conserved order parameter.

The mechanism of dimensional reduction was explained above for the antiferromagnet. Adapting this discussion for the ferromagnet, note that the ground state is now the usual ordered ground state of parallel spins. This breaks the  $SO(3)$  symmetry to  $SO(2)$  as before. And again, massless Goldstone modes can exist only for infinite extent  $\beta$  of the third dimension. There is a subtle difference from the antiferromagnetic case when one maps the 3-d  $O(3)$  model to a 2-d lattice  $O(3)$  model. The size  $a'$  of the spatial extent of the blocks that are used for the block spin transformation has to be chosen in accordance with the non-relativistic dispersion relation of the ferromagnet. To determine a suitable value for  $a'$ , consider the dispersion relation,  $E = (\rho_s/S)p^2$ . This implies  $1/\beta = (\rho_s/S)(1/a'^2)$ , and hence  $a' = \sqrt{\rho_s\beta/S}$ . The lattice spacing of the new theory is therefore equal to  $\sqrt{\rho_s\beta/S}$ , which, again, is different from the lattice spacing of the original quantum Heisenberg model (see figure 2-1). The correlation length of the 2-d  $O(3)$  model in lattice units is given by  $\xi/a' \propto g^2 \exp(2\pi/g^2)$ . The value of  $a'$  was found above, and we can identify the

coupling constant as  $1/g^2 = \rho_s \beta$ . We thus arrive at the expression for the dependence of the correlation length on the extent  $\beta$  of the additional dimension,

$$\xi \propto (\rho_s \beta)^{-1/2} \exp(2\pi \rho_s \beta). \quad (2.35)$$

This relation was first found by Kopietz and Chakravarty in [11], where they used the same techniques of renormalization-group analysis that had been used for the anti-ferromagnet in [8, 9] by Chakravarty, Halperin and Nelson. As before, the correlation length diverges exponentially as  $\beta \rightarrow \infty$ , i.e. as the extent of the third dimension becomes large. This leads to the continuum limit of the theory. At the same time, the correlation length is exponentially larger than the extent of the third direction,  $\xi \gg \sqrt{\rho_s \beta / S}$ , and the theory thus undergoes dimensional reduction.

## Chapter 3

# $U(N)$ Quantum Spins and Quantum Links

In the following chapters we will be considering models whose fundamental degrees of freedom in the conventional formulation are elements of unitary  $N \times N$  matrices. In D-theory these fields are replaced by quantum operators, so that we have matrices whose entries are operators rather than complex numbers. However, we still want the Hamiltonian constructed from these operators to be invariant under the same symmetries as the conventional action. In particular, we will be considering two models, the principal chiral model with a global  $SU(N)_L \times SU(N)_R \times U(1)_{L=R}$  symmetry, and Yang-Mills theory with a local  $SU(N)$  symmetry. In order to construct the appropriate D-theory Hamiltonians, let us consider the operators associated with just one lattice site in the case of the principal chiral model, or just one link in the case of Yang-Mills theory. Dropping the site/link indices, we denote these operators by  $\hat{U}^{ij}$ . The appropriate symmetries follow by construction if we also associate with each site/link an  $SU(N)_L \times SU(N)_R$  algebra, generated by  $\{\hat{L}^a\}$  and  $\{\hat{R}^a\}$  which satisfy the commutation relations

$$[\hat{L}^a, \hat{L}^b] = 2if_{abc}\hat{L}^c, \quad [\hat{R}^a, \hat{R}^b] = 2if_{abc}\hat{R}^c, \quad [\hat{L}^a, \hat{R}^b] = 0. \quad (3.1)$$



Here  $f_{abc}$  are the usual structure constants of  $SU(N)$ . We then require that the site/link operator variables transform as

$$\widehat{U}' = \exp\left(-i\alpha^a \widehat{L}^a - i\beta^a \widehat{R}^a\right) \widehat{U} \exp\left(i\alpha^b \widehat{L}^b + i\beta^b \widehat{R}^b\right) = \exp\left(-i\alpha^a \lambda^a\right) \widehat{U} \exp\left(i\beta^b \lambda^b\right), \quad (3.2)$$

where the  $\lambda^a$  are the Hermitian generators of  $SU(N)$  in the fundamental representation. These generators satisfy

$$[\lambda^a, \lambda^b] = 2if_{abc}\lambda^c, \quad \text{Tr}\lambda^a\lambda^b = 2\delta^{ab}. \quad (3.3)$$

The transformation rule (3.2) is implied by the following commutation relations,

$$[\widehat{L}^a, \widehat{U}^{ij}] = \lambda_{ik}^a \widehat{U}^{kj}, \quad [\widehat{R}^a, \widehat{U}^{ij}] = -\widehat{U}^{ik} \lambda_{kj}^a. \quad (3.4)$$

All of these commutation relations can be satisfied by embedding the operators in an  $SU(2N)$  algebra. In particular, the aforementioned  $SU(N)_L \times SU(N)_R$  algebra is embedded diagonally, while the  $\widehat{U}^{ij}$  operators fill in the off-diagonal blocks. To summarise, we get the full set of commutation relations

$$\begin{aligned} [\widehat{L}^a, \widehat{L}^b] &= 2if_{abc}\widehat{L}^c, & [\widehat{R}^a, \widehat{R}^b] &= 2if_{abc}\widehat{R}^c, \\ [\widehat{R}^a, \widehat{U}^{ij}] &= -\widehat{U}^{ik} \lambda_{kj}^a, & [\widehat{L}^a, \widehat{U}^{ij}] &= \lambda_{ik}^a \widehat{U}^{kj}, & [\widehat{T}, \widehat{U}^{ij}] &= -2\widehat{U}^{ij}, \\ [\widehat{R}^a, \widehat{L}^b] &= [\widehat{T}, \widehat{L}^a] = [\widehat{T}, \widehat{R}^a] = [\widehat{U}^{ij}, \widehat{U}^{k\ell}] &= 0, \\ [\widehat{U}^{ij}, \widehat{U}^{k\ell}] &= [\widehat{U}^{ij}, (\widehat{U}^\dagger)^{\ell k}] = \delta^{j\ell} \widehat{L}^{ik} - \delta^{ik} \widehat{R}^{\ell j}. \end{aligned} \quad (3.5)$$

Here,  $\widehat{T}$  generates an extra  $U(1)$  symmetry. (Later, this symmetry needs to be broken explicitly to obtain an  $SU(N)$  rather than  $U(N)$  Yang-Mills theory.) We can choose a representation of  $SU(2N)$  by introducing a set of fermionic creation and annihilation operators, which we call rishons [19, 20] (we use the summation convention for Greek

indices),

$$\begin{aligned}
\widehat{S}^{ij} = \widehat{L}^{ij} &= \left( \widehat{\ell}^{i\alpha\dagger} \widehat{\ell}^{j\alpha} - \frac{M}{2} \delta^{ij} \right), & \widehat{S}^{N+i, N+j} = \widehat{R}^{ij} &= \left( \widehat{r}^{i\alpha\dagger} \widehat{r}^{j\alpha} - \frac{M}{2} \delta^{ij} \right), \\
\widehat{S}^{i, N+j} = \widehat{U}^{ij} &= \widehat{\ell}^{i\alpha\dagger} \widehat{r}^{j\alpha}, & \widehat{S}^{N+i, j} = \left( \widehat{U}^\dagger \right)^{ij} &= \left( \widehat{U}^{ji} \right)^\dagger = \widehat{r}^{i\alpha\dagger} \widehat{\ell}^{j\alpha}, \\
\widehat{T} = \sum_i &\left( \widehat{r}^{i\alpha\dagger} \widehat{r}^{i\alpha} - \widehat{\ell}^{i\alpha\dagger} \widehat{\ell}^{i\alpha} \right), & \sum_i &\left( \widehat{\ell}^{i\alpha\dagger} \widehat{\ell}^{i\beta} + \widehat{r}^{i\alpha\dagger} \widehat{r}^{i\beta} \right) = \delta^{\alpha\beta} N, \quad (3.6)
\end{aligned}$$

where  $\alpha = 1, \dots, M$  is a rishon flavor index and  $i, j = 1, \dots, N$ . For convenience, we have chosen these generators not to be traceless. The rishon operators  $\widehat{\ell}^{i\alpha}$  and  $\widehat{r}^{i\alpha}$  obey canonical anticommutation relations

$$\begin{aligned}
\{ \widehat{\ell}^{i\alpha}, \widehat{\ell}^{j\beta\dagger} \} &= \{ \widehat{r}^{i\alpha}, \widehat{r}^{j\beta\dagger} \} = \delta^{ij} \delta^{\alpha\beta}, \\
\{ \widehat{\ell}^{i\alpha}, \widehat{\ell}^{j\beta} \} &= \{ \widehat{\ell}^{i\alpha\dagger}, \widehat{\ell}^{j\beta\dagger} \} = 0, & \{ \widehat{r}^{i\alpha}, \widehat{r}^{j\beta} \} &= \{ \widehat{r}^{i\alpha\dagger}, \widehat{r}^{j\beta\dagger} \} = 0, \\
\{ \widehat{\ell}^{i\alpha}, \widehat{r}^{j\beta} \} &= \{ \widehat{\ell}^{i\alpha}, \widehat{r}^{j\beta\dagger} \} = 0, & \{ \widehat{\ell}^{i\alpha\dagger}, \widehat{r}^{j\beta} \} &= \{ \widehat{\ell}^{i\alpha\dagger}, \widehat{r}^{j\beta\dagger} \} = 0. \quad (3.7)
\end{aligned}$$

We then have  $\widehat{L}^a = \lambda_{ij}^a \widehat{L}^{ji}$  and  $\widehat{R}^a = \lambda_{ij}^a \widehat{R}^{ji}$ . The constraint (3.6) fixes the representation for  $SU(2N)$  to the one corresponding to a rectangular Young tableau with  $N$  rows and  $M$  columns, as shown in figure 3-1(a). Interestingly, this constraint also imposes Gauss' law for the  $U(M)$  gauge symmetry of the  $\widehat{\Phi}$ -model discussed in chapter 6. We come back to this point in section 6.3. Here, we explain how this constraint actually leads to the correct representation. At each lattice site/link we have introduced a set of  $\widehat{\ell}$  and  $\widehat{r}$  rishons with colour and rishon flavour indices. Hence, there are a total of  $2NM$  rishons. We can form at most  $(2NM)^2$  independent bilinears of these, which are all the generators of a  $SU(2NM)$  algebra. Since the rishons obey fermionic anticommutation relations, the representation of  $SU(2NM)$  we have thus chosen is necessarily totally antisymmetric. If we decide to put a total of  $NM$  rishons on each lattice site/link, then we obtain the representation of figure 3-1(c) for  $SU(2NM)$ . Embedded in  $SU(2NM)$ , we have the algebra of  $SU(2N) \times SU(M)$ , and we want to fix the representation for  $SU(2N)$ . The constraint of eq.(3.6) fixes the representation of  $SU(M)$  to be a singlet representation (and this is where Gauss' law comes

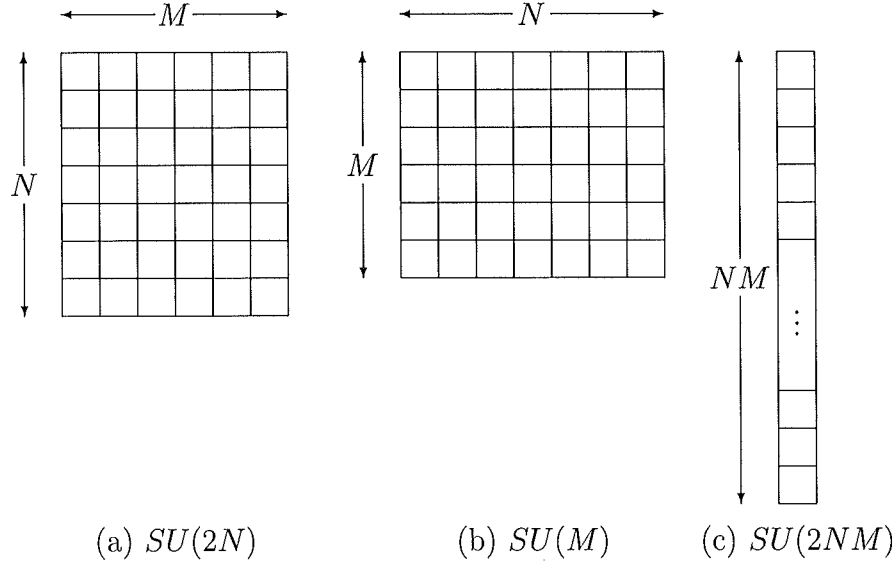


Figure 3-1: Choosing a representation for  $SU(2N)$  by which states at each lattice site/link transform.

in as mentioned above). The singlet representation of  $SU(M)$  which is built of  $NM$  fermions is the one shown in figure 3-1(b). Regarding the rishons as indistinguishable particles for a moment, and the Young tableaux of figure 3-1 as representations of the permutation group, we know that a totally antisymmetric representation (figure 3-1(c)) is only contained in the direct product of a certain representation (figure 3-1(b)) with its associate (figure 3-1(a)), which has the reflected Young tableau. In other words, in order to get a totally antisymmetric representation from a representation which symmetric in colour and antisymmetric in flavour, we need to take its direct product with a representation which is symmetric in flavour and antisymmetric in colour.

For the purpose of deriving systems of coherent states, which is described in detail in appendix B, it will be convenient to introduce the following notation,

$$\tilde{c}^{i\alpha} = \begin{cases} \hat{\ell}^{i\alpha}, & \text{for } 1 \leq i \leq N; \\ \hat{r}^{i\alpha}, & \text{for } N + 1 \leq i \leq 2N. \end{cases} \quad (3.8)$$

We then have

$$\widehat{S}^{ij} = \hat{c}^{i\alpha\dagger} \hat{c}^{j\alpha} - \frac{M}{2} \delta^{ij}, \quad \sum_i \hat{c}^{i\alpha\dagger} \hat{c}^{i\beta} = \delta^{\alpha\beta} N. \quad (3.9)$$

Notice that the labels  $i$  and  $j$  now run from 1 to  $2N$ .

As mentioned above, in the next three chapter we will consider a number of models, whose degrees of freedom are described by quantum operators that are elements of the  $SU(2N)$  algebra just described. The Hamilton operators are of course different in each case, but we now outline the general procedure for setting up a coherent state path integral for such models. The procedure is completely analogous to the one described in chapter 2 for the spin coherent state path integral. Furthermore, we make use of the results of appendix B regarding systems of coherent states for  $SU(2N)$ . The starting point is the quantum partition function, which is given by

$$Z = \text{Tr} \exp(-\beta \widehat{H}) = \text{Tr} \left( e^{-\epsilon \widehat{H}} e^{-\epsilon \widehat{H}} \dots e^{-\epsilon \widehat{H}} \right). \quad (3.10)$$

The Hamilton operator is an expression formed of the generators of  $SU(2N)$ , and the trace is taken in the Hilbert space in which these generators act. We divide the interval  $\beta$  up into  $N_\epsilon$  smaller intervals of size  $\epsilon$  as indicated in (3.10). We then insert a resolution of the identity operator in between each pair of factors  $\exp(-\epsilon \widehat{H})$ . Assuming smooth field configurations, matrix elements  $\langle q(\tau + \epsilon) | \exp(-\epsilon \widehat{H}) | q(\tau) \rangle$  are expanded to linear order in  $\epsilon$ . Using the identity (B.25), we can evaluate the diagonal matrix elements of the Hamilton operator between coherent states,

$$\mathcal{H}[Q] = \langle q | \widehat{H} | q \rangle. \quad (3.11)$$

We thus obtain a path integral over classical variables  $Q$  of the form

$$Z = \int \mathcal{D}Q \exp(-S[Q]). \quad (3.12)$$

The action is now given by

$$S[Q] = S_B + \int_0^\beta d\tau \mathcal{H}[Q]. \quad (3.13)$$

Here,  $S_B$  is a Berry phase term which is discussed in detail in appendix B.

## Chapter 4

# The Principal Chiral Model

In nature, the two lightest quarks, the up and the down quarks, are so light that to a first approximation they can be treated as massless. If the up and down quarks had the same nonzero mass, the QCD Lagrangian would be invariant under a global  $SU(2)$  isospin rotation, which mixes the two. If we set their masses equal to zero, then the left- and right-handed parts of the up and down quarks can be rotated independently. Hence, the Lagrangian has an  $SU(2)_L \times SU(2)_R$  symmetry. However, even in the approximation  $m_u = m_d = 0$ , the vacuum does not have this symmetry. In a world of massless quarks, it is very easy to create  $q\bar{q}$  excitations because they cost virtually nothing. In fact, the vacuum is a condensate of quark-antiquark pairs, characterised by the order parameter

$$\langle \bar{q}q \rangle = \langle \bar{q}_L q_R + \bar{q}_R q_L \rangle \neq 0. \quad (4.1)$$

This nonvanishing expectation value now acts as an effective mass term for the quarks. Hence, as excitations of the vacuum the up and down quarks are no longer massless. Still, the effective mass is the same for both quarks, and so the global  $SU(2)_L \times SU(2)_R$  symmetry is broken to  $SU(2)_{L=R}$ . By Goldstone's theorem, there are massless excitations associated with this symmetry breaking. These particles live in the coset space  $(SU(2)_L \times SU(2)_R)/SU(2)_{L=R} \simeq SU(2)$  and are identified with the pions. In nature, pions are not truly massless, because the up and down quarks have small,

nonvanishing masses. Thus, the described symmetry breaking mechanism is only approximate, leading to an observed pion mass of 140 MeV. Pions are therefore often referred to as pseudo-Goldstone bosons.

## 4.1 The Standard Formulation

Using chiral perturbation theory, one can find the effective low-energy Lagrangian governing the physics of Goldstone bosons that are associated with chiral symmetry breaking. The degrees of freedom entering the Lagrangian are  $SU(2)$  matrices,

$$U(x) = \exp \left( \frac{i}{F_\pi} \left( \pi_1(x) \frac{\sigma_1}{2} + \pi_2(x) \frac{\sigma_2}{2} + \pi_3(x) \frac{\sigma_3}{2} \right) \right). \quad (4.2)$$

The  $\sigma_i$  are the Pauli matrices, which are the generators of  $SU(2)$ . The physical basis one takes for the pions is  $\pi_0 = \pi_3$ , and  $\pi_\pm = (1/\sqrt{2})(\pi_1 \mp i\pi_2)$ . The action for these fields is given by

$$S[U] = \frac{F_\pi^2}{4} \int d^4x \operatorname{Tr} [\partial_\mu U^\dagger(x) \partial_\mu U(x)]. \quad (4.3)$$

The coupling constant,  $F_\pi$ , is the pion decay constant. In order to make phenomenological predictions based on this model,  $F_\pi$  first needs to be determined from experimental results, or calculated directly from QCD by means of numerical simulations.

This action can be generalised to describe the symmetry breaking pattern of  $(SU(N)_L \times SU(N)_R) \rightarrow SU(N)_{L=R}$ . In this case, the matrices  $U(x)$  are simply  $SU(N)$  matrices. Letting the matrices be elements of  $U(N)$  instead, the action is invariant under an additional  $U(1)$  symmetry. The general model is called the principal chiral model. It has been studied in different dimensions, notably in 2 and 4 dimension, and also in 3 dimensions. In 2 dimensions the  $SU(N)_L \times SU(N)_R$  symmetry cannot break spontaneously at finite temperature, due to the Mermin-Wagner theorem. Quite apart from the context of chiral symmetry breaking, the 2-dimensional principal chiral model has been studied as a toy model for QCD due to the fact that it is asymptotically free and has a massgap. In the following, we concentrate on the 2 dimensional case.

The action of the 2-d principal chiral model in the continuum is given by

$$S[U] = \frac{1}{2g^2} \int d^2x \operatorname{Tr} [\partial_\mu U^\dagger(x) \partial_\mu U(x)], \quad (4.4)$$

where, as noted above, the  $U(x)$  are unitary  $N \times N$  matrices. In Wilson's approach to regularising the theory, space-time is discretised by introducing a regular lattice. Derivatives are replaced by finite differences to obtain an action of the form

$$S[U] = -\frac{1}{g^2} \sum_{\langle xy \rangle} \operatorname{Tr} [U_x^\dagger U_y]. \quad (4.5)$$

Here,  $\langle xy \rangle$  denotes summation over nearest neighbour pairs. The target theory has a global  $SU(N)_L \times SU(N)_R \times U(1)_{L=R}$  symmetry of the form  $U_x \rightarrow U'_x = L U_x R^\dagger$ , where  $L$  and  $R$  are unitary matrices. It is known that this symmetry breaks to an  $SU(N)$  vector symmetry ( $L = R$ ) at  $g = 0$ . Due to the Mermin-Wagner theorem, however, the symmetry cannot break for  $g > 0$ .

## 4.2 The Quantum Spin Formulation

Let us now replace the classical fields  $U_x^{ij}$  by quantum operators  $\hat{U}_x^{ij}$  and write down a D-theory Hamiltonian, which evolves the two-dimensional system in an additional Euclidean time direction,

$$\hat{H} = 2J \sum_{x,\mu} \operatorname{Re} \operatorname{Tr} \hat{U}_x \hat{U}_{x+\hat{\mu}}^\dagger = J \sum_{x,\mu} \left[ \hat{U}_x^{ij} \left( \hat{U}_{x+\hat{\mu}}^{ij} \right)^\dagger + \hat{U}_{x+\hat{\mu}}^{ij} \left( \hat{U}_x^{ij} \right)^\dagger \right]. \quad (4.6)$$

We would like this Hamiltonian to have an  $SU(N)_L \times SU(N)_R \times U(1)_{L=R}$  symmetry, i.e.  $[\hat{G}_L^a, \hat{H}] = [\hat{G}_R^a, \hat{H}] = [\hat{T}, \hat{H}] = 0$ , where  $\hat{G}_L^a$  and  $\hat{G}_R^a$  are mutually commuting sets of  $SU(N)$  generators and  $\hat{T}$  generates a  $U(1)$  symmetry. As we saw in chapter 3, this can be realised by embedding  $SU(N)_L \times SU(N)_R \times U(1)_{L=R}$  diagonally in  $SU(2N)$ . In particular, we have an  $SU(2N)$  algebra of the form (3.6) at each lattice site. Labeling the generators that correspond to the lattice site  $x$  by a subscript  $x$ , we can write the generators of the algebra of the global  $SU(N)_L \times SU(N)_R \times U(1)_{L=R}$



symmetry group as  $\widehat{G}_L^a = \sum_x \widehat{L}_x^a$ ,  $\widehat{G}_R^a = \sum_x \widehat{R}_x^a$  and  $\widehat{T} = \sum_x \widehat{T}_x$ . It then follows that  $[\widehat{G}_L^a, \widehat{H}] = [\widehat{G}_R^a, \widehat{H}] = [\widehat{G}_L^a, \widehat{G}_R^b] = [\widehat{T}, \widehat{H}] = 0$ .

For  $J < 0$  this model is ferromagnetic. We choose the same representation of  $SU(2N)$  for the generators at each site of the lattice, namely the one mentioned above, with a rectangular Young tableau as shown in figure 3-1. Note that the properties of the system defined by the Hamiltonian  $\widehat{H}$  are completely determined, once a representation of  $SU(2N)$  has been specified.

### 4.3 The Low-Energy Effective Theory of the Quantum Spin Model

Using the results of appendix B, we set up a coherent state path integral for the partition function. The Berry phase term of the corresponding action was calculated in (B.44). The other term in the action is given by

$$\begin{aligned} \int_0^\beta d\tau \mathcal{H}(Q(\tau)) &= \frac{JM^2}{4} \int_0^\beta d\tau \sum_{x,\mu} \left[ V_x^{ij} (V_{x+\hat{\mu}}^{ij})^* + (V_{x+\hat{\mu}}^{ij})^* V_x^{ij} \right] \\ &= \frac{JM^2}{4} \int_0^\beta d\tau \sum_{x,\mu} \text{Tr} \left[ V_x V_{x+\hat{\mu}}^\dagger + V_{x+\hat{\mu}} V_x^\dagger \right]. \end{aligned} \quad (4.7)$$

As discussed in appendix B, we have  $V = -U \sin(2B)$  and we define  $S \equiv \sin(2B)$ . Then  $V = -US$ , where  $S = S^\dagger$ . We can think of  $S$  as the radial component and of  $U$  as the phase of the matrix  $V$ . Due to the sine function and the fact that we can cover the coset space  $SU(2N)/[SU(N)_L \times SU(N)_R \times U(1)_{L=R}]$  by limiting the matrix  $B$  to have eigenvalues between 0 and  $\pi/2$ , the eigenvalues of  $S$  are constrained to lie between 0 and 1. Substituting the above coset decomposition into (4.7), we obtain

$$\frac{JM^2}{4} \int_0^\beta d\tau \sum_{x,\mu} \text{Tr} \left[ U_x S_x S_{x+\hat{\mu}} U_{x+\hat{\mu}}^\dagger + U_{x+\hat{\mu}} S_{x+\hat{\mu}} S_x U_x^\dagger \right]. \quad (4.8)$$

We would like to expand this action around its minimum. Since the eigenvalues of  $S$  are bounded by 1, the minimum of the action occurs when  $S = \mathbb{1}$  (up to a global

$SU(N)_L \times SU(N)_R \times U(1)_{L=R}$  rotation), and  $U$  is constant across the lattice. We now introduce fluctuations in the fields, defining the forward lattice derivative,

$$\Delta_\mu^f U_x = \frac{U_{x+\hat{\mu}} - U_x}{a}, \quad (4.9)$$

and writing

$$S_x = \sin(2B_x) = \sin\left(2\left(\frac{\pi}{4} + aE_x\right)\right) \approx \mathbb{1} - 2a^2 E_x^2. \quad (4.10)$$

Substituting these expressions into (4.8), expanding to quadratic order in  $a$  and dropping an irrelevant constant, we obtain

$$S = S_B - \frac{JM^2}{4} \int_0^\beta d\tau \sum_{x,\mu} a^2 \text{Tr} [(\Delta_\mu^f U_x^\dagger) (\Delta_\mu^f U_x) + 8E_x^2]. \quad (4.11)$$

Next, we turn to the Berry phase term, and expand the fields in the same way as before,

$$\begin{aligned} S_B &= - \sum_x \frac{M}{2} \int_0^\beta d\tau \text{Tr} (\cos(2B_x) U_x^\dagger \partial_\tau U_x) \\ &\approx - \sum_x \frac{M}{2} \int_0^\beta d\tau \text{Tr} \left[ \left( \cos\left(\frac{\pi}{2}\right) - 2aE_x \sin\left(\frac{\pi}{2}\right) \right) U_x^\dagger \partial_\tau U_x \right] \\ &= \sum_x \frac{M}{2} \int_0^\beta d\tau \text{Tr} [2aE_x U_x^\dagger \partial_\tau U_x] \end{aligned} \quad (4.12)$$

We thus obtain the final expression for the action in the continuum limit,

$$\begin{aligned} S &= \int_0^\beta d\tau \int d^2x \text{Tr} \left[ -\frac{JM^2}{4} \partial_\mu U^\dagger(x) \partial_\mu U(x) - 4JM^2 E^2(x) + \frac{M}{a} E(x) U^\dagger(x) \partial_\tau U(x) \right] \\ &= \int_0^\beta d\tau \int d^2x \frac{\rho_s}{2} \text{Tr} \left[ \partial_\mu U^\dagger \partial_\mu U + \frac{1}{c^2} \partial_\tau U^\dagger \partial_\tau U \right]. \end{aligned} \quad (4.13)$$

In this equation, we have integrated out the shifted field

$$E'(x) = E(x) - \frac{1}{8JM a} U(x)^\dagger \partial_\tau U(x). \quad (4.14)$$

The spin stiffness is given by  $\rho_s = |J|M^2/2$ , and  $c = 2Ma|J|$  is the spin wave velocity.

Notice that we now have a theory with a relativistic dispersion relation. The ferromagnetic  $SO(3)$  spin model has a non-relativistic dispersion relation [38], because the order parameter commutes with the Hamiltonian. In the ferromagnetic principal chiral model on the other hand, the order parameter  $\hat{U}^{ij} = \sum_x \hat{U}_x^{ij}$  does not commute with  $\hat{H}$ , so a relativistic dispersion relation comes as no surprise.

The three-dimensional system will dimensionally reduce if the correlation length is much larger than the extent of the third dimension,  $\xi \gg \beta c$ . If we assume that this is the case, then the fields  $U$  will have no dependence on  $\tau$ , and the integration over  $\tau$  becomes trivial,

$$S = \frac{\beta \rho_s}{2} \int d^2x \text{Tr} [\partial_\mu U^\dagger \partial_\mu U]. \quad (4.15)$$

From [39] we know that the correlation for the two-dimensional principal chiral model is given by

$$\xi \propto \exp\left(\frac{2\pi}{g^2 N}\right) = \exp\left(\frac{2\pi\beta\rho_s}{N}\right). \quad (4.16)$$

When performing a blockspin transformation in the way described for the Heisenberg ferromagnet, the new lattice spacing for a system with a relativistic dispersion relation, such as the present one, is  $a' = \beta c$ . Equation (4.16) is consistent with  $\xi \gg \beta c$  in the zero temperature ( $\beta \rightarrow \infty$ ) limit, so dimensional reduction does indeed occur.

# Chapter 5

## Gauge Theory

A concept of supreme importance in particle physics is that of local or gauge symmetry. In fact, the standard model is an  $SU(3) \times SU(2) \times U(1)$  gauge theory, and even its best understood part, electromagnetism, is still a  $U(1)$  gauge theory. In a globally symmetric theory, like the Heisenberg or principal chiral models, there is one symmetry transformation associated with the entire lattice. In gauge symmetric models on the other hand, the action remains invariant under a whole set of different symmetry transformations, one for each lattice site. In this section, we review the standard formulation of gauge theories on the lattice. We derive those properties, that are important for the discussions in this and the next chapter. We then reformulate  $SU(N)$  gauge theories as quantum link models. Finally, we show how quantum link models reduce to the standard formulation for large enough representations of the quantum link algebra after dimensional reduction occurs.

### 5.1 The Standard Formulation

The propagation of free quarks in Minkowski space is described by Dirac's famous action,

$$S[\psi] = \int d^4x \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi \quad (5.1)$$

Here,  $\Psi$  and  $\bar{\Psi}$  are independent anti-commuting Grassmann numbers. In four-dimensional Minkowski space the matrices  $\gamma_\mu$  satisfy the anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad (5.2)$$

where  $g^{\mu\nu}$  is the Minkowski metric tensor,  $g = \text{diag}(1, -1, -1, -1)$ . In the chiral basis, the gamma matrices are

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3. \quad (5.3)$$

In order to describe quark propagation on a lattice, the first step is to do a Wick rotation to Euclidean space-time by taking time  $t$  to  $it$ . We then replace partial derivatives by finite differences. However, at this point a fundamental problem arises, which has serious implications for the description of fermion physics on a lattice at all levels. Namely, a set of degenerate fermions appear at all corners of the Brillouin zone, with lattice momenta  $p$  such that  $p_\mu = 0$  or  $\pi$ . All of these doubler fermions have the same mass and dispersion relation as the physical fermion that has  $p = 0$ . Wilson solved this problem by adding a term to the action, which corresponds to a second order derivative and vanishes in the continuum limit. At the same time, this term lifts the degeneracy of the doublers and makes them decouple from the theory in the continuum limit. The Wilson action for fermions is given by

$$\begin{aligned} S[\Psi] = & \frac{1}{2} \sum_{x,\mu} (\bar{\Psi}_x \gamma_\mu \Psi_{x+\hat{\mu}} - \bar{\Psi}_{x+\hat{\mu}} \gamma_\mu \Psi_x) + m \sum_x \bar{\Psi}_x \Psi_x \\ & + \frac{r}{2} \sum_{x,\mu} (2\bar{\Psi}_x \Psi_x - \bar{\Psi}_x \Psi_{x+\hat{\mu}} - \bar{\Psi}_{x+\hat{\mu}} \Psi_x). \end{aligned} \quad (5.4)$$

The first two terms are simply the discretisation of the continuum Dirac action for fermions. The term proportional to  $r$  is the Wilson term just described. The Euclidean gamma matrices in (5.4), satisfy the following anticommutation relations,

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}. \quad (5.5)$$

In the chiral basis, they are the following matrices,

$$\gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \quad i = 1, 2, 3, \quad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (5.6)$$

The fifth gamma matrix, which anticommutes with all the others is given by,

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5.7)$$

Suppose now, that (5.4) is an action that describes the physics of  $N$  fermions which all have the same mass. Thus, the fermion field  $\Psi$  carries an index  $\Psi^i$ , where  $i = 1, \dots, N$ . We shall call the index  $a$  the colour index so that  $N$  is the number of colours. The action thus has a global  $SU(N)$  symmetry, which rotates the vector of fermions by the same matrix across the entire lattice, say  $\Psi_x^i \rightarrow \Psi_x^{i'} = G^{ij} \Psi_x^j$ . The rotation matrix  $G$  is a special unitary matrix, satisfying  $G^\dagger G = \mathbb{1}$  and  $\det G = 1$ . A more powerful symmetry would be one that allows a different rotation of the fermion-vector at each lattice site,  $\Psi_x^i \rightarrow \Psi_x^{i'} = G_x^{ij} \Psi_x^j$ . Here, the rotation matrices  $G_x^{ij}$  are associated with lattice sites  $x$ , and are in general different for different lattice sites. To obtain an action which is invariant under such local transformations requires a non-trivial modification of (5.4). While terms such as  $\bar{\Psi}_x^i \Psi_x^i$  are invariant under such local  $SU(N)$  transformations, other terms, such as  $\bar{\Psi}_x^i \gamma_\mu \Psi_{x+\hat{\mu}}^i$  are clearly not. However, we can introduce a new matrix-valued field,  $U_{x,\mu}$ , which by definition transforms in the adjoint representation of  $SU(N)$ , i.e.

$$U_{x,\mu}^\dagger = G_x U_{x,\mu} G_{x+\hat{\mu}}^\dagger. \quad (5.8)$$

Such a field now lives on the links of the lattice, and it also satisfies  $U_{x+\hat{\mu},-\mu} = U_{x,\mu}^\dagger$ . In order for the theory to be well-defined,  $U_{x,\mu}$  must be a compact variable, and we restrict ourselves to elements of  $SU(N)$ . The matrices  $U_{x,\mu}$  are also called parallel transporter matrices, because they transport the field  $\Psi_x$  by one lattice site, so that  $U_{x,\mu}^\dagger \Psi_x$  transforms in the same way as  $\Psi_{x+\hat{\mu}}$ . Making use of such fields, we can write

down a gauge invariant action [40],

$$\begin{aligned}
S[\Psi] = & -\frac{1}{g^2} \sum_{x,\mu \neq \nu} \text{Re Tr} \left[ U_{x,\mu} U_{x+\hat{\mu}} U_{x+\hat{\nu}}^\dagger U_{x,\nu}^\dagger \right] \\
& \frac{1}{2} \sum_{x,\mu} \left( \bar{\Psi}_x \gamma_\mu U_{x,\mu} \Psi_{x+\hat{\mu}} - \bar{\Psi}_{x+\hat{\mu}} \gamma_\mu U_{x,\mu}^\dagger \Psi_x \right) + m \sum_x \bar{\Psi}_x \Psi_x \\
& + \frac{r}{2} \sum_{x,\mu} \left( 2\bar{\Psi}_x \Psi_x - \bar{\Psi}_x U_{x,\mu} \Psi_{x+\hat{\mu}} - \bar{\Psi}_{x+\hat{\mu}} U_{x,\mu}^\dagger \Psi_x \right). \tag{5.9}
\end{aligned}$$

In this action we have also included a plaquette term that multiplies link matrices around a lattice plaquette. This term is a kinetic term for the link fields. For the action to be hermitian the sum over  $\mu \neq \nu$  must include positive and negative lattice directions. Then each plaquette is traversed once in a clockwise, and once in a counter-clockwise direction.

Now consider the continuum limit of this action. The plaquette term is minimised when the link fields are constant and Abelian across the entire lattice. Using the gauge symmetry of the action, we can always rotate this constant field configuration so that the fields are equal to the identity matrix, at least in a simply-connected patch of the lattice. It may not be possible to do this globally for certain topologies such as tori. But perturbation theory always starts from a configuration  $U = \mathbb{1}$  in a certain lattice region. The lowest-order excitations are small perturbations around this classical vacuum. Near the identity we can write  $SU(N)$  matrices as exponential of an algebra-valued field,

$$U_{x,\mu} = \exp(-iaA_{x,\mu}). \tag{5.10}$$

Here,  $A_{x,\mu} = gA_{x,\mu}^a \lambda^a$ , and the  $\lambda^a$  are just the generators of the algebra  $SU(N)$  (see appendix A). Expanding the  $U_{x,\mu}$  matrices to quartic order in  $a$  and taking the continuum limit then leads to the continuum action

$$S[\psi, A_\mu] = \int d^4x \left( -\frac{1}{4} \text{Tr}[F_{\mu\nu} F^{\mu\nu}] - \bar{\Psi}(\gamma^\mu D_\mu + m)\Psi \right). \tag{5.11}$$

In this equation, we have dropped an irrelevant constant, and introduced the notation

$$\begin{aligned}
F_{\mu\nu} &= F_{\mu\nu}^a \lambda^a, \\
F_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c, \\
D_\mu &= \partial_\mu - igA_\mu.
\end{aligned}
\tag{5.12}$$

In the following, we exclude quarks from the discussion for a moment, and take a look at some of the properties of pure gauge theories on the lattice in Wilson's formulation. This discussion follows [41] for the most part. In the next section, we then formulate this theory as a quantum link model. The inclusion of quarks as domain wall fermions in the quantum link model is also discussed briefly. The proof that quantum link models reduce to ordinary Yang-Mills theory after dimensional reduction is then carried out without the inclusion of quarks.

The Wilson action without quarks is simply,

$$S[U] = -\frac{1}{g^2} \sum_{x,\mu \neq \nu} \text{Re Tr} \left[ U_{x,\mu} U_{x+\hat{\mu},\nu} U_{x+\hat{\nu},\mu}^\dagger U_{x,\nu}^\dagger \right].
\tag{5.13}$$

The partition function for this system is given by

$$Z = \int \mathcal{D}U \exp(-S[U]).
\tag{5.14}$$

Physical observables are calculated as expectation values of operators  $\mathcal{O}[U]$ ,

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}U \mathcal{O}[U] \exp(-S[U]).
\tag{5.15}$$

The measure in the path integral is the Haar measure over the group  $SU(N)$ , which is shift invariant,

$$\int dU = \int d(UV^{-1}),
\tag{5.16}$$

where  $V \in SU(N)$ . This property, combined with the gauge invariance of the action can be used to show that observables have to be gauge invariant operators. The



simplest such operators is the Wilson loop. This is the trace of a product of  $U$ -variables around a closed loop  $\mathcal{C}$  on the lattice,

$$\mathcal{W}(\mathcal{C}) = \langle \text{Tr } U(\mathcal{C}) \rangle. \quad (5.17)$$

The plaquette term in the action is the smallest possible Wilson loop, which encloses just a single plaquette. The importance of Wilson loops is that any gauge-invariant observable that depends continuously on the field  $U$  can be expanded as a series in Wilson loops [42],

$$\langle \mathcal{O} \rangle = \sum_{n \geq 0} \sum_{\mathcal{C}_1, \dots, \mathcal{C}_n} a(\mathcal{C}_1, \dots, \mathcal{C}_n) \text{Tr } U(\mathcal{C}_1), \dots, \text{Tr } U(\mathcal{C}_n). \quad (5.18)$$

Let us now consider the calculation of an observable which serves as an order parameter for confinement: The static quark potential. As we shall see, this is also readily obtained from a Wilson loop. We use this order parameter in chapter 6 to investigate confinement in  $\widehat{\Phi}$ -models.

**Static quark potential** The calculation of the potential between two static quarks is best done in the Hamiltonian formulation of lattice gauge theory. Since gauge transformations commute with the Hamiltonian, we can restrict our Hamiltonian to act on gauge invariant wave functions, and those gauge invariant wave functions will remain gauge invariant. Hence, imposing Gauss' law once guarantees that it will remain in effect for all future times. There is therefore no longer any need for the Lagrange multiplier field  $A_4$ , which is the means for enforcing Gauss' law in the Lagrangian formulation. We set  $A_4 = 0$ , or equivalently,  $U_{t+1,t} = \mathbf{1}$ . The Hamiltonian now commutes with spatial gauge transformations, and physical states are invariant under such transformations.

A wave function that corresponds to a quark-antiquark excitation is not invariant under gauge transformations. Suppose we have an excitation that transforms in the fundamental representation of the gauge group at spatial position  $\mathbf{x}$ , and an excitation that transforms in the antifundamental representation at  $\mathbf{y}$ . The corresponding wave

function transforms as

$$\psi^{ij}[U'] = G_{\mathbf{x}}^{ik} (G_{\mathbf{y}}^{\dagger})^{\ell j} \psi^{k\ell}[U]. \quad (5.19)$$

To consider such excitations, we can restrict ourselves to the Hilbert space of all wave functions, which transform according to (5.19). Let us denote this Hilbert space by  $\mathcal{H}_{\mathbf{xy}}$ . Furthermore, let  $\{|n\rangle\}$  denote a basis for  $\mathcal{H}_{\mathbf{xy}}$  consisting of a complete set of eigenstates of the Hamiltonian  $\hat{H}$ . In particular, let  $|0\rangle$  denote the lowest energy eigenstate of  $\hat{H}$ . Then for any state  $|\psi\rangle$  which has non-zero overlap with  $|0\rangle$ , we have

$$\begin{aligned} \langle\psi|\exp(-T\hat{H})|\psi\rangle &= \sum_n |\langle\psi|n\rangle|^2 \exp(-TE_n) \\ &\rightarrow |\langle\psi|0\rangle|^2 \exp(-TE_0) \quad \text{as } T \rightarrow \infty. \end{aligned} \quad (5.20)$$

We define the lowest energy value  $E_0$  to be the static quark potential. This potential will of course be a function of the lattice points  $\mathbf{x}$  and  $\mathbf{y}$  of the excitations. If these points can be joined by a straight line along one of the directions of the lattice, and they are separated by a distance  $R$ , then the potential  $E_0 = V(R)$  is a function of  $R$ , by lattice translation invariance of the Hamiltonian. As a particular choice of the function  $\psi$  of (5.20) above, we take

$$\psi^{ij}(U) = U^{ij}(\mathbf{x}, \mathbf{y}) \Omega(U). \quad (5.21)$$

Here,  $\Omega(U)$  is the gauge invariant ground state function, i.e. the eigenfunction of  $\hat{H}$  with the lowest eigenvalue, belonging to the Hilbert space of gauge invariant functions.  $U^{ij}(\mathbf{x}, \mathbf{y})$  is the parallel transporter for a straight path between points  $\mathbf{x}$  and  $\mathbf{y}$ . It is obvious that  $\psi^{ij}$  has the right transformation properties, and is therefore an element of  $\mathcal{H}_{\mathbf{xy}}$ . For this test function, the matrix element in (5.20) can be rewritten as

$$\begin{aligned} \langle\psi^{ij}|\exp(-T\hat{H})|\psi_{ij}\rangle &= \langle\Omega|\hat{U}^{ij\dagger}(\mathbf{x}, \mathbf{y}) \exp(-T\hat{H})\hat{U}_{ij}(\mathbf{x}, \mathbf{y})|\Omega\rangle \\ &= \frac{1}{Z} \int \mathcal{D}U U^{ij*}(\mathbf{x} + T\hat{\mathbf{0}}, \mathbf{y} + T\hat{\mathbf{0}}) U^{ij}(\mathbf{x}, \mathbf{y}) \exp(-S[U]). \end{aligned} \quad (5.22)$$

Here, we have switched back to the action formulation to evaluate the vacuum ex-

pectation value. Recall that we are still in  $U_{t+1,t} = \mathbb{1}$  gauge. Hence,

$$U^{ij*}(\mathbf{x} + T\hat{\mathbf{0}}, \mathbf{y} + T\hat{\mathbf{0}})U^{ij}(\mathbf{x}, \mathbf{y}) = \text{Tr } U(\mathcal{C}_{R,T}), \quad (5.23)$$

where  $\mathcal{C}_{R,T}$  is a closed rectangular loop of spatial extent  $R$  and temporal extent  $T$ . The left hand side of eq.(5.23) is manifestly gauge invariant, and so in any gauge we have that

$$\begin{aligned} \langle \psi^{ij} | \exp(-T\hat{H}) | \psi^{ij} \rangle &= \frac{1}{Z} \int \mathcal{D}U \text{Tr } U(\mathcal{C}_{R,T}) \exp(-S[U]) \\ &= W(\mathcal{C}_{R,T}). \end{aligned} \quad (5.24)$$

This is simply the expectation value of the rectangular loop  $\mathcal{C}_{R,T}$ , or the Wilson loop  $W(\mathcal{C}_{R,T})$ . It then follows from eq.(5.20) that the static quark potential is related to the Wilson loop by

$$V(R) = - \lim_{T \rightarrow \infty} \frac{1}{T} \log W(\mathcal{C}_{R,T}). \quad (5.25)$$

Equivalently, for large  $T$ , such a Wilson loop behaves like

$$W(\mathcal{C}_{R,T}) \stackrel{T \rightarrow \infty}{\sim} C \exp(-TV(R)). \quad (5.26)$$

Seiler showed in [43], that the static quark potential cannot increase faster than linearly with the distance  $R$  of spatial separation of the two sources. To describe the behaviour of  $V(R)$  with increasing  $R$ , we define the string tension  $\sigma$  as

$$\sigma \equiv \lim_{R \rightarrow \infty} \frac{1}{R} V(R) = - \lim_{R,T \rightarrow \infty} \frac{1}{RT} \log W(\mathcal{C}_{R,T}). \quad (5.27)$$

If the string tension is non-zero, then the potential increases linearly with  $R$  at long distances. In this case, there is a constant force  $\sigma$  acting between widely separated quarks, a phenomenon known as static quark confinement. Thus, confinement manifests itself as an area dependence of a Wilson loop,

$$W(\mathcal{C}_{R,T}) \stackrel{R,T \rightarrow \infty}{\sim} C \exp(-\sigma RT). \quad (5.28)$$

## 5.2 The Quantum Link Formulation

Let us now consider the D-theory formulation of  $SU(N)$  gauge theories. Proceeding as we did in the cases of the Heisenberg and principal chiral models in chapters 2 and 4, we first replace the classical fields that make up the entries  $U_{x,\mu}^{ij}$  of the parallel transporter matrices  $U_{x,\mu}$  in the action by quantum operators  $\widehat{U}_{x,\mu}^{ij}$ , to obtain a quantum Hamilton operator that evolves the system in an additional Euclidean direction. The Hamilton operator takes the form

$$H = -J \sum_{x,\mu \neq \nu} \left[ \widehat{U}_{x,\mu}^{ij} \widehat{U}_{x+\hat{\mu},\nu}^{jk} \left( \widehat{U}_{x+\hat{\nu},\mu}^{\ell k} \right)^\dagger \left( \widehat{U}_{x,\nu}^{i\ell} \right)^\dagger + \text{h.c.} \right] - J' \sum_{x,\mu} [\det \widehat{U}_{x,\mu} + \det \widehat{U}_{x,\mu}^\dagger]. \quad (5.29)$$

The determinant term is understood to mean

$$\det \widehat{U}_{x,\mu} = \frac{1}{N!} \epsilon_{i_1 i_2 \dots i_N} \epsilon_{j_1 j_2 \dots j_N} \widehat{U}_{x,\mu}^{i_1 j_1} \widehat{U}_{x,\mu}^{i_2 j_2} \dots \widehat{U}_{x,\mu}^{i_N j_N}. \quad (5.30)$$

It has been introduced into the Hamiltonian to break an extra  $U(1)$  symmetry that would otherwise be present and lead to a  $U(N)$  rather than an  $SU(N)$  gauge invariant model.

This Hamilton operator has to be invariant under gauge transformations, i.e., we require that

$$[\widehat{H}, \widehat{G}_x^a] = 0, \quad (5.31)$$

where  $\widehat{G}_x^a$  are the generators of an  $SU(N)$  algebra at each lattice site  $x$ , obeying the commutation relations

$$[\widehat{G}_x^a, \widehat{G}_y^b] = 2i \delta_{xy} f_{abc} \widehat{G}_x^c. \quad (5.32)$$

In the Hilbert space, a general gauge transformation is represented by the operator  $\prod_x \exp(i\alpha_x^a \widehat{G}_x^a)$ . We can construct gauge covariant transformations of the fields by

requiring that

$$\begin{aligned}\widehat{U}'_{x,\mu} &= \prod_y \exp\left(-i\alpha_y^a \widehat{G}_y^a\right) \widehat{U}_{x,\mu} \prod_z \exp\left(i\alpha_z^b \widehat{G}_z^b\right) \\ &= \exp\left(-i\alpha_x^a \lambda^a\right) \widehat{U}_{x,\mu} \exp\left(i\alpha_{x+\hat{\mu}}^b \lambda^b\right).\end{aligned}\quad (5.33)$$

This implies commutation relations of the form

$$[\widehat{G}_x^a, \widehat{U}_{y,\mu}] = -\delta_{x,y+\hat{\mu}} \widehat{U}_{y,\mu} \lambda^a + \delta_{x,y} \lambda^a \widehat{U}_{y,\mu}.\quad (5.34)$$

In order to satisfy these relations, we write

$$\widehat{G}_x^a = \sum_{\mu} (\widehat{R}_{x-\hat{\mu},\mu}^a + \widehat{L}_{x,\mu}^a),\quad (5.35)$$

where  $\widehat{R}_{x,\mu}^a$  and  $\widehat{L}_{x,\mu}^a$  are generators of left and right gauge transformations of the link variable  $\widehat{U}_{x,\mu}$ . They generate a  $SU(N)_R \times SU(N)_L$  algebra on each link, which can be embedded diagonally in the algebra of  $SU(2N)$ , with the commutation relations as given in (3.5).

We choose representations for the  $SU(2N)$  algebra with rectangular Young tableaux, as we already did in the case of the principal chiral model. In particular, we can use the rishon representation of (3.6)-(3.9). In contrast to the principal chiral model, operators now live on the links and not on the lattice sites. The notation is the following,

$$\begin{aligned}\widehat{S}_{x,\mu}^{ij} &= \widehat{L}_{x,\mu}^{ij} = \left(\widehat{c}_{x,\mu}^{i\alpha\dagger} \widehat{c}_{x,\mu}^{j\alpha} - \frac{M}{2} \delta^{ij}\right), & \widehat{S}_{x,\mu}^{N+i,N+j} &= \widehat{R}_{x,\mu}^{ij} = \left(\widehat{c}_{x+\hat{\mu},-\mu}^{i\alpha\dagger} \widehat{c}_{x+\hat{\mu},-\mu}^{j\alpha} - \frac{M}{2} \delta^{ij}\right), \\ \widehat{S}_{x,\mu}^{i,N+j} &= \widehat{U}_{x,\mu}^{ij} = \widehat{c}_{x,\mu}^{i\alpha\dagger} \widehat{c}_{x+\hat{\mu},-\mu}^{j\alpha}, & \widehat{S}_{x,\mu}^{N+i,j} &= \left(\widehat{U}_{x,\mu}^{ji}\right)^\dagger = \widehat{c}_{x+\hat{\mu},-\mu}^{i\alpha\dagger} \widehat{c}_{x,\mu}^{j\alpha}, \\ \widehat{T}_{x,\mu} &= \sum_i \left(\widehat{c}_{x+\hat{\mu},-\mu}^{i\alpha\dagger} \widehat{c}_{x+\hat{\mu},-\mu}^{i\alpha} - \widehat{c}_{x,\mu}^{i\alpha\dagger} \widehat{c}_{x,\mu}^{i\alpha}\right), & \sum_i \left(\widehat{c}_{x,\mu}^{i\alpha\dagger} \widehat{c}_{x,\mu}^{i\beta} + \widehat{c}_{x+\hat{\mu},-\mu}^{i\alpha\dagger} \widehat{c}_{x+\hat{\mu},-\mu}^{i\beta}\right) &= \delta^{\alpha\beta} N,\end{aligned}\quad (5.36)$$

where  $\alpha, \beta = 1, \dots, M$  and  $i, j = 1, \dots, N$ . In chapter 3 we mentioned that the

operators  $\widehat{T}_{x,\mu}$  generate an extra  $U(1)$  gauge transformation. To explain, we define

$$\widehat{G}_x = \frac{1}{2} \sum_{\mu} \left( \widehat{T}_{x-\hat{\mu},\mu} - \widehat{T}_{x,\mu} \right). \quad (5.37)$$

The link operators  $\widehat{U}_{x,\mu}$  transform under corresponding  $U(1)$  gauge transformations as follows,

$$\widehat{U}'_{x,\mu} = \prod_y \exp \left( -i\alpha_y \widehat{G}_y \right) \widehat{U}_{x,\mu} \prod_z \exp \left( i\alpha_z \widehat{G}_z \right) = \exp \left( -i\alpha_x \right) \widehat{U}_{x,\mu} \exp \left( i\alpha_{x+\hat{\mu}} \right). \quad (5.38)$$

It is this  $U(1)$  gauge symmetry which is explicitly broken by the determinant term in the Hamilton operator.

The inclusion of quarks in the quantum link model is fairly straightforward. Wilson's approach to a lattice theory of quarks can readily be adapted to quantum link models, with only a slight change in the boundary conditions. However, Wilson fermions suffer from a number of practical problems. Taking the continuum limit requires careful fine-tuning of the bare quark mass. Furthermore, the Wilson term, which removes the doublers from the theory, breaks chiral symmetry in a maximal way. Since we already have an additional fifth Euclidean dimension, the stage is set for the natural inclusion of quarks as domain wall fermions. Shamir's modification of Kaplan's original proposal is particularly simple and well-suited for this purpose. We review the basic ideas, following [19, 16]. To obtain the Hamilton operator with quarks, we keep the gauge part unchanged from (5.29). The quark sector is constructed from Wilson's action for full QCD (5.9). It is important to note that the Hamilton operator describes the evolution of the system in a fifth direction, and so

we need to replace  $\bar{\Psi}_x$  by  $\hat{\Psi}_x^\dagger \gamma_5$ . This leads to,

$$\begin{aligned}
\hat{H} = & -J \sum_{x, \mu \neq \nu} \text{Tr} \left[ \hat{U}_{x, \mu} \hat{U}_{x+\hat{\mu}, \nu} \hat{U}_{x+\hat{\nu}, \mu}^\dagger \hat{U}_{x, \nu}^\dagger \right] - J' \sum_{x, \mu} \left[ \det \hat{U}_{x, \mu} + \det \hat{U}_{x, \mu}^\dagger \right] \\
& + \frac{1}{2} \sum_{x, \mu} \left[ \hat{\Psi}_x^\dagger \gamma_5 \gamma_\mu \hat{U}_{x, \mu} \hat{\Psi}_{x+\hat{\mu}} - \hat{\Psi}_{x+\hat{\mu}}^\dagger \gamma_5 \gamma_\mu \hat{U}_{x, \mu}^\dagger \hat{\Psi}_x \right] + m \sum_x \hat{\Psi}_x^\dagger \gamma_5 \hat{\Psi}_x \\
& + \frac{r}{2} \sum_{x, \mu} \left[ 2 \hat{\Psi}_x^\dagger \gamma_5 \hat{\Psi}_x - \hat{\Psi}_x^\dagger \gamma_5 \hat{U}_{x, \mu} \hat{\Psi}_{x+\hat{\mu}} - \hat{\Psi}_{x+\hat{\mu}}^\dagger \gamma_5 \hat{U}_{x, \mu}^\dagger \hat{\Psi}_x \right]. \tag{5.39}
\end{aligned}$$

Indices have been suppressed in this expression. The  $\hat{\Psi}_x^{ia\dot{\alpha}\dagger}$  and  $\hat{\Psi}_x^{ia\dot{\alpha}}$  are fermionic quark creation and annihilation operators, which obey canonical anticommutation relations. They carry a Dirac index  $a$ , a colour index  $i$  and a quark flavour index  $\dot{\alpha}$ . While this operator looks very much as if it might describe Wilson fermions, it is the boundary conditions, that are crucial in the domain wall fermion formulation. Furthermore, the sign of  $r$  in the Wilson term is different from that in the standard Wilson action. As explained below, we require  $r < 0$ .

Generators of gauge transformations need to be suitably amended to include transformations of quarks. They now take the form

$$\hat{G}_x^a = \sum_{\mu} \left( \hat{R}_{x-\hat{\mu}, \mu}^a + \hat{L}_{x, \mu}^a \right) + \hat{\Psi}_x^\dagger \lambda^a \hat{\Psi}_x. \tag{5.40}$$

It is straightforward to show that commutators of  $\hat{H}$  with gauge generators  $\hat{G}_x^a$  vanish for all lattice sites  $x$ .

In the Kaplan-Shamir proposal for gauge theories with domain wall fermions, one requires gauge fields to be constant in the fifth dimension. In our case, the fifth dimension is of course also a dynamical dimension for gluonic degrees of freedom. However, due to the exponentially large correlation length, which induces dimensional reduction to four dimensions, physical gluons are essentially constant along the fifth dimension. Hence, the domain wall construction can straightforwardly be applied to quantum link models. In order to implement the correct boundary conditions, the

partition function now takes the form

$$Z = \text{Tr} \langle 0 | \exp(-\beta \hat{H}) | 0 \rangle. \quad (5.41)$$

The trace is only over the gluonic Hilbert space, which induces periodic boundary conditions for the gluons. Open boundary conditions for fermions are induced by the Fock state  $|0\rangle$ . Writing quarks in terms of their chiral components

$$\hat{\Psi}_x = \begin{pmatrix} \hat{\Psi}_{Rx} \\ \hat{\Psi}_{Lx}^\dagger \end{pmatrix}, \quad (5.42)$$

this state is chosen so that it is annihilated by all  $\hat{\Psi}_{Rx}$  and all  $\hat{\Psi}_{Lx}$ . Hence, there are no left-handed quarks at  $x_5 = 0$ , and no right-handed quarks at  $x_5 = \beta$ .

We give a brief discussion of free domain wall fermions without gauge fields or interactions to illustrate the basic ideas of chiral symmetry restoration. The important feature is that the physical quark mass is exponentially small in the extent  $\beta$  of the fifth dimension, and vanishes as  $\beta \rightarrow \infty$ . This limit corresponds to the continuum limit and it is of course the same limit in which the quantum link model undergoes dimensional reduction. The evolution of free quarks with spatial lattice momentum  $\mathbf{p}$  in physical time (and not in the fifth Euclidean dimension) is governed by the Hamiltonian

$$H(\mathbf{p}) = \gamma_4 \left( \gamma_5 \partial_5 + M + \frac{r}{2} \sum_i \left( 2 \sin \left( \frac{p_i}{2} \right) \right)^2 - i \sum_i \gamma_i \sin p_i \right). \quad (5.43)$$

In order to obtain the wave function for a quark with momentum  $\mathbf{p}$ , we need to solve the eigenvalue problem

$$H(\mathbf{p})\Psi(\mathbf{p}, x_5) = E(\mathbf{p})\Psi(\mathbf{p}, x_5). \quad (5.44)$$

The quark mass is obtained from the energy of a quark at rest. As mentioned before, in lattice theories static quarks are characterised by momenta with components  $p_i = 0$



or  $\pi$ . While  $\mathbf{p} = \mathbf{0}$  corresponds to a physical quark at rest, the other such momentum states, with at least one non-zero momentum component equal to  $\pi$ , are the notorious doubler fermions. For such states,  $\sin p_i = 0$ , and so the above eigenvalue equations read

$$\begin{aligned} -\partial_5 \Psi_L(\mathbf{x}, x_5) + (M + 2nr) \Psi_L(\mathbf{p}, x_5) &= E(\mathbf{p}) \Psi_R(\mathbf{p}, x_5), \\ \partial_5 \Psi_R(\mathbf{x}, x_5) + (M + 2nr) \Psi_R(\mathbf{p}, x_5) &= E(\mathbf{p}) \Psi_L(\mathbf{p}, x_5). \end{aligned} \quad (5.45)$$

Here, we denote the number of non-zero components of the momentum vector by  $n$ ,  $n = \sum_i \sin^2(p_i/2)$ . Recall that the boundary conditions are  $\Psi_L(\mathbf{p}, 0) = \Psi_R(\mathbf{p}, \beta) = 0$ . We therefore make the ansatz

$$\Psi_L(\mathbf{p}, x_5) = A \sinh \alpha x_5, \quad \Psi_R(\mathbf{p}, x_5) = \pm A \sinh \alpha(x_5 - \beta). \quad (5.46)$$

Substituting these expressions in to eqn.(5.45), we obtain the following conditions for  $\alpha$  and  $E(\mathbf{p})$ ,

$$\alpha = (M + 2nr) \tanh \alpha\beta, \quad M + 2nr = \pm E(\mathbf{p}) \cosh \alpha\beta. \quad (5.47)$$

In order to get a non-trivial solution ( $\alpha \neq 0$ ), we need  $M + 2nr > 0$ . Then for large  $\beta$ , we have  $\tanh \alpha\beta \approx 1$ , and hence  $\alpha \approx (M + 2nr)$ . Hence, the energy of such states is given by

$$E(\mathbf{p}) \approx \pm 2(M + 2nr) \exp(-(M + 2nr)\beta). \quad (5.48)$$

We can rid ourselves of doublers by insisting that  $M + 2nr < 0$  for all  $n \geq 1$ . Hence, letting  $r < -M/2$  ensures that the only non-trivial solutions correspond to the physical static quark. The mass of the physical quark is thus given by

$$\mu = |E(\mathbf{0})| = 2M \exp(-M\beta). \quad (5.49)$$

Note, that this mass is exponentially small in  $\beta$ .

The correlation length governing the scale at which confinement takes place in the

quantum link model can be determined from the 1-loop  $\beta$ -function coefficient,

$$\xi \propto \exp\left(\frac{24\pi^2\beta}{(11N - 2N_f)e^2}\right). \quad (5.50)$$

Hence, for  $M > 24\pi^2/[(11N - 2N_f)e^2]$  the chiral limit is attained at the same time as dimensional reduction occurs as  $\beta$  gets large. For a given  $r$ , the range of  $M$  is restricted by  $M < 2|r|$ . However, we can choose the coupling  $J$  of the quantum link model, and hence the coupling  $1/e^2$  of the effective theory, in such a way that the above condition can always be satisfied.

Quarks are given a physical mass in this theory by suitably adjusting the boundary conditions. In particular, let

$$\begin{aligned} \Psi_L(x, x_5 = 0) &= -\frac{m_q}{2M} \Psi_L(x, x_5 = \beta) \\ \Psi_R(x, x_5 = \beta) &= -\frac{m_q}{2M} \Psi_R(x, x_5 = 0). \end{aligned} \quad (5.51)$$

Here,  $m_q$  is a mass parameter, and if  $m_q = 0$ , we obtain the same boundary conditions as above. Solving the differential equations (5.45) with these boundary conditions, we indeed obtain a physical quark mass equal to  $m_q$  in the continuum limit  $\beta \rightarrow \infty$ , just as long as  $m_q \ll M$ . In [17] it was shown that in the interacting theory,  $m_q$  only gets multiplicatively renormalised.

On the level of the partition function for the quantum link model, the above boundary conditions are implemented by the inclusion of a suitable operator in the path integral,

$$Z = \text{Tr} \left( \exp \left( -\beta \hat{H} \right) \hat{\mathcal{P}}(m_q) \right). \quad (5.52)$$

The operator  $\hat{\mathcal{P}}(m_q)$ , which depends on the mass parameter  $m_q$  is defined as

$$\hat{\mathcal{P}}(m_q) = \prod_x \left( \hat{\Psi}_{Rx} \hat{\Psi}_{Rx}^\dagger + \frac{m_q}{2M} \hat{\Psi}_{Rx}^\dagger \hat{\Psi}_{Rx} \right) \left( \hat{\Psi}_{Lx} \hat{\Psi}_{Lx}^\dagger + \frac{m_q}{2M} \hat{\Psi}_{Lx}^\dagger \hat{\Psi}_{Lx} \right). \quad (5.53)$$

For  $m_q = 0$ , this reduces to the same open boundary conditions as above.

We now return to quantum link models without quarks and prove that they indeed

undergo dimensional reduction for large enough representations.

### 5.3 The Low-Energy Effective Theory of the Quantum Link Model

To determine the low-energy effective theory of the quantum link model, we set up a coherent state path integral as discussed in appendix B and chapter 3. We will consider the analogue of (B.28) and the coset decomposition of (B.39) to determine some properties of the matrix  $V$  that appears in the action of the coherent state path integral. We have

$$\widehat{U}_{x,\mu}^{ij} = \widehat{c}_{x,\mu}^{i\alpha\dagger} \widehat{c}_{x+\widehat{\mu},-\mu}^{j\alpha}. \quad (5.54)$$

From (B.28) we obtain,

$$\langle q | \widehat{U}_{x,\mu}^{ij} | q \rangle = \frac{M}{2} V_{x,\mu}^{ij}. \quad (5.55)$$

Now consider

$$\begin{aligned} \frac{M}{2} V_{x+\widehat{\mu},-\mu}^{ij} &= \langle q | \widehat{U}_{x+\widehat{\mu},-\mu}^{ij} | q \rangle = \langle q | \widehat{c}_{x+\widehat{\mu},-\mu}^{i\alpha\dagger} \widehat{c}_{x,\mu}^{j\alpha} | q \rangle \\ &= \langle q | \left( \widehat{U}_{x,\mu}^{ji} \right)^\dagger | q \rangle = \langle q | \widehat{U}_{x,\mu}^{ji} | q \rangle^* = \frac{M}{2} V_{x,\mu}^{ji*}, \end{aligned} \quad (5.56)$$

and hence we see that

$$V_{x+\widehat{\mu},-\mu} = V_{x,\mu}^\dagger. \quad (5.57)$$

The coset decomposition is  $V_{x,\mu} = -U_{x,\mu} S_{x,\mu}$ , where  $S = S^\dagger$  and  $UU^\dagger = \mathbb{1}$ . Taken together with (5.57), this leads to

$$\begin{aligned} V_{x+\widehat{\mu},-\mu} &= -U_{x+\widehat{\mu},-\mu} S_{x+\widehat{\mu},-\mu} = V_{x,\mu}^\dagger \\ &= -S_{x,\mu} U_{x,\mu}^\dagger = -U_{x,\mu}^\dagger U_{x,\mu} S_{x,\mu} U_{x,\mu}^\dagger, \end{aligned} \quad (5.58)$$

and we deduce that

$$U_{x+\widehat{\mu},-\mu} = U_{x,\mu}^\dagger, \quad S_{x+\widehat{\mu},-\mu} = U_{x,\mu} S_{x,\mu} U_{x,\mu}^\dagger. \quad (5.59)$$

At this point, the complete action in the coherent state path integral is

$$S = S_B - \frac{JM^4}{16} \int_0^\beta d\tau \sum_{x,\mu \neq \nu} 2\text{Re Tr} \left[ U_{x,\mu} S_{x,\mu} U_{x+\hat{\mu},\nu} S_{x+\hat{\mu},\nu} S_{x+\hat{\nu},\mu} U_{x+\hat{\nu},\mu}^\dagger S_{x,\nu} U_{x,\nu}^\dagger \right] \\ - J' \int_0^\beta d\tau \sum_{x,\mu} \left[ \det \left( \frac{M}{2} U_{x,\mu} S_{x,\mu} \right) + \det \left( \frac{M}{2} S_{x,\mu} U_{x,\mu}^\dagger \right) \right]. \quad (5.60)$$

Here,

$$S_B = -\frac{M}{2} \sum_{x,\mu} \int_0^\beta d\tau \text{Tr} \left( \cos(2B_{x,\mu}) U_{x,\mu}^\dagger \partial_\tau U_{x,\mu} \right), \quad (5.61)$$

which follows from (B.44).

We now want to do a semi-classical expansion around the minimum of the action. As explained in chapter 2, this requires the existence of an intermediate cut-off scale  $\Lambda$ , such that

$$\xi^{-1} \ll \Lambda \ll \Lambda_{\text{B.Z.}} \quad (5.62)$$

As we will see below, such an intermediate cut-off scale is guaranteed to exist for large enough  $M$ , i.e. for a large enough representation of  $SU(2N)$ . From (5.60), the action is minimized when the eigenvalues of  $S$  are largest, i.e. equal to one, and the field  $U$  is constant for all links on the lattice. We can use a gauge transformation to rotate these constant fields to the identity matrix. The expansion for  $S$  is the same as in section 4.3,

$$S_{x,\mu} = \sin(2B_{x,\mu}) = \sin \left( 2 \left( \frac{\pi}{4} + a^2 E_{x,\mu} \right) \right) \approx \mathbf{1} - 2a^4 E_{x,\mu}^2. \quad (5.63)$$

We substitute this expression into the action, dropping terms of order  $E_{x,\mu}^3$  and higher. After some rearrangement and relabeling of the summed indices, we find

$$S \approx S_B - \frac{JM^4}{16} \int_0^\beta d\tau \sum_{x,\mu \neq \nu} 2\text{Re Tr} \left[ \left( \mathbf{1} - 4a^4 (E_{x,\mu}^2 + E_{x,\nu}^2) \right) U_{x,\mu} U_{x+\hat{\mu},\nu} U_{x+\hat{\nu},\mu}^\dagger U_{x,\nu}^\dagger \right] \\ - 2J' \left( \frac{M}{2} \right)^N \int_0^\beta d\tau \sum_{x,\mu} \left( 1 - 2a^4 \text{Tr} E_{x,\mu}^2 \right) \cos(a^2 \theta_{x,\mu}). \quad (5.64)$$

Here, we have defined  $\exp(-ia^2\theta_{x,\mu}) \equiv \det U_{x,\mu}$ . If we take  $J' > 0$ , then the minimum of the action will occur for  $\theta_{x,\mu} = 0$ . Thus, the matrices  $U_{x,\mu}$  will have determinant equal to one, and belong to  $SU(N)$  rather than  $U(N)$ .

In order to be able to take the continuum limit we need to express the unitary matrix field  $U_{x,\mu}$ , assumed to be close to the identity in our expansion, as the exponential of algebra-valued matrix fields. So let

$$A_{x,\mu} = -A_{x,\mu}^a \lambda^a \quad (5.65)$$

be a Lie algebra valued vector field,  $\lambda^a$  denoting the generators of  $SU(N)$ , and write

$$U_{x,\mu} \equiv \exp(-ia^2(\theta_{x,\mu}/N)\mathbb{1} - ia A_{x,\mu}). \quad (5.66)$$

The next step is to expand the action to order  $a^4$ , using (5.66). This leads to

$$S = S_B + \int_0^\beta d\tau \left( \frac{JM^4}{16} \sum_{x,\mu \neq \nu} a^4 \text{Tr} [F_{\mu\nu} F_{\mu\nu}] + \sum_{x,\mu} a^4 \text{Tr} \left[ \gamma E_{x,\mu}^2 + 2J' \left( \frac{M}{2} \right)^N \theta_{x,\mu}^2 \right] \right), \quad (5.67)$$

where we have dropped a constant, and

$$\gamma = 3JM^4 + 4J' \left( \frac{M}{2} \right)^N. \quad (5.68)$$

Of course, there aren't any terms linear in  $E$  in (5.67), and  $\gamma > 0$  since we are expanding about a minimum.

We use the same expansion to manipulate the Berry phase term,

$$\begin{aligned}
S_B &= -\frac{M}{2} \int_0^\beta d\tau \sum_{x,\mu} \text{Tr} [\cos(2B_{x,\mu}) U_{x,\mu}^\dagger \partial_\tau U_{x,\mu}] \\
&\approx -\frac{M}{2} \int_0^\beta d\tau \sum_{x,\mu} \text{Tr} \left[ \left( \cos\left(\frac{\pi}{2}\right) - 2a^2 \sin\left(\frac{\pi}{2}\right) E_{x,\mu} \right) \right. \\
&\quad \left. \times (\mathbb{1} + ia A_{x,\mu}) \partial_\tau (\mathbb{1} - ia A_{x,\mu}) \right] \\
&= -i \frac{M}{2} \int_0^\beta d\tau \sum_{x,\mu} a^4 \text{Tr} \left[ \frac{2}{a} E_{x,\mu} \partial_\tau A_{x,\mu} \right]. \tag{5.69}
\end{aligned}$$

The complete action in the limit  $\sum_x a^4 \rightarrow \int d^4x$  is now

$$\begin{aligned}
S &= \frac{1}{2e^2} \int_0^\beta d\tau \int d^4x \text{Tr} \left[ F_{\mu\nu} F_{\mu\nu} + 2e^2 \gamma E_\mu E_\mu - 2ie^2 \frac{M}{a} E_\mu \partial_\tau A_\mu \right. \\
&\quad \left. - 4e^2 J' \left(\frac{M}{2}\right)^N \theta_\mu \theta_\mu \right] \\
&= \frac{1}{2e^2} \int_0^\beta d\tau \int d^4x \text{Tr} \left[ F_{\mu\nu} F_{\mu\nu} + \frac{1}{c^2} \partial_\tau A_\mu \partial_\tau A_\mu + 2e^2 \gamma E'_\mu E'_\mu \right. \\
&\quad \left. - 4e^2 J' \left(\frac{M}{2}\right)^N \theta_\mu \theta_\mu \right], \tag{5.70}
\end{aligned}$$

where  $e^2 = 8/(M^4 J)$  and  $c = (Ma/2)\sqrt{\gamma J}$ . We have completed the square in order to integrate out the shifted field

$$E'_\mu = E_\mu - i \frac{M}{2\gamma a} \partial_\tau A_\mu, \tag{5.71}$$

as well as the  $\theta$ -field, obtaining

$$S = \frac{1}{2e^2} \int_0^\beta d\tau \int d^4x \text{Tr} \left[ F_{\mu\nu} F_{\mu\nu} + \frac{1}{c^2} \partial_\tau A_\mu \partial_\tau A_\mu \right]. \tag{5.72}$$

If we now again assume that the correlation length is much larger than the extent of the fifth dimension, we can perform the trivial integration over  $\tau$ , to obtain

$$S = \frac{\beta}{2e^2} \int d^4x \text{Tr} [F_{\mu\nu} F_{\mu\nu}]. \tag{5.73}$$

It was argued in [15], that a finite correlation length

$$\xi \propto \exp\left(\frac{24\pi^2\beta}{11Ne^2}\right) \quad (5.74)$$

is expected to be generated non-perturbatively. Again, the continuum limit in which the correlation length diverges is achieved by taking the extent of the extra dimension  $\beta$  to infinity. In this limit, we also find that the extent of the extra dimension in physical units is much smaller than the correlation length,  $\beta c \ll \xi$ . Thus, the theory undergoes dimensional reduction. Notice further, that  $1/e^2 \propto M^4$ . Thus, already for finite  $\beta$ , the semi-classical approximation is valid if the size  $M$  of the representation for  $SU(2N)$  is large enough.

In the next chapter we will investigate the large  $N$  limit of quantum link models. At this point, let us determine how the coupling constant  $J$ , which appears in (5.29) scales with  $N$ . We do this by requiring that the 5-dimensional low-energy effective theory in (5.72) have a well-defined 't Hooft large  $N$  limit. In the 't Hooft limit,

$$\frac{1}{e^2 N} = \frac{M^4 J}{8N} \quad (5.75)$$

is fixed as  $N$  gets large. Thus,  $J \sim N/M^4$ . For the 5-dimensional term in the action,

$$\frac{1}{Ne^2 c^2} = \frac{1}{6NM^2 J a^2} \quad (5.76)$$

is kept fixed, and so  $J \sim 1/(NM^2)$ . Hence, we find that for a meaningful 't Hooft limit to exist, we need  $M \sim N$ , and thus  $J \sim 1/N^3$ .

# Chapter 6

## Large $N$ Yang-Mills Theory

In this chapter we rewrite the quantum link model as a theory of colour singlet fields, which we denote by  $\Phi$ . The resulting  $\Phi$ -model Hamiltonian depends on the number of colours in a trivial way, thus offering a new approach to the large  $N$  limit of gauge theories. We investigate some of the properties of this model, and most importantly, we show that the  $\Phi$ -model confines colour and rishon flavour in the strong coupling limit.

In the first section of this chapter, we review the basic ideas for considering the large  $N$  limit of standard QCD. In section 6.2 we describe the strong coupling expansion of Wilson's formulation of gauge theories, a technique that we adapt for the  $\Phi$ -model in section 6.4. In section 6.3 we first derive the  $\Phi$ -model from the quantum link model. It is a model of quantum operators acting in a Hilbert space, and we then use coherent state techniques to derive a path integral expression for the partition function. The degrees of freedom in the resulting action are elements of the coset space  $U(2dM)/[U(dM) \times U(dM)]$ . In section 6.4, we discuss the strong coupling expansion of  $\Phi$ -models and demonstrate confinement.

### 6.1 The Large $N$ Limit of Standard QCD

Let us briefly review the basic results of standard QCD in the large  $N$  limit following the discussion in [44]. In order to investigate this limit, it is crucial to keep track of



colour indices in the model. With colour indices explicitly displayed, the imaginary-time Lagrangian is

$$\mathcal{L} = \frac{N}{g^2} \left( -\frac{1}{4} F_{\mu\nu}^i F_{\mu\nu}^j + \bar{\psi}_i (\delta_j^i \partial_\mu + A_{\mu j}^i) \gamma^\mu \psi^j - m \bar{\psi}_i \psi^i \right). \quad (6.1)$$

The scaling of the coupling constant is chosen in such a way, that  $g$  is kept fixed as  $N$  goes to infinity. This is the only way of obtaining a meaningful expansion in powers of  $1/N$ . From the Lagrangian in (6.1), it is straightforward to derive Feynman rules to be used in perturbative calculations of expectation values of observables. The quark propagator is given by

$$\langle 0 | T \psi^i(x) \bar{\psi}_j(y) | 0 \rangle = \delta_j^i S_F(x - y). \quad (6.2)$$

Here,  $S_F(x - y)$  is just the Feynman propagator of a single Dirac fermion. Hence, a propagating quark does not change colour, and the indices at the beginning of a quark propagation line are the same as those at the end of it (see figure 6-1).

The gluon propagator is also straightforwardly derived from the Lagrangian,

$$\langle 0 | T A_{\mu j}^i(x) A_{\nu \ell}^k(y) | 0 \rangle = \left( \delta_\ell^i \delta_j^k - \frac{1}{N} \delta_j^i \delta_\ell^k \right) D_{\mu\nu}(x - y). \quad (6.3)$$

In this equation,  $D_{\mu\nu}(x - y)$  is simply the photon propagator, i.e. the one corresponding to a single gauge boson. The term proportional to  $1/N$  is there, because the gluons live in the adjoint representation of  $SU(N)$  rather than  $U(N)$ , and the corresponding generators are therefore traceless matrices. However, in the limit as  $N \rightarrow \infty$ , this term vanishes, and we cannot distinguish between an  $SU(N)$  and a  $U(N)$  gauge theory. Without this term, it is also clear that the index structure of the gluon propagator is now particularly simple. There are two completely separate upper-lower pairs of indices, which do not mix. In fact, it is as if there were a quark-antiquark pair propagating — though only as far as the colour index structure is concerned. 't Hooft introduced a new representation of gluon Feynman propagators, in which they are drawn as double lines, indicating the separate propagation of the

indices [30]. This is shown in figure 6-1, which was created using the L<sup>A</sup>T<sub>E</sub>X package for Feynman rules by Michael Levine [45]. It is also shown how at vertices, colour indices are “conserved”, i.e. the colour index of an incoming quark becomes the colour index associated with the outgoing line of the double line gluon propagator, and similarly, the outgoing quark index is the same as that of the incoming line of the gluon propagator. At gluon-gluon vertices, the incoming colour index of one propagator becomes the outgoing colour index of another propagator. It is perfectly possible for a standard Feynman diagram to permit a number of ways of assigning indices to it. In this case, there is one corresponding new Feynman diagram for each of these cases.

This colour propagation structure makes the counting of powers of  $1/N$  of a given Feynman graph quite easy. First, we see from the Lagrangian that each propagator, quark and gluon, introduces a factor of  $1/N$ . Each vertex on the other hand introduces a factor of  $N$ . Finally, each closed loop involves the trace of the identity matrix, and thus gives a factor of  $N$  also.

A vacuum diagram can be viewed as an orientable surface with edges, vertices and faces. Such an object is called a polyhedral complex. Each propagator corresponds to an edge of the complex, and hence the diagram is proportional to  $N^E$ , where  $E$  is the number of edges. Each interaction vertex is a vertex of the complex, and if we denote the number of vertices by  $V$ , then there is a contribution to the diagram of  $N^V$ . Finally, each face of the complex corresponds to a closed loop, and we get a factor of  $N^F$ , where  $F$  is the number of faces. Thus, the power of  $N$  associated with such a diagram is given by

$$N^{F-E+V} = N^\chi. \tag{6.4}$$

We have introduced  $\chi$ , which is the Euler characteristic of the complex. By Euler’s theorem it is equal to

$$\chi = F - E + V = 2 - 2H - B. \tag{6.5}$$

Here,  $H$  is the number of handles of the complex, and  $B$  is the number of holes (the boundary). It thus follows, that the lowest order diagrams are those, which

• This is the first item of the list emphasized in this way. Text within such a segment given additional emphasis with Roman type. Italic type loses its ability to emphasize simply distracting when used excessively.

It is important to understand that the use of the BOLD and ITALIC commands from here on will not work if you are using a line where it might otherwise be used. The only difference is between the "Mr." and "Jones" in "Mr. Jones", or within a word—especially when the word is a symbol like  $x^2$  or  $y^2$ . The only difference is between the "Mr." and "Jones" in "Mr. Jones", or within a word—especially when the word is a symbol like  $x^2$  or  $y^2$ .

Text allows you to set the amount of space between the  $x$  and  $y$  in  $x^2$  or  $y^2$ . The amount of space between the  $x$  and  $y$  in  $x^2$  or  $y^2$  is set by the  $\text{&#x2013}$  command. A displayed formula is one line long and is centered on the page. The  $\text{&#x2013}$  command is used to generate a simple document of your own.

## 2 Displayed Text $x^2 + y^2 = z^2$

Text is set from the left margin. Quotations are common in text. There are short quotations. The ends of words and sentences are marked by spaces. It doesn't matter how many spaces you use. The end of a paragraph is marked by a blank line and a paragraph indentation.

Since any number of consecutive spaces are treated like a single one, the formatting of text is not affected by the number of spaces. The only difference is between the "Mr." and "Jones" in "Mr. Jones", or within a word—especially when the word is a symbol like  $x^2$  or  $y^2$ . The only difference is between the "Mr." and "Jones" in "Mr. Jones", or within a word—especially when the word is a symbol like  $x^2$  or  $y^2$ .

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<sup>1</sup>This is an example of a footnote.

Figure 6-1: Double Line representation of Feynman diagrams

have no boundary and no handles. A boundary corresponds to a fermion propagator, because each gluon propagator corresponds to an edge between two adjacent faces and therefore cannot lie on a boundary. Conversely, since a fermion propagator consists of only one line, it cannot be on the edge between two adjacent faces. Hence, a boundary always corresponds to a fermion loop, and we conclude that fermion loops are suppressed in the large  $N$  limit.

Since handles are also suppressed, the only vacuum diagrams that contribute at the leading order are planar graphs made up entirely of gluons.

## 6.2 Standard Yang-Mills Theory in the Strong Coupling Limit

The strong coupling expansion of lattice gauge theories is discussed in [41] as an expansion in group characters. Here, we adapt this discussion to a level which can be applied to  $\Phi$ -models. Consider Wilson's action for a  $U(N)$  gauge theory,

$$S = \sum_{\square} S_{\square}(U_{\square})$$

$$S_{\square} = -\frac{\beta}{2N} \left( \text{Tr} U_{\square} + \text{Tr} U_{\square}^{\dagger} \right) = -\frac{\beta}{N} \text{Re Tr} U_{\square}. \quad (6.6)$$

Here,  $\square$  stands for a lattice plaquette, and  $U_{\square}$  is the product of link matrices around the plaquette. We have renamed the coupling constant,  $\beta = 2N/g^2$ , so that the limit  $g \rightarrow \infty$  now corresponds to  $\beta \rightarrow 0$ . This will make it easier to keep track of terms in the strong coupling expansion. The idea behind this expansion is based on the fact that as  $\beta \rightarrow 0$ , the exponential  $\exp(-S_{\square}(U_{\square})) \rightarrow 1$  uniformly, because  $-\beta \leq S_{\square} \leq 0$  for all  $\beta$ . If we write  $\exp(-S_{\square}(U_{\square})) = 1 + f_{\square}(U_{\square})$ , then in the limit as  $\beta \rightarrow 0$ , we have  $f_{\square}(U_{\square}) \rightarrow 0$  uniformly. To first order,  $f_{\square}(U_{\square}) \sim \mathcal{O}(\beta)$ ,

$$f_{\square}(U_{\square}) = \sum_{n=1}^{\infty} \left( \frac{\beta}{2N} \right)^n \frac{\left( \text{Tr}(U_{\square}) + \text{Tr}(U_{\square}^{\dagger}) \right)^n}{n!}$$

$$= \sum_{n=1}^{\infty} \left( \frac{\beta}{2N} \right)^n \left( \sum_{k=0}^n \frac{(\text{Tr} U_{\square})^k (\text{Tr} U_{\square}^{\dagger})^{n-k}}{(n-k)!k!} \right). \quad (6.7)$$

Thus, higher powers of  $f_{\square}$  imply higher leading powers of  $\beta$ . In terms of the functions  $f_{\square}(U_{\square})$ , the integrand of the partition function can be written as

$$\begin{aligned}
\exp(-S) &= \prod_{\square} (1 + f_{\square}) = (1 + f_{\square_1})(1 + f_{\square_2}) \cdots \\
&= 1 + \sum_{\square} f_{\square} + \sum_{(\square, \square')} f_{\square} f_{\square'} + \cdots \\
&= \sum_{\mathcal{P}} \prod_{\square \in \mathcal{P}} f_{\square}.
\end{aligned} \tag{6.8}$$

Here,  $\mathcal{P}$  denotes an element of the power set of the set of plaquettes. The partition function itself now reads

$$Z = \int \mathcal{D}U \exp(-S) = \sum_{\mathcal{P}} \int \mathcal{D}U \prod_{\square \in \mathcal{P}} f_{\square}(U_{\square}). \tag{6.9}$$

This expression suggests an expansion of the partition function in terms of the number of plaquettes in  $\mathcal{P}$ . The more plaquettes in  $\mathcal{P}$ , the higher the power of  $\beta$  of such a term, and the smaller the contribution to the path integral in the limit as  $\beta$  gets small. We need to determine the lowest order non-vanishing contributions to the partition function, for as it turns out, a lot of terms in the sum (6.9) are zero. This requires a discussion of integrals over  $U(N)$  matrices, which is given in appendix C. The important results are that

$$\int dU = 1 \tag{6.10}$$

and if  $F(U, U^{\dagger})$  is a monomial in  $U$  and  $U^{\dagger}$ , then the integral

$$\int dU F(U, U^{\dagger}) \tag{6.11}$$

is non-zero only if  $U$  and  $U^{\dagger}$  appear with equal powers. This implies that an element of  $\mathcal{P}$  that makes a non-zero contribution to the partition function (6.9) needs to contain plaquettes in such a way that each lattice link contains an even number of plaquette edges. So the graph formed from such a collection of plaquettes cannot have a boundary, that would be formed by single plaquette edges. Furthermore,

$$\int dU \left[ \text{Diagram of two adjacent plaquettes sharing a link } U \text{ and } U^\dagger \right] = \frac{1}{N} \left[ \text{Diagram of a single plaquette with a factor of } 1/N \right]$$

Figure 6-2: Integrating over a common link of adjacent plaquettes

the plaquettes must be oriented in such a way so that the orientation of a pair of plaquette edges, which share the same link, is in opposing directions. This is a direct consequence of the shift invariance of the Haar measure, which requires that only singlets under shift transformations can make a non-zero contribution to the integral in (6.11). Furthermore, the following property can be verified using the formulae of appendix C,

$$\int dU \text{Tr}(VU) \text{Tr}(U^\dagger W) = \frac{1}{N} \text{Tr}(VW). \quad (6.12)$$

Graphically, this implies that if two plaquettes share a link, so that the common link variables are oriented in opposing directions, then integration over the common link variable dissolves the link between the two plaquettes and introduces a factor of  $1/N$ . This is shown in figure 6-2.

Let us now consider how to calculate observables in the strong coupling expansion. The expectation value of the gauge invariant operator  $\mathcal{O}$  is

$$\begin{aligned} \langle \mathcal{O} \rangle &= \frac{1}{Z} \int \mathcal{D}U \mathcal{O} \exp(-S[U]) \\ &= \frac{\sum_{\mathcal{P}} \int \mathcal{D}U \mathcal{O} \prod_{\square \in \mathcal{P}} f_{\square}}{\sum_{\mathcal{P}'} \int \mathcal{D}U' \prod_{\square' \in \mathcal{P}'} f_{\square'}} \\ &= \sum_{\tilde{\mathcal{P}}} \int \mathcal{D}U \mathcal{O} \prod_{\square \in \tilde{\mathcal{P}}} f_{\square}. \end{aligned} \quad (6.13)$$

Here,  $\tilde{\mathcal{P}}$  is the set of all those collections of plaquettes, which are connected and include the variables  $U$  that make up the operator  $\mathcal{O}$ . In this context, two plaquettes are connected if they share a common edge, but are disconnected if, for example, they only share a common vertex. So the lowest order contribution to the Wilson

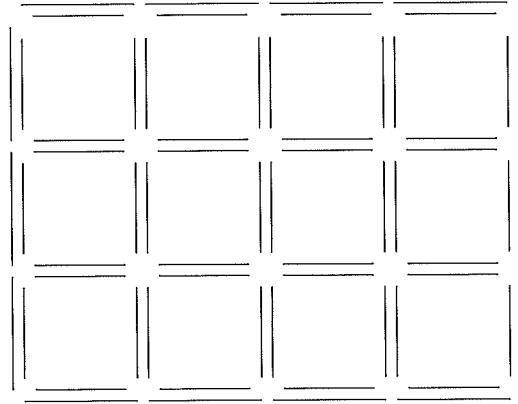


Figure 6-3: Wilson's gauge theory: Lowest order contribution to the Wilson loop in the strong coupling expansion

loop, for instance, is given in figure 6-3. Let us calculate its value, so we can extract the lowest order contribution to the string tension in the strong coupling limit. Each plaquette in figure 6-3 contributes a factor  $\beta/2N$  from the leading term in  $f_{\square}$ . There are  $RT$  plaquettes in this Wilson loop of dimensions  $R$  by  $T$ . Using the integration rule of eq.(6.12), we can eliminate all the vertical links in the interior of the Wilson loop at the expense of factors of  $1/N$ . There are  $(R-1)T$  such vertical links. We can now use the same integration rule to eliminate one horizontal link from each row of horizontal links, thus introducing a factor of  $(1/N)^{T-1}$ . All the other horizontal links in the interior of the Wilson loop now disappear, because products of a unitary matrix with its conjugate give just the identity. This leaves us with the perimeter of the Wilson loop. Here we can integrate over one link to obtain a factor  $1/N$ , and then the remaining links, except for one, again disappear. Finally, we get a trace over the identity matrix, which gives a factor of  $N$ . Thus, to lowest order

$$W_{R,T} = \left(\frac{\beta}{2N}\right)^{RT} \left(\frac{1}{N}\right)^{(R-1)T+(T-1)+1-1} = \frac{1}{N} \left(\frac{\beta}{2N^2}\right)^{RT}. \quad (6.14)$$

Hence, the string tension is given by

$$\sigma = - \lim_{R,T \rightarrow \infty} \frac{1}{RT} \log W_{R,T} = - \log \left(\frac{\beta}{2N^2}\right). \quad (6.15)$$

As shown in section 6.1, the 't Hooft limit of large  $N$  requires a different scaling of the coupling constant with  $N$ . In this limit,  $\beta = 2N^2/g^2$ , and  $g$  is kept fixed as  $N \rightarrow \infty$ . This leads to a finite limit of the string tension,

$$\sigma \rightarrow -\log \frac{2}{g^2}, \quad \text{as } N \rightarrow \infty. \quad (6.16)$$

### 6.3 The $\Phi$ -Model

We want to consider the quantum link model in the large  $N$  limit. As we saw in eq.(6.3) the difference between gluon propagators in  $U(N)$  and  $SU(N)$  gauge theories is of order  $1/N$ , and hence can be ignored in the large  $N$  limit. This suggests discarding the determinant term in the Hamilton operator and considering a  $U(N)$  rather than an  $SU(N)$  gauge-symmetric quantum link model. The full Hamilton operator including quarks is

$$\begin{aligned} \hat{H} = & -J \sum_{x,\mu \neq \nu} \left[ \hat{U}_{x,\mu}^{ij} \hat{U}_{x+\hat{\mu},\nu}^{jk} \left( \hat{U}_{x+\hat{\nu},\mu}^{lk} \right)^\dagger \left( \hat{U}_{x,\nu}^{il} \right)^\dagger \right] \\ & + \frac{1}{2} \sum_{x,\mu} \left[ \left( \hat{\Psi}_x^{i\dot{\alpha}} \right)^\dagger \gamma_5 \gamma_\mu \hat{U}_{x,\mu}^{ij} \hat{\Psi}_{x+\hat{\mu}}^{j\dot{\alpha}} - \left( \hat{\Psi}_{x+\hat{\mu}}^{i\dot{\alpha}} \right)^\dagger \gamma_5 \gamma_\mu \left( \hat{U}_{x,\mu}^{ji} \right)^\dagger \hat{\Psi}_x^{j\dot{\alpha}} \right] \\ & + M \sum_x \left( \hat{\Psi}_x^{i\dot{\alpha}} \right)^\dagger \gamma_5 \hat{\Psi}_x^{i\dot{\alpha}} \\ & + \frac{r}{2} \sum_{x,\mu} \left[ 2 \left( \hat{\Psi}_x^{i\dot{\alpha}} \right)^\dagger \gamma_5 \hat{\Psi}_x^{i\dot{\alpha}} - \left( \hat{\Psi}_x^{i\dot{\alpha}} \right)^\dagger \gamma_5 \hat{U}_{x,\mu}^{ij} \hat{\Psi}_{x+\hat{\mu}}^{j\dot{\alpha}} - \left( \hat{\Psi}_{x+\hat{\mu}}^{i\dot{\alpha}} \right)^\dagger \gamma_5 \left( \hat{U}_{x,\mu}^{ji} \right)^\dagger \hat{\Psi}_x^{j\dot{\alpha}} \right] \\ = & J \sum_{x,\mu \neq \nu} \left[ \left( \hat{c}_{x,\nu}^{i\delta\dagger} \hat{c}_{x,\mu}^{i\alpha} \right)^\dagger \left( \hat{c}_{x+\hat{\mu},-\mu}^{j,\alpha\dagger} \hat{c}_{x+\hat{\mu},\nu}^{j\beta} \right) \left( \hat{c}_{x+\hat{\mu}+\hat{\nu},-\nu}^{k\beta\dagger} \hat{c}_{x+\hat{\mu}+\hat{\nu},-\mu}^{k\gamma} \right) \left( \hat{c}_{x+\hat{\nu},\mu}^{\ell\gamma\dagger} \hat{c}_{x+\hat{\nu},-\nu}^{\ell\delta} \right) \right] \\ & + \frac{1}{2} \sum_{x,\mu} \left[ \left( \hat{\Psi}_x^{i\dot{\alpha}\dagger} \hat{c}_{x,\mu}^{i\beta} \right) \gamma_5 \gamma_\mu \left( \hat{c}_{x+\hat{\mu},-\mu}^{j\beta\dagger} \hat{\Psi}_{x+\hat{\mu}}^{j\dot{\alpha}} \right) - \left( \hat{\Psi}_{x+\hat{\mu}}^{i\dot{\alpha}\dagger} \hat{c}_{x+\hat{\mu},-\mu}^{i\beta} \right) \gamma_5 \gamma_\mu \left( \hat{c}_{x,\mu}^{j\beta\dagger} \hat{\Psi}_x^{j\dot{\alpha}} \right) \right] \\ & + M \sum_x \left( \hat{\Psi}_x^{i\dot{\alpha}\dagger} \gamma_5 \hat{\Psi}_x^{i\dot{\alpha}} \right) + \frac{r}{2} \sum_{x,\mu} \left[ 2 \hat{\Psi}_x^{i\dot{\alpha}\dagger} \gamma_5 \hat{\Psi}_x^{i\dot{\alpha}} - \left( \hat{\Psi}_x^{i\dot{\alpha}\dagger} \hat{c}_{x,\mu}^{i\beta} \right) \gamma_5 \left( \hat{c}_{x+\hat{\mu},-\mu}^{j\beta} \hat{\Psi}_{x+\hat{\mu}}^{j\dot{\alpha}} \right) \right. \\ & \quad \left. - \left( \hat{\Psi}_{x+\hat{\mu}}^{i\dot{\alpha}\dagger} \hat{c}_{x+\hat{\mu},-\mu}^{i\beta} \right) \gamma_5 \left( \hat{c}_{x,\mu}^{j\beta\dagger} \hat{\Psi}_x^{j\dot{\alpha}} \right) \right] \quad (6.17) \end{aligned}$$

In this and the following equations it is important to remember that repeated indices are summed over. Indices  $i, j, \dots = 1, \dots, N$  are colour indices. Quark flavour indices



are denoted by  $\dot{\alpha} = 1, \dots, N_f$  and rishon flavour indices are written as  $\alpha, \beta, \dots = 1, \dots, M$ . Dirac indices are suppressed in this expression. In the last part of eq.(6.17), we used parentheses to group together terms that form colour singlets. We thus observe that the Hamilton operator can be expressed entirely in terms of colour singlet operators [19]. Remarkably, all of these operators live on the sites of the lattice, in contrast to the quantum link operators we started out with, which lived on the links of the lattice. For the gauge part of the Hamilton operator we define colour singlet  $\widehat{\Phi}$ -operators,

$$\widehat{\Phi}_{x,\mu,\nu}^{\alpha\beta} \equiv \widehat{c}_{x,\mu}^{i\alpha\dagger} \widehat{c}_{x,\nu}^{i\beta} - \delta_{\mu\nu} \delta^{\alpha\beta} \frac{N}{2}. \quad (6.18)$$

These operators carry Lorentz indices  $\mu$  and  $\nu$ , which — somewhat peculiarly — vary over the range  $-d, \dots, -1, 1, \dots, d$  of positive and negative lattice directions. They also carry rishon flavour indices  $\alpha$  and  $\beta$ .

In the quark sector of the model we have meson operators,

$$\widehat{M}_x^{ab,\dot{\alpha}\dot{\beta}} \equiv \widehat{\Psi}_x^{ia\dot{\alpha}\dagger} \widehat{\Psi}_x^{ib\dot{\beta}}. \quad (6.19)$$

These operators carry quark flavour indices  $\dot{\alpha}$  and  $\dot{\beta}$ . They also carry Dirac indices  $a$  and  $b$ , which vary between 1 and the dimension  $d_c$  of the Euclidean Lorentz algebra in  $d$  space-time dimensions. In particular,  $d_c$  is the dimension of the spinor representation of  $SO(d)$ , which is equal to  $2^{d/2}$  if  $d$  is even and  $2^{(d-1)/2}$  if  $d$  is odd.

Finally, there are constituent quark operators, defined as

$$\widehat{Q}_{x,\mu}^{a\alpha\dot{\alpha}} \equiv \widehat{c}_{x,\mu}^{i\alpha\dagger} \widehat{\Psi}_x^{ia\dot{\alpha}}. \quad (6.20)$$

These operators carry a Dirac index,  $a$ , a rishon flavour index,  $\alpha$ , and a quark flavour index  $\dot{\alpha}$ .

With these definitions, we can rewrite the Hamilton operator in terms of the new

colour singlet operators as

$$\begin{aligned}
\hat{H} = & J \sum_{x, \mu \neq \nu} \left[ \hat{\Phi}_{x, \nu, \mu}^{\delta\alpha} \hat{\Phi}_{x+\hat{\mu}, -\mu, \nu}^{\alpha\beta} \hat{\Phi}_{x+\hat{\mu}+\hat{\nu}, -\nu, -\mu}^{\beta\gamma} \hat{\Phi}_{x+\hat{\nu}, \mu, -\nu}^{\gamma\delta} \right] \\
& + \frac{1}{2} \sum_{x, \mu} \left[ \hat{Q}_{x, \mu}^{\beta\dot{\alpha}\dagger} \gamma_5 \gamma_\mu \hat{Q}_{x+\hat{\mu}, -\mu}^{\beta\dot{\alpha}} - \hat{Q}_{x+\hat{\mu}, -\mu}^{\beta\dot{\alpha}\dagger} \gamma_5 \gamma_\mu \hat{Q}_{x, \mu}^{\beta\dot{\alpha}} \right] \\
& + M \sum_{x, \mu} \text{Tr} \left[ \widehat{M}_x^{\dot{\alpha}\dot{\alpha}} \gamma_5 \right] \\
& + \frac{r}{2} \sum_{x, \mu} \left[ 2 \text{Tr} \left[ \widehat{M}_x^{\dot{\alpha}\dot{\alpha}} \gamma_5 \right] - \hat{Q}_{x, \mu}^{\beta\dot{\alpha}\dagger} \gamma_5 \hat{Q}_{x+\hat{\mu}, -\mu}^{\beta\dot{\alpha}} - \hat{Q}_{x+\hat{\mu}, -\mu}^{\beta\dot{\alpha}\dagger} \gamma_5 \hat{Q}_x^{\beta\dot{\alpha}} \right]. \tag{6.21}
\end{aligned}$$

In this expression explicit traces are over Dirac indices, that have been suppressed.

To investigate the algebraic structure defined by this new set of operators, let us determine their commutation relations. Clearly, operators at different lattice sites commute, and we can concentrate on commutation relations of operators corresponding to the same lattice site. The commutators of two  $\hat{\Phi}$ -operators are

$$\left[ \hat{\Phi}_{\mu\nu}^{\alpha\beta}, \hat{\Phi}_{\rho\sigma}^{\gamma\delta} \right] = \delta^{\gamma\beta} \delta_{\rho\nu} \hat{\Phi}_{\mu\sigma}^{\alpha\delta} - \delta^{\alpha\delta} \delta_{\mu\sigma} \hat{\Phi}_{\rho\nu}^{\gamma\beta}. \tag{6.22}$$

These operators generate a  $U(2dM)$  algebra. To see this more explicitly, we can treat the indices in a unified manner by defining

$$\hat{\Phi}^{AB} = \hat{\Phi}^{\alpha+M\mu, \beta+M\nu} \equiv \hat{\Phi}_{\mu\nu}^{\alpha\beta}. \tag{6.23}$$

With this new index convention, the commutation relations take the more familiar form

$$\left[ \hat{\Phi}^{AB}, \hat{\Phi}^{CD} \right] = \delta^{CB} \hat{\Phi}^{AD} - \delta^{AD} \hat{\Phi}^{CB}. \tag{6.24}$$

The same trick could be used to treat the indices of the remaining operators in a unified manner. The commutation relations of two meson operators at the same lattice site are given by

$$\left[ \widehat{M}^{ab, \dot{\alpha}\dot{\beta}}, \widehat{M}^{cd, \dot{\gamma}\dot{\delta}} \right] = \delta^{cb} \delta^{\dot{\gamma}\dot{\beta}} \widehat{M}^{ad, \dot{\alpha}\dot{\delta}} - \delta^{ad} \delta^{\dot{\alpha}\dot{\delta}} \widehat{M}^{cb, \dot{\gamma}\dot{\beta}}. \tag{6.25}$$

Hence, these operators generate a  $U(d_c N_f)$  algebra at each lattice site. The glueball and meson operators commute,  $[\widehat{M}, \widehat{\Phi}] = 0$ , so thus far we have a  $U(2dM) \times U(d_c N_f)$  algebra at each lattice site. The constituent quark operators complete this algebra to the full algebra of  $U(2dM + d_c N_f)$ . The commutators involving these operators are

$$\begin{aligned}
[\widehat{Q}_\mu^{a\alpha\dot{\alpha}}, \widehat{Q}_\nu^{b\beta\dot{\beta}\dagger}] &= \delta^{ab} \delta^{\dot{\alpha}\dot{\beta}} \widehat{\Phi}_{\mu\nu}^{\alpha\beta} - \delta^{\alpha\beta} \delta_{\mu\nu} \widehat{M}^{ba, \dot{\beta}\dot{\alpha}}, \\
[\widehat{Q}_\mu^{a\alpha\dot{\alpha}}, \widehat{Q}_\nu^{b\beta\dot{\beta}}] &= [\widehat{Q}_\mu^{a\alpha\dot{\alpha}\dagger}, \widehat{Q}_\nu^{b\beta\dot{\beta}\dagger}] = 0; \\
[\widehat{\Phi}_{\mu\nu}^{\alpha\beta}, \widehat{Q}_\rho^{c\gamma\dot{\gamma}}] &= \delta_{\nu\rho} \delta^{\beta\gamma} \widehat{Q}_\mu^{c\alpha\dot{\gamma}}; \\
[\widehat{M}^{ab, \dot{\alpha}\dot{\beta}}, \widehat{Q}_\rho^{c\gamma\dot{\gamma}}] &= -\delta^{ac} \delta^{\dot{\alpha}\dot{\gamma}} \widehat{Q}_\rho^{b\gamma\dot{\beta}}.
\end{aligned} \tag{6.26}$$

As explained in section 6.1, the leading order vacuum diagrams in the standard large  $N$  limit of QCD only involve gluons. Quarks contribute to the vacuum only in next-to-leading order. Therefore, we will consider the case of pure gauge theory without quarks in the following discussion. The Hamiltonian takes the form

$$\widehat{H} = J \sum_{x, \mu \neq \nu} \left[ \widehat{\Phi}_{x, \nu, \mu}^{\delta\alpha} \widehat{\Phi}_{x+\hat{\mu}, -\mu, \nu}^{\alpha\beta} \widehat{\Phi}_{x+\hat{\mu}+\hat{\nu}, -\nu, -\mu}^{\beta\gamma} \widehat{\Phi}_{x+\hat{\nu}, \mu, -\nu}^{\gamma\delta} \right]. \tag{6.27}$$

Although the  $U(N)$  colour gauge symmetry of the original quantum link model Hamiltonian is now hidden in the  $\widehat{\Phi}$  operators, this Hamiltonian has a manifest  $U(M)$  gauge symmetry of rishon flavour. The corresponding gauge transformations live on the links of the lattice, unlike the colour gauge transformations, which live on the lattice sites. We now construct the generators of this  $U(M)$  gauge symmetry. The  $\widehat{\Phi}$  operators need to transform in a covariant way under such gauge transformations,

$$\begin{aligned}
\widehat{\Phi}'_{x, \mu, \nu} &= \left[ \prod_{y, \rho > 0} \exp \left( (-\alpha_{y, \rho}^{\alpha\beta} + \alpha_{y, \rho}^{\beta\alpha*}) \widehat{\mathcal{G}}_{y, \rho}^{\beta\alpha} \right) \right] \widehat{\Phi}_{x, \mu, \nu} \left[ \prod_{z, \sigma > 0} \exp \left( (\alpha_{z, \sigma}^{\gamma\delta} - \alpha_{z, \sigma}^{\delta\gamma*}) \widehat{\mathcal{G}}_{z, \sigma}^{\delta\gamma} \right) \right] \\
&= \exp \left( -\alpha_{x, \mu} + \alpha_{x, \mu}^\dagger \right) \widehat{\Phi}_{x, \mu, \nu} \exp \left( \alpha_{x, \nu} - \alpha_{x, \nu}^\dagger \right), \quad \text{for } \mu, \nu > 0.
\end{aligned} \tag{6.28}$$

In this equation we have suppressed the rishon flavour indices. Since there is exactly one gauge transformation on each link, we have denoted generators of gauge trans-

formations by  $\widehat{\mathcal{G}}_{y,\rho}$  restricting ourselves to  $\rho > 0$ . After working out the commutation relation between  $\widehat{\Phi}_{x,\mu,\nu}$  and  $\widehat{\mathcal{G}}_{y,\rho}$ , we extend the definition of gauge transformation to include  $\rho < 0$  by requiring that  $\widehat{\mathcal{G}}_{y+\hat{\rho},-\rho} = \widehat{\mathcal{G}}_{y,\rho}$ . The transformation properties of eq.(6.28) imply the commutation relation,

$$\left[ \widehat{\mathcal{G}}_{y,\rho}^{\alpha\beta}, \widehat{\Phi}_{x,\mu,\nu}^{\gamma\delta} \right] = \delta_{x,y} \delta_{\rho,\mu} \delta^{\beta\gamma} \widehat{\Phi}_{x,\mu,\nu}^{\alpha\delta} - \delta_{x,y} \delta_{\rho,\nu} \delta^{\alpha\delta} \widehat{\Phi}_{x,\mu,\nu}^{\gamma\beta}, \quad (6.29)$$

where  $\mu, \nu, \rho > 0$ . Requiring that  $\widehat{\mathcal{G}}_{y+\hat{\rho},-\rho} = \widehat{\mathcal{G}}_{y,\rho}$ , this commutator can be realised by defining

$$\widehat{\mathcal{G}}_{x,\mu}^{\alpha\beta} \equiv \widehat{\Phi}_{x,\mu,\mu}^{\alpha\beta} + \widehat{\Phi}_{x+\hat{\mu},-\mu,-\mu}^{\alpha\beta} \quad (6.30)$$

for all values of  $\mu$ . It is straightforward to show that the Hamiltonian commutes with each generator  $\widehat{\mathcal{G}}_{x,\mu}^{\alpha\beta}$ . In fact, as discussed in chapter 3, imposing Gauss' law for this  $U(M)$  gauge symmetry of the  $\widehat{\Phi}$ -model is equivalent to choosing a representation for the  $U(2N)$  quantum link algebra (cf. 3-1),

$$\begin{aligned} \sum_{i=1}^N \left( c_{x,\mu}^{i\alpha\dagger} c_{x,\mu}^{i\beta} + c_{x+\hat{\mu},-\mu}^{i\alpha\dagger} c_{x+\hat{\mu},-\mu}^{i\beta} \right) &= \delta^{\alpha\beta} N \\ \Leftrightarrow \widehat{\Phi}_{x,\mu,\mu}^{\alpha\beta} + \widehat{\Phi}_{x+\hat{\mu},-\mu,-\mu}^{\alpha\beta} &= 0. \end{aligned} \quad (6.31)$$

In terms of the  $\widehat{\Phi}$ -model, this constraint introduces a dependence of the  $\widehat{\Phi}$ -operators of neighbouring lattice sites on each other. Below, in the strong coupling expansion of this model, we replace this explicit constraint by adding a term to the action, which encourages the relation (6.31), without strictly enforcing it. This will enable us to calculate the string tension of the model in the strong coupling limit analytically.

In the construction of the original quantum link model we chose not to impose Gauss' law for  $U(N)$  gauge transformations, so as not to obtain the  $A_5$  Lagrange multiplier field that would enforce it. This made the argument for dimensional reduction simpler. However, even if we choose to impose Gauss' law, we can argue that the quantum link model still undergoes dimensional reduction and leads to the desired target theory.

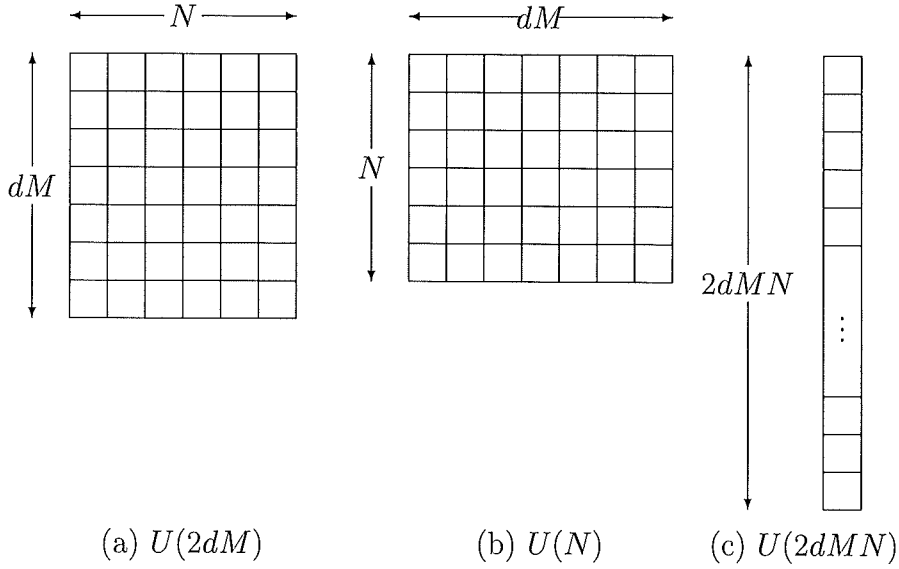


Figure 6-4: Choosing a representation for the  $U(2dM)$  algebra of the  $\widehat{\Phi}$ -model.

Imposing Gauss' law for the quantum link model actually corresponds to choosing a representation for the  $\widehat{\Phi}$ -operator algebra  $U(2dM)$  at each lattice site. This is in complete analogy with the discussion of chapter 3, where we showed that imposing Gauss' law for the  $U(M)$  gauge symmetry of the  $\widehat{\Phi}$ -model is equivalent to fixing the representation of the  $U(2N)$  quantum link algebra. Here, we have a total of  $2dMN$  rishons associated with each lattice site, which carry rishon flavour, colour and lattice direction indices. As all possible bilinear combinations of these rishons we obtain the generators of a  $U(2dMN)$  algebra in a totally antisymmetric representation, as the rishons are fermions. Since the operator for the number of rishons on a lattice site commutes with the Hamilton operator, we can choose the number of rishons per lattice site to be  $dMN$ . This gives a totally antisymmetric representation of  $U(2dMN)$  with a Young tableau of one column that is  $dMN$  boxes long (see figure 6-4(c)). Let us consider the  $U(N) \times U(2dM)$  subalgebra of  $U(2dMN)$ . Imposing Gauss' law for the quantum link model is to choose a singlet representation for  $U(N)$ , as shown in figure 6-4(b). This fixes the representation of  $U(2dM)$  to be the associated representation with reflected Young tableau as shown in figure 6-4(a),

$$\begin{aligned} \sum_{\mu} \left( \hat{c}_{x,\mu}^{i\alpha\dagger} \hat{c}_{x,\mu}^{j\alpha} + \hat{c}_{x,-\mu}^{i\alpha\dagger} \hat{c}_{x,-\mu}^{j\alpha} \right) &= \delta^{ij} dM \\ \Leftrightarrow \sum_{\mu} \left( \hat{L}_{x,\mu}^{ij} + \hat{R}_{x-\hat{\mu},\mu}^{ij} \right) &= 0. \end{aligned} \quad (6.32)$$

Here,  $\hat{L}^{ij}$  and  $\hat{R}^{ij}$  are the generators of left and right gauge transformations of section 5.2. The representation we have thus chosen for the generators  $\hat{\Phi}$  is just the one with rectangular Young tableau with  $dM$  rows and  $N$  columns. So taking the limit as  $N$  gets large simply means that we are letting the representation of the  $\hat{\Phi}$  generators get large.

In order to investigate the dynamics of the  $\hat{\Phi}$ -operator model, we can derive a Euclidean quantum field theory using coherent states. We make use of the results of appendix B and set up a coherent state path integral in much the same way as was done in chapters 4 and 5. Operators  $\hat{\Phi}$  are replaced by complex matrix elements  $\Phi$  according to

$$\langle q | \hat{\Phi}_{\mu,\nu}^{\alpha\beta} | q \rangle = \frac{N}{2} \Phi_{\mu,\nu}^{\alpha\beta}. \quad (6.33)$$

The matrix  $\Phi$  is a  $2dM \times 2dM$  matrix, given by

$$\Phi = \exp \begin{pmatrix} 0 & q^\dagger \\ -q & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1}_d & 0 \\ 0 & -\mathbb{1}_d \end{pmatrix} \exp \begin{pmatrix} 0 & -q^\dagger \\ q & 0 \end{pmatrix}. \quad (6.34)$$

The entries of this matrix are arranged into  $(2d)^2$  blocks, each of size  $M \times M$ , which are indexed by  $\mu\nu$ . The convention is that indices  $\mu$  and  $\nu$  run from  $-d$  to  $d$  (excluding 0). The entries within a  $\mu\nu$  block are given by  $\Phi_{\mu\nu}^{\alpha\beta}$ , where  $\alpha, \beta = 1, \dots, M$ . In eq.(6.34), the matrices  $q$  are  $GL(dM, \mathbb{C})$  matrices. We thus obtain the path integral,

$$Z = \int \mathcal{D}\Phi \exp(-S[\Phi]). \quad (6.35)$$

The action in this path integral is given by

$$S[\Phi] = -\frac{N}{4} \int_0^\beta d\tau \int_0^1 dv \sum_x \text{Tr} \left[ \Phi \frac{\partial \Phi}{\partial v} \frac{\partial \Phi}{\partial \tau} \right] + \int_0^\beta d\tau \mathcal{H}[\Phi]. \quad (6.36)$$

Here,  $\mathcal{H}$  is obtained from the Hamilton operator  $\widehat{H}$  by replacing every occurrence of  $\widehat{\Phi}_{x,\mu,\nu}^{\alpha\beta}$  in (6.27) by  $(N/2)\Phi_{x,\mu,\nu}^{\alpha\beta}$ . Hence,

$$\mathcal{H}[\Phi] = \frac{JN^4}{16} \sum_{x,\mu \neq \nu} \left[ \Phi_{x,\nu,\mu}^{\delta\alpha} \Phi_{x+\hat{\mu},-\mu,\nu}^{\alpha\beta} \Phi_{x+\hat{\mu}+\hat{\nu},-\nu,-\mu}^{\beta\gamma} \Phi_{x+\hat{\nu},\mu,-\nu}^{\gamma\delta} \right] \quad (6.37)$$

In the Berry phase term of the action, the field  $\Phi(\tau, v)$  is an extrapolation of the field  $\Phi(\tau)$  in an additional direction as discussed in appendix B. The additional direction is parametrised by  $v \in [0, 1]$ , and the corresponding boundary conditions are

$$\begin{aligned} \Phi(\tau, 0) &= \Phi(\tau', 0) \quad \text{for all } \tau, \tau'; \\ \Phi(\tau, 1) &= \Phi(\tau); \quad \Phi(0, v) = \Phi(\beta, v). \end{aligned} \quad (6.38)$$

In section 5.3 we showed that the coupling constant  $J$  scales with  $N$  as  $J \sim 1/N^3$ . In the  $\Phi$ -model action of (6.36) the Berry phase term is explicitly proportional to  $N$ . With the correct scaling of  $J$  we see that the coupling of the other term is also proportional to  $N$ . Finally, we let  $\lambda \sim N$ .

We saw in the operator formulation of the model that it has a  $U(M)$  gauge symmetry living on the links of the lattice. This symmetry is now expressed in terms of transformations of the matrix  $\Phi$  that leave the action invariant. In particular, at each site we define  $\mathcal{G} = \text{diag}(\mathcal{G}_{-d}, \mathcal{G}_{-d+1}, \dots, \mathcal{G}_d)$  to be a matrix of diagonal blocks of  $M \times M$  unitary matrices  $\mathcal{G}_i$ . Consider  $\Phi' = \mathcal{G}\Phi\mathcal{G}^\dagger$ . Thus,  $\Phi'$  also lives in the coset space  $U(2dM)/[U(dM) \times U(dM)]$ . Furthermore, its  $M \times M$  block elements are related to those of  $\Phi$  by

$$\Phi'_{\mu\nu} = \mathcal{G}_\mu \Phi_{\mu\nu} \mathcal{G}_\nu^\dagger. \quad (6.39)$$

Clearly, such transformations leave the Berry phase term invariant, and the other part of the action given in (6.37) transforms as

$$\begin{aligned} \mathcal{H}[\Phi'] &= \frac{JN^4}{16} \sum_{x,\mu \neq \nu} \text{Tr} \left[ \mathcal{G}_{x,\nu} \Phi_{x,\nu,\mu} \mathcal{G}_{x,\mu}^\dagger \mathcal{G}_{x+\hat{\mu},-\mu} \Phi_{x+\hat{\mu},-\mu,\nu} \mathcal{G}_{x+\hat{\mu},\nu}^\dagger \right. \\ &\quad \left. \times \mathcal{G}_{x+\hat{\mu}+\hat{\nu},-\nu} \Phi_{x+\hat{\mu}+\hat{\nu},-\nu,-\mu} \mathcal{G}_{x+\hat{\mu}+\hat{\nu},-\mu}^\dagger \mathcal{G}_{x+\hat{\nu},\mu} \Phi_{x+\hat{\nu},\mu,-\nu} \mathcal{G}_{x+\hat{\nu},-\nu}^\dagger \right]. \end{aligned} \quad (6.40)$$

Hence, the action is invariant under such transformations if  $\mathcal{G}_{x+\hat{\mu},-\mu} = \mathcal{G}_{x,\mu}$  for each  $x$  and  $\mu$ . These are the gauge transformations on the links mentioned above.

One of the main focus points of the first part of this thesis is the dimensional reduction of D-theory models to give rise to continuous field theory models. At this point, the spectrum of  $\Phi$ -models is not understood well enough to address the issue of dimensional reduction of these models directly. However, since they are just a reformulation of quantum link models they should describe the same physics. And since quantum link models undergo dimensional reduction, the same should be true of  $\Phi$ -models. Furthermore, we can argue that in the strong coupling and large  $N$  limits, we do not even need to introduce an additional dimension to set up a coherent state path integral. In particular, consider the quantum partition function in the strong coupling limit. In this limit, the operator  $\exp(-\beta\hat{H})$  should uniformly approach the unit operator as  $\beta \rightarrow 0$ . Thus, in the partition function we can expand the exponential as a power series in powers of  $\beta$ ,

$$Z = \text{Tr} \exp\left(-\beta\hat{H}[\hat{\Phi}]\right) = \int dq \langle q | \sum_{i=0}^{\infty} \frac{1}{i!} (-\beta\hat{H}[\hat{\Phi}])^i | q \rangle, \quad (6.41)$$

To obtain the leading order behaviour, we consider only the smallest relevant terms in the calculation of expectation values of observables. The trace in the partition function is taken by introducing a resolution of the identity in terms of coherent states. It is important to note that

$$\langle q | \hat{\Phi}_{x_1, \mu_1, \nu_1}^{\alpha_1 \beta_1} \dots \hat{\Phi}_{x_m, \mu_m, \nu_m}^{\alpha_m \beta_m} | q \rangle = \frac{N^m}{2^m} (\Phi_{x_1, \mu_1, \nu_1}^{\alpha_1 \beta_1} \dots \Phi_{x_m, \mu_m, \nu_m}^{\alpha_m \beta_m} + \mathcal{O}(1/N)). \quad (6.42)$$

There are no corrections to the first term on the right-hand side of eqn. (6.42) if all the operators in the matrix element commute, while any corrections due to non-commuting operators are suppressed by a factor  $1/N$ . Thus, in the limit as  $N \rightarrow \infty$ , the partition function in (6.41) simply becomes

$$Z = \int \mathcal{D}\Phi \exp(-\beta\mathcal{H}[\Phi]), \quad (6.43)$$



where  $\mathcal{H}[\Phi]$  is obtained from  $\widehat{H}[\widehat{\Phi}]$  by replacing  $\widehat{\Phi}_{x,\mu,\nu}^{\alpha\beta}$  by  $(N/2)\Phi_{x,\mu\nu}^{\alpha\beta}$ . This is a just the path integral of a  $d$ -dimensional theory. It has no Berry phase term, and is the same result we obtain by postulating dimensional reduction and trivially integrating over the  $\tau$ -dependence of the  $\Phi$ -fields. It is the  $d$ -dimensional model of eq.(6.43) which we take as our starting point for the strong coupling calculations of the next section.

We also introduce an approximation to the constraint in eq.(6.31) between neighbouring  $\Phi$ -variables by adding an additional term to the action,

$$\mathcal{H}[\Phi] = \frac{JN^4}{16} \sum_{x,\mu \neq \nu} \text{Tr} [\Phi_{x,\nu,\mu} \Phi_{x+\hat{\mu},-\mu,\nu} \Phi_{x+\hat{\mu}+\hat{\nu},-\nu,-\mu} \Phi_{x+\hat{\nu},\mu,-\nu}] + \lambda \sum_{x,\mu} \text{Tr} [(\Phi_{x,\mu,\mu} + \Phi_{x+\hat{\mu},-\mu,-\mu})^2]. \quad (6.44)$$

It is easily checked, that this term is invariant under the  $U(M)$  gauge transformations of eq.(6.39). For  $\lambda = \infty$ , this constraint is exact, but in the following section, we shall soften the requirement by allowing  $\lambda$  to be finite. We are thus able to calculate the string tension in the strong coupling expansion analytically.

## 6.4 The Strong Coupling Expansion of the $\Phi$ -Model

The idea for the strong coupling expansion in the  $\Phi$ -model is the same as in Wilson's theory. For convenience, we introduce a graphical notation for the  $\Phi$ -variables. Since  $\Phi_{x,\mu,\nu}^{\alpha\beta}$  carries two direction indices  $\mu$  and  $\nu$ , we denote the index structure of such a variable by

$$\Phi_{\mu\nu} = \Phi_{\perp} = \perp, \quad \Phi_{\mu\mu} = \Phi_{\subset} = \subset. \quad (6.45)$$

This notation is somewhat ambiguous, because usually it doesn't matter which particular values for  $\mu$  and  $\nu$  we are talking about. However, a product such as  $\perp \perp$  denotes two adjacent  $\Phi$ -variables, with matrix multiplication over rishon flavour indices implied, i.e.  $\Phi_{\mu,\nu}^{\alpha\beta} \Phi_{-\nu,\mu}^{\beta\gamma}$ . There is also a direction associated with such a variable, and it is a priori not clear whether  $\perp$  stands for  $\Phi_{\mu,\nu}$  or  $\Phi_{\nu,\mu}$ . If the direction is important, we shall make this explicit, if it is not, then one of the two possible directions may be

chosen at will, but will need to be considered consistently. We can write the action of the  $\Phi$ -model as

$$S = \sum_{\square} S_{\square}(\Phi_{\square}) + \sum_{\blacksquare} S_{\blacksquare}(\Phi_{\blacksquare}). \quad (6.46)$$

Here,  $\square$  denotes the plaquettes of the lattice, and  $S_{\square}(\Phi_{\square})$  is the part of the action which multiplies four  $\Phi$ s around a plaquette,

$$S_{\square}(\Phi_{\square}) = \frac{JN^4}{2^4} (\text{Tr}(\Phi_{\square}) + \text{Tr}(\Phi_{\square}^{\dagger})), \quad \Phi_{\square} = \Phi_{\square}^{\downarrow} \Phi_{\square}^{\rightarrow} \Phi_{\square}^{\uparrow} \Phi_{\square}^{\leftarrow}. \quad (6.47)$$

The links of the lattice are denoted by  $\blacksquare$ , and  $S_{\blacksquare}(\Phi_{\blacksquare})$  is the part of the action which multiplies two  $\Phi$ s on opposite corners of a link,

$$S_{\blacksquare} = \frac{\lambda N^2}{2^2} \text{Tr}(\Phi_{\blacksquare}), \quad \Phi_{\blacksquare} = (\Phi_{\square}^{\leftarrow} + \Phi_{\square}^{\rightarrow})^2. \quad (6.48)$$

As  $\beta \rightarrow 0$  the exponentials  $\exp(-\beta S_{\square}) \rightarrow 1$  and  $\exp(-\beta S_{\blacksquare}) \rightarrow 1$  uniformly. Alternatively, we write

$$\exp(-\beta S_{\square}(\Phi_{\square})) = 1 + f_{\square}(\Phi_{\square}), \quad \exp(-\beta S_{\blacksquare}(\Phi_{\blacksquare})) = 1 + g_{\blacksquare}(\Phi_{\blacksquare}). \quad (6.49)$$

Hence, in the limit as  $\beta \rightarrow 0$ , we have  $f_{\square} \rightarrow 0$  and  $g_{\blacksquare} \rightarrow 0$  uniformly. Expressions for  $f_{\square}$  and  $g_{\blacksquare}$  are

$$\begin{aligned} f_{\square}(\Phi_{\square}) &= \sum_{n=1}^{\infty} \left( -\frac{\beta JN^4}{2^4} \right)^n \left( \sum_{k=0}^n \frac{(\text{Tr}(\Phi_{\square}))^k (\text{Tr}(\Phi_{\square}^{\dagger}))^{n-k}}{(n-k)!k!} \right), \\ g_{\blacksquare}(\Phi_{\blacksquare}) &= \sum_{n=1}^{\infty} \left( -\frac{\beta \lambda N^2}{2^2} \right)^n (\text{Tr}(\Phi_{\blacksquare}))^n. \end{aligned} \quad (6.50)$$

We can thus write the integrand of the partition function as

$$\begin{aligned}
\exp(-\beta S) &= \left( \prod_{\square} (1 + f_{\square}) \right) \left( \prod_{\underline{-}} (1 + g_{\underline{-}}) \right) \\
&= (1 + f_{\square_1})(1 + f_{\square_2}) \cdots (1 + g_{\underline{-}_1})(1 + g_{\underline{-}_2}) \cdots \\
&= \sum_{\mathcal{P}} \left( \prod_{\square \in \mathcal{P}} f_{\square} \right) \left( \prod_{\underline{-} \in \mathcal{P}} g_{\underline{-}} \right). \tag{6.51}
\end{aligned}$$

Here,  $\mathcal{P}$  are the elements of the power set of the set of all plaquettes and links of the lattice. The partition function can now be written as

$$Z = \int \mathcal{D}\Phi \exp(-S) = \sum_{\mathcal{P}} \int \mathcal{D}\Phi \left( \prod_{\square \in \mathcal{P}} f_{\square} \right) \left( \prod_{\underline{-} \in \mathcal{P}} g_{\underline{-}} \right). \tag{6.52}$$

In order to calculate observables, we need to determine the rules that govern integration over  $\Phi$ -variables. Recall that  $\Phi = U\Lambda U^\dagger$ , where  $U$  is a  $2dM \times 2dM$  unitary matrix, and  $\Lambda = \text{diag}(1, \dots, 1, -1, \dots, -1)$ . So integration over  $\Phi$  is induced from integration over unitary matrices  $U$ , where the measure is the Haar measure. In appendix C the process of determining integrals over  $\Phi$ -variables is explained in detail, and the relevant results are listed in table 6.1. In order for a given collection of plaquette and link terms to make a non-zero contribution to the partition function there needs to be an even number of  $\Phi$ -variables on every lattice site. This is due to the fact that for every  $\Phi$  there is a  $-\Phi$ , so that integration over an odd number of them is zero from symmetry.

Let us now consider how to calculate observables in the strong coupling expansion of the  $\Phi$ -model. Since the integration measure of the path integral is induced from the shift invariant Haar measure it follows that  $d\Phi = d(\mathcal{G}\Phi\mathcal{G}^\dagger)$  for unitary  $\mathcal{G}$ . The action has a  $U(M)$  rishon flavour gauge symmetry, so that physical observables need to be gauge invariant quantities. Thus, let  $\mathcal{O}$  be a gauge invariant functional of the

picture	integral	value
	$\int d\Phi \Phi_{\mu,\nu}^{\alpha\beta} \Phi_{\nu,\mu}^{\gamma\delta}$	$\frac{2dM}{(2dM)^2 - 1} \delta^{\alpha\delta} \delta^{\gamma\beta}$
	$\int d\Phi \Phi_{\mu,\nu}^{\alpha\beta} \Phi_{\nu,-\mu}^{\gamma\delta} \Phi_{-\mu,-\nu}^{\epsilon\rho} \Phi_{-\nu,\mu}^{\sigma\eta}$	$-\frac{2dM}{((2dM)^2 - 1)((2dM)^2 - 9)} \delta^{\alpha\eta} \delta^{\gamma\beta} \delta^{\epsilon\delta} \delta^{\sigma\rho}$
	$\int d\Phi \Phi_{\mu,\nu}^{\alpha\beta} \Phi_{\nu,-\nu}^{\gamma\delta} \Phi_{-\nu,\mu}^{\epsilon\rho} \Phi_{\nu,\nu}^{\sigma\eta}$	$-\frac{1}{((2dM)^2 - 1)((2dM)^2 - 9)} \times (3\delta^{\alpha\rho} \delta^{\gamma\beta} \delta^{\epsilon\delta} \delta^{\sigma\eta} - 2dM \delta^{\alpha\rho} \delta^{\gamma\eta} \delta^{\epsilon\delta} \delta^{\sigma\beta})$
	$\int d\Phi \Phi_{\mu,\mu}^{\alpha\beta} \Phi_{\mu,\nu}^{\gamma\delta} \Phi_{\nu,\mu}^{\epsilon\rho} \Phi_{\nu,\nu}^{\sigma\eta}$	$-\frac{2}{((2dM)^2 - 1)((2dM)^2 - 9)} \times (3\delta^{\alpha\rho} \delta^{\gamma\beta} \delta^{\epsilon\delta} \delta^{\sigma\eta} + 3\delta^{\alpha\beta} \delta^{\gamma\delta} \delta^{\epsilon\eta} \delta^{\sigma\rho} - 2dM \delta^{\alpha\beta} \delta^{\gamma\rho} \delta^{\epsilon\delta} \delta^{\sigma\eta} - 2dM \delta^{\alpha\rho} \delta^{\gamma\beta} \delta^{\epsilon\eta} \delta^{\sigma\delta})$

Table 6.1: Integrals of the Strong Coupling Expansion

$\Phi$ -field. Then,

$$\begin{aligned}
\langle \mathcal{O} \rangle &= \frac{1}{Z} \int \mathcal{D}\Phi \mathcal{O} \exp(-S[\Phi]) \\
&= \frac{\sum_{\mathcal{P}} \int \mathcal{D}\Phi \mathcal{O} \left( \prod_{\square \in \mathcal{P}} f_{\square} \right) \left( \prod_{\blacksquare \in \mathcal{P}} g_{\blacksquare} \right)}{\sum_{\mathcal{P}'} \int \mathcal{D}\Phi \left( \prod_{\square \in \mathcal{P}'} f_{\square} \right) \left( \prod_{\blacksquare \in \mathcal{P}'} g_{\blacksquare} \right)} \\
&= \sum_{\tilde{\mathcal{P}}} \int \mathcal{D}\Phi \mathcal{O} \left( \prod_{\square \in \tilde{\mathcal{P}}} f_{\square} \right) \left( \prod_{\blacksquare \in \tilde{\mathcal{P}}} g_{\blacksquare} \right) \tag{6.53}
\end{aligned}$$

Here,  $\tilde{\mathcal{P}}$  is the set of connected diagrams, including those  $\Phi$ -variables which belong to  $\mathcal{O}$ . Since the  $\Phi$ -variables live on the sites, two neighbouring plaquette terms are connected to each other if they touch at a corner. This is different from Wilson's theory, where neighbouring plaquettes need to share a common link in order to be connected to each other, as illustrated in figure 6-5.

From gauge invariance of the measure and the action, a given graph can make a non-vanishing contribution only if the  $\Phi$ s associated with each lattice site can combine to give a singlet under  $U(M)$  gauge transformations. Graphically this implies that

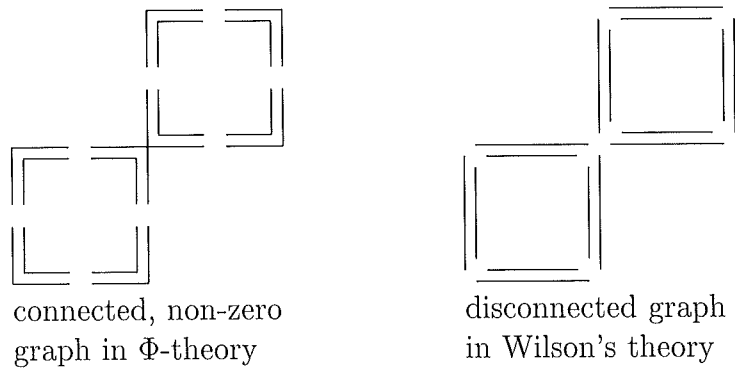


Figure 6-5: Connected and disconnected graphs in  $\Phi$ -theory and Wilson's theory

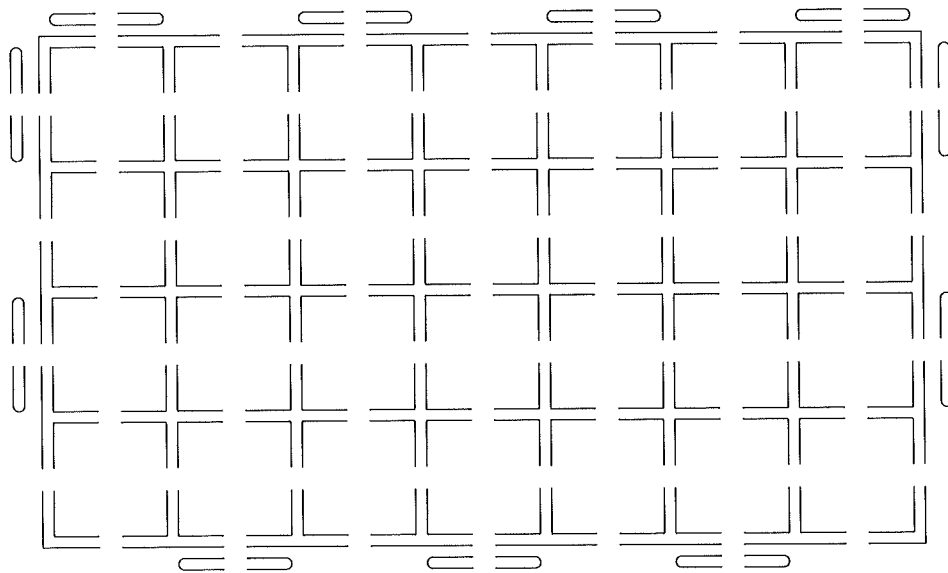


Figure 6-6:  $\Phi$ -model: Lowest order contribution to the Wilson loop in the strong coupling expansion

each link-index attached to a site needs to be traversed at least twice in opposite directions, if it is to contribute to the integral.

We can now compute the lowest order contribution to the string tension in the strong coupling expansion. For this we need to determine the lowest order contribution to the Wilson loop. Counting powers of  $\beta$ , the first non-vanishing contribution to this expectation value is the graph which tiles all the plaquettes inside of a Wilson loop. This is shown in figure 6-6. Notice that a non-zero contribution requires an

even number of  $\Phi$ -variables at each lattice site. This requires a certain number of terms that are proportional to  $\lambda$  along the edges of the Wilson loop. Such terms do not contribute to the string tension, since their presence is merely an edge effect, negligible compared to area effects. Let us therefore determine those factors of  $W(\mathcal{C}_{R,T})$  that go with the area of the Wilson loop. First, for each plaquette, there is a factor of  $(-\beta JN^4/2^4)$  from the action. A Wilson loop of dimension  $R \times T$  is tiled by  $RT$  plaquettes. Each internal lattice site contributes an integration

$$\frac{\begin{array}{|c|} \hline \text{---} \\ \hline \end{array}}{\begin{array}{|c|} \hline \text{---} \\ \hline \end{array}} = -\frac{2dM}{((2dM)^2 - 1)((2dM)^2 - 9)} \quad (6.54)$$

from table 6.1. There are a total of  $(R-1)(T-1)$  internal lattice sites. Finally, there is a factor of  $M$  for each link belonging to the interior of the Wilson loop. This is due to the contraction of indices induced by Kronecker  $\delta$ s on the two ends of a link,

$$\frac{\delta^{\alpha\alpha'}}{\begin{array}{|c|} \hline \alpha \uparrow \downarrow \alpha' \\ \hline \end{array}} \delta^{\alpha\alpha'} \quad (6.55)$$

There are altogether  $2RT - R - T$  such links on the interior of the loop. The part of the Wilson loop that grows with the area is therefore

$$W(\mathcal{C}_{R,T}) \sim \left( \frac{\beta JN^4}{2^4} \frac{2dM}{((2dM)^2 - 1)((2dM)^2 - 9)} M^2 \right)^{RT}. \quad (6.56)$$

Hence, the string tension is given by

$$\begin{aligned} \sigma &= - \lim_{R,T \rightarrow \infty} \frac{1}{RT} \log W(\mathcal{C}_{R,T}) \\ &= - \log \left( \frac{\beta JN^4}{2^4} \frac{2dM}{((2dM)^2 - 1)((2dM)^2 - 9)} M^2 \right). \end{aligned} \quad (6.57)$$

Setting  $M = N$  and taking  $N$  large, where  $JN^3$  is assumed to be fixed (see chapter

5), we find that the string tension approaches the finite limit

$$\sigma \rightarrow -\log \left( \frac{\beta(JN^3)}{2^4(2d)^3} \right). \quad (6.58)$$

This result is different from the one obtained for Wilson's theory in eq.(6.16). But of course, the quantum link model is not the same model as Wilson's lattice gauge theory, and the two are only agree in the continuum limit. Unfortunately, it is unknown how to extrapolate strong coupling results to the continuum limit for any theory, since the strong coupling expansion makes maximum use of the lattice.

## 6.5 The $\Phi$ -Model Formulation of the Principal Chiral Model

In chapter 4 of this thesis we formulated the principal chiral model as a quantum spin model with  $SU(N) \times SU(N) \times U(1)$  symmetry. The ideas developed in that chapter were then adapted to the discussion of quantum link models in chapter 5. In the present chapter we found that quantum link models can be re-expressed in terms of a colour singlet field, the  $\Phi$  field. To round things off, we show in this section, that the D-theory formulation of the principal chiral model can also be written as a  $\Phi$ -model with rather peculiar properties.

For simplicity, let us consider a  $U(N) \times U(N)$  symmetric quantum spin model, which is the same model as the one constructed in chapter 4, except that there are no tracelessness constraints for the  $\hat{L}^{ij}$  and  $\hat{R}^{ij}$  operators. The D-theory Hamiltonian of the principal chiral model, expressed in terms of rishons, is

$$\begin{aligned} \hat{H} &= \sum_{x,\mu} \left( \hat{\ell}_x^{i\alpha\dagger} \hat{r}_x^{j\alpha} \hat{r}_{x+\hat{\mu}}^{j\beta\dagger} \hat{\ell}_{x+\hat{\mu}}^{i\beta} + \hat{\ell}_{x+\hat{\mu}}^{i\alpha\dagger} \hat{r}_{x+\hat{\mu}}^{j\alpha} \hat{r}_x^{j\beta\dagger} \hat{\ell}_x^{i\beta} \right) \\ &= -J \sum_{x,\mu} \left( (\hat{\ell}_x^{i\alpha\dagger} \hat{\ell}_{x+\hat{\mu}}^{i\beta}) (\hat{r}_{x+\hat{\mu}}^{j\beta\dagger} \hat{r}_x^{j\alpha}) + (\hat{\ell}_{x+\hat{\mu}}^{i\alpha\dagger} \hat{\ell}_x^{i\beta}) (\hat{r}_x^{j\beta\dagger} \hat{r}_{x+\hat{\mu}}^{j\alpha}) \right). \end{aligned} \quad (6.59)$$

We have grouped together colour singlet expressions, for which we introduce the

following notation,

$$\widehat{\Phi}_{x,y}^{\alpha\beta} \equiv \widehat{\ell}_x^{i\alpha\dagger} \widehat{\ell}_y^{i\beta} - \frac{N}{2} \delta_{xy} \delta^{\alpha\beta}, \quad \widehat{\Psi}_{x,y}^{\alpha\beta} \equiv \widehat{r}_x^{i\alpha\dagger} \widehat{r}_y^{i\beta} - \frac{N}{2} \delta_{xy} \delta^{\alpha\beta}. \quad (6.60)$$

These new colour-singlet operators satisfy the commutation relations

$$\begin{aligned} [\widehat{\Phi}_{xy}^{\alpha\beta}, \widehat{\Phi}_{vw}^{\gamma\delta}] &= \delta_{yv} \delta^{\beta\gamma} \widehat{\Phi}_{xw}^{\alpha\delta} - \delta_{wx} \delta^{\alpha\delta} \widehat{\Phi}_{vy}^{\gamma\beta}, & [\widehat{\Phi}_{xy}^{\alpha\beta}, \widehat{\Psi}_{vw}^{\gamma\delta}] &= 0, \\ [\widehat{\Psi}_{xy}^{\alpha\beta}, \widehat{\Psi}_{vw}^{\gamma\delta}] &= \delta_{yv} \delta^{\beta\gamma} \widehat{\Psi}_{xw}^{\alpha\delta} - \delta_{wx} \delta^{\alpha\delta} \widehat{\Psi}_{vy}^{\gamma\beta}. \end{aligned} \quad (6.61)$$

The operators  $\widehat{\Phi}$  and  $\widehat{\Psi}$  are therefore the generators of a  $U(MV) \times U(MV)$  algebra, where  $V$  is the total number of lattice sites and  $M$  is the number of rishon flavours.

We rewrite the Hamilton operator as

$$\widehat{H} = -J \sum_{x,\mu} \left( \widehat{\Phi}_{x,x+\mu}^{\alpha\beta} \widehat{\Psi}_{x+\mu,x}^{\beta\alpha} + \widehat{\Phi}_{x+\mu,x}^{\alpha\beta} \widehat{\Psi}_{x,x+\mu}^{\beta\alpha} \right). \quad (6.62)$$

Notice, that the global  $U(N) \times U(N)$  symmetry is all but hidden in the  $\widehat{\Phi}$  and  $\widehat{\Psi}$  operators. However, this Hamiltonian has a manifest local  $U(M)$  symmetry, generated by

$$\widehat{\mathcal{G}}_x^{\alpha\beta} \equiv \widehat{\Phi}_{xx}^{\alpha\beta} + \widehat{\Psi}_{xx}^{\alpha\beta}. \quad (6.63)$$

The operators  $\widehat{\mathcal{G}}_x$  commute with the Hamiltonian for any  $x$ . This is the property that allowed us to choose the representation for the  $U(2N)$  quantum spin algebra at each lattice site, by imposing Gauss' law for this  $U(M)$  symmetry (cf. figure 3-1),

$$\begin{aligned} \sum_i \left( \widehat{\ell}_x^{i\alpha\dagger} \widehat{\ell}_x^{i\beta} + \widehat{r}_x^{i\alpha\dagger} \widehat{r}_x^{i\beta} \right) &= N \delta^{\alpha\beta} \\ \Leftrightarrow \widehat{\Phi}_{xx}^{\alpha\beta} + \widehat{\Psi}_{xx}^{\alpha\beta} &= 0. \end{aligned} \quad (6.64)$$

As in the case of the quantum link model, we soften this constraint when we consider the strong coupling expansion of this model. We can choose a representation for the  $U(MV) \times U(MV)$  algebra of the  $\widehat{\Phi}$  and  $\widehat{\Psi}$  generators by imposing the analogue of



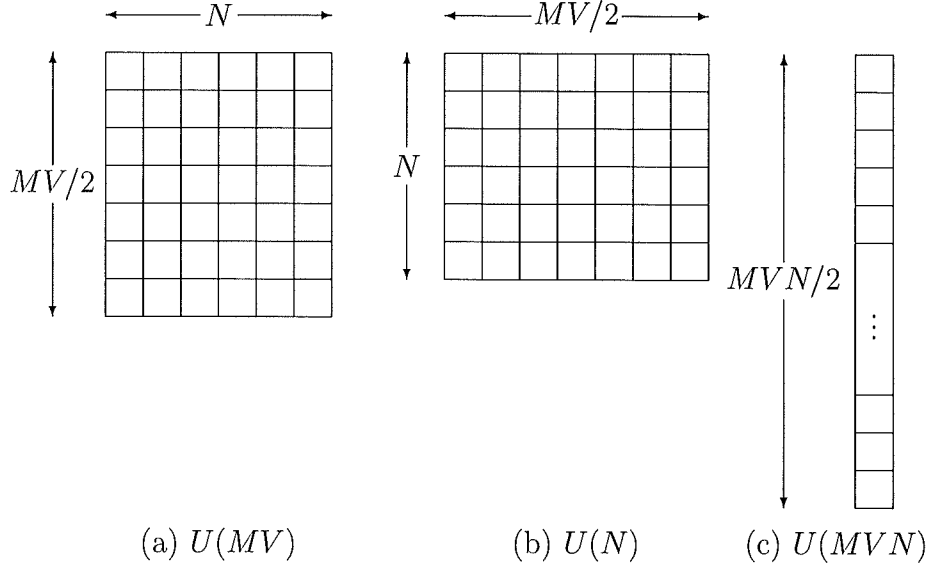


Figure 6-7: Choosing a representation for the  $U(MV)$  algebra of  $\hat{\Phi}$ -operators.

Gauss' law for the  $U(N) \times U(N)$  symmetry of the principal chiral model,

$$\begin{aligned} \hat{L}^{ij} &= \sum_x \hat{\ell}_x^{i\alpha\dagger} \hat{\ell}_x^{j\alpha} = \frac{MV}{2} \delta^{ij}, \\ \hat{R}^{ij} &= \sum_x \hat{r}_x^{i\alpha\dagger} \hat{r}_x^{j\alpha} = \frac{MV}{2} \delta^{ij}, \end{aligned} \tag{6.65}$$

where we assume that  $MV$  is an even number for simplicity<sup>1</sup>. By the same mechanism as the one explained in chapter 3, this fixes the representation for the  $U(MV) \times U(MV)$  algebra to be the one corresponding to a rectangular Young tableau with  $MV/2$  rows and  $N$  columns.

Using coherent states, we can set up a Euclidean path integral as described in appendix B,

$$\int d\Phi d\Psi \exp(-S). \tag{6.66}$$

Interestingly, since for all lattice sites  $x$  and  $y$ , the  $\hat{\Phi}_{x,y}$  and  $\hat{\Psi}_{x,y}$  operators are generators of a single  $U(MV) \times U(MV)$  algebra, the degrees of freedom in the path integral

---

<sup>1</sup>Instead of  $MV/2$  we could choose any integer between 1 and  $MV - 1$  in eq.(6.65). However, the resulting coset spaces for  $\Phi$  and  $\Psi$  would be different in each case, and so would the integrals over elements of the coset spaces. With our assumption, we can make use of the results of appendix C.

are two big matrices  $\Phi$  and  $\Psi$ , each of size  $MV \times MV$ . The individual elements  $\Phi_{x,y}^{\alpha\beta}$  and  $\Psi_{x,y}^{\alpha\beta}$  are all elements of these two matrices. The action in (6.66) is given by

$$S(\Phi, \Psi) = S_B + \int_0^\beta d\tau \mathcal{H}(\Phi, \Psi). \quad (6.67)$$

Here,  $S_B$  is a Berry phase term,

$$S_B = -\frac{N}{4} \int_0^\beta d\tau \int_0^1 dv \left[ \text{Tr} \left( \Phi(\tau, v) \frac{\partial \Phi(\tau, v)}{\partial v} \frac{\partial \Phi(\tau, v)}{\partial \tau} \right) + \text{Tr} \left( \Psi(\tau, v) \frac{\partial \Psi(\tau, v)}{\partial v} \frac{\partial \Psi(\tau, v)}{\partial \tau} \right) \right]. \quad (6.68)$$

The other term in the action is obtained from the Hamilton operator by replacing every occurrence of  $\widehat{\Phi}_{x,y}^{\alpha\beta}$  with  $(N/2)\Phi_{x,y}^{\alpha\beta}$ , and each  $\widehat{\Psi}_{x,y}^{\alpha\beta}$  with  $(N/2)\Psi_{x,y}^{\alpha\beta}$ ,

$$\mathcal{H}(\Phi) = -\frac{JN^2}{4} \sum_{x,\mu} \text{Tr} (\Phi_{x,x+\hat{\mu}} \Psi_{x+\hat{\mu},x} + \Phi_{x+\hat{\mu},x} \Psi_{x,x+\hat{\mu}}). \quad (6.69)$$

We showed in chapter 4 that the  $SU(N) \times SU(N) \times U(1)$  quantum spin model undergoes dimensional reduction in the limit as the extent  $\beta$  of the third dimension is taken large. In the present model, as in the  $\Phi$ -model formulation of gauge theory, we do not understand the theory well enough to argue for dimensional reduction on dynamical grounds. However, assuming that the physics is the same as in the quantum spin model, dimensional reduction does occur and the fields are effectively constant in the third direction. Hence, the Berry phase term vanishes, and we can perform the trivial integration over  $\tau$ . In eqs.(6.41–6.42) we also gave an argument why the dimensionally reduced theory is in fact a valid description in the strong coupling expansion of  $\Phi$ -models.

In order to implement the constraint (6.64), we add an additional term to the action,

$$\mathcal{H}(\Phi) = -\frac{JN^2}{4} \sum_{x,\mu} \text{Tr} (\Phi_{x,x+\hat{\mu}} \Psi_{x+\hat{\mu},x} + \Phi_{x+\hat{\mu},x} \Psi_{x,x+\hat{\mu}}) + \lambda \sum_x \text{Tr} (\Phi_{x,x} + \Psi_{x,x})^2. \quad (6.70)$$

For  $\lambda = \infty$ , the constraint (6.64) is exact, but we shall only approximate it by allowing  $\lambda$  to be finite in the following discussion.

To examine the strong coupling expansion, we write the integrand of the partition function as

$$\exp(-S) = \left( \prod_{\underline{-}} \exp(-\beta S_{\underline{-}}) \right) \left( \prod_{\bullet} \exp(-\beta S_{\bullet}) \right). \quad (6.71)$$

Here,  $\underline{-}$  denotes lattice links and  $\bullet$  denotes lattice sites. In particular,

$$\begin{aligned} S_{\underline{-}} &= -\frac{JN^2}{4} \text{Tr} (\Phi_{x,x+\hat{\mu}} \Psi_{x+\hat{\mu},x} + \Phi_{x+\hat{\mu},x} \Psi_{x,x+\hat{\mu}}), \\ S_{\bullet} &= \frac{\lambda N^2}{4} \text{Tr} (\Phi_{x,x} + \Psi_{x,x})^2. \end{aligned} \quad (6.72)$$

Since  $\Phi$  and  $\Psi$  are bounded variables,  $\exp(-\beta S_{\underline{-}}) \rightarrow 1$  and  $\exp(-\beta S_{\bullet}) \rightarrow 1$ , both uniformly in the limit as  $\beta \rightarrow 0$ . Hence, we can expand the partition function in a power series in  $\beta$ . Likewise, we can expand expectation values of observables in powers of  $\beta$  and calculate leading order contributions. Let us set up the calculation of spin-spin correlation functions in the strong coupling limit. We will find that we won't be able to complete this calculation, because we have so far not been able to find a formula for integrals over an arbitrary number of  $\Phi$  and  $\Psi$ -variables. But at least we can extract the leading behaviour of correlation functions with the coupling  $\beta J$ . Consider then

$$\begin{aligned} \langle \widehat{U}_x^{ij} \widehat{U}_y^{ij\dagger} \rangle &= - \langle \widehat{\Phi}_{xy}^{\alpha\beta} \widehat{\Psi}_{yx}^{\beta\alpha} \rangle \\ &= \frac{N^2}{4} \frac{1}{Z} \int d\Phi d\Psi \Phi_{xy}^{\alpha\beta} \Psi_{yx}^{\beta\alpha} \exp(-S). \end{aligned} \quad (6.73)$$

To a first approximation, the partition function is  $Z = 1$ , since the integrals over  $\Phi$  and  $\Psi$  are suitably normalised. We can now expand the exponential in eq.(6.73) in powers of  $\beta$  until we get the first non-zero contribution. As discussed in detail in appendix C, integrals over an odd number of  $\Phi$ s or an odd number of  $\Psi$ s vanish. Furthermore, the measure is invariant under unitary transformations,  $d\Phi \rightarrow d(\mathcal{G}^\dagger \Phi \mathcal{G})$  (and similarly for  $\Psi$ ), so that the integrand must contain the trivial representation of

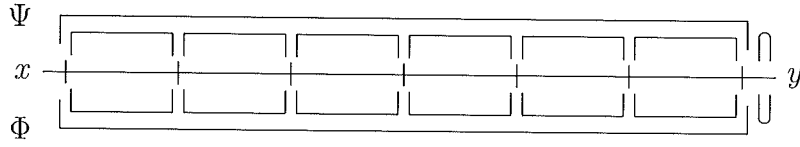


Figure 6-8: Lowest order contribution to the spin-spin correlation function in the strong coupling expansion.

$U(MV)$  in order to be non-zero. For simplicity, let us take  $x$  and  $y$  to be on the same lattice axis, and  $|y - x| = R$ , say. Then in order to get a non-zero contribution to the expectation value in eq.(6.73) we need to include  $R$  link terms from the action, as indicated in figure 6-8. Notice further, that in order to get an even number of  $\Phi$  and  $\Psi$  variables, we may have to include a term from the action which is proportional to  $\lambda$ , depending on whether  $R$  is even or odd. As mentioned before, we cannot evaluate the necessary integrals over  $\Phi$  and  $\Psi$  for arbitrarily large  $R$ , but the dependence of the correlation function on  $\beta J$  is given by

$$\langle \Phi_{xy}^{\alpha\beta} \Psi_{yx}^{\beta\alpha} \rangle \sim (J\beta)^R. \quad (6.74)$$

Thus, the associated mass is

$$m \sim -\log |J\beta| + \text{other terms}. \quad (6.75)$$

# Chapter 7

## Conclusions

*Sich allen Abend ernstlich zu befragen, was man an dem Tage Neues gelernt hat.*

G. C. Lichtenberg [46]

In the D-theory formulation of quantum field theories, a field Lagrangian is replaced by a Hamilton operator and continuous classical fields are replaced by operator fields. The Hamilton operator evolves the system in an additional Euclidean direction. Guided by symmetry considerations, we have formulated the principal chiral model as such a quantum spin system. We then went on to show that with a particular choice of representation for the operators in the Hamiltonian, the theory reduces to a Wilsonian lattice principal chiral model. From numerical simulations we know that the  $SU(N)_L \times SU(N)_R \times U(1)_{L=R}$  symmetry of such a model breaks spontaneously to  $SU(N)_{L=R}$  at  $\beta = \infty$  [28, 29]. We chose representations with rectangular Young tableaux, with  $N$  rows and  $M$  columns, where  $M$  was taken to be large. The Goldstone modes arising from the spontaneous symmetry breaking cause the system to undergo dimensional reduction when we make the extent of the third dimension finite and we thus recover the 2-d principal chiral model. We have seen that the continuous degrees of freedom of the low-energy effective theory, which is the same as the standard formulation of the principal chiral model, arise as collective excitations of the discrete degrees of freedom in the D-theory formulation of the model.

We also showed that the quantum link model in  $(4 + 1)$ -d undergoes dimensional reduction to 4-d Yang Mills theory. We chose the quantum link operators to be in the same type of representation that we considered for the operators in the principal chiral model, namely large representations with rectangular Young tableaux. The mechanism for dimensional reduction is different in this case. Instead of Goldstone modes arising from a spontaneously broken global symmetry, the massless modes we need for dimensional reduction result from the fact that a  $(4 + 1)$ -d gauge theory can exist in a non-Abelian Coulomb phase. We showed that for the aforementioned representations the low-energy effective theory of the D-theory is a 5-d Wilson-type lattice gauge theory. It is known from numerical simulations that such a theory is indeed in the non-Abelian Coulomb phase when the extent of the fifth dimension is infinite [13, 14]. At finite temperature the gauge bosons form glueballs and acquire mass, due to the confinement hypothesis. The correlation length, however, is exponential in the extent of the fifth dimension, hence leading to dimensional reduction. Again, the continuous fields of the low-energy effective theory after dimensional reduction arise as collective excitations of discrete variables.

In order to be able to get an analytic handle on the behavior of the D-theory formulations of the principal chiral model and non-Abelian gauge theory, we had to consider large representations for the quantum operators in the Hamiltonian. On the other hand, to develop more efficient algorithms for simulating such theories one would like to consider smaller representations, so that each variable can assume only a few discrete values. It is not clear at this point if the mechanism of dimensional reduction also occurs for small representations. Numerical studies are needed to answer this question.

While D-theory opens up the way to the development of cluster algorithms for simulating many important field theories, it also offers new analytical insight into such theories. In particular, we showed that by expressing quantum spin/link operators in terms of constituent rishons operators we could separate the colour indices of the quantum spins and quantum links. Colour indices could then be recombined to form colour neutral operators, which we called  $\widehat{\Phi}$ -operators. We thus transformed the

principal chiral model, which has a global  $U(N) \times U(N)$  symmetry, into a model with a local  $U(M)$  symmetry. However, the operator algebra of the resulting  $\widehat{\Phi}$ -model is no longer local, but includes all operators across the entire lattice. They can be embedded in a  $U(MV) \times U(MV)$  algebra, where  $V$  is the number of lattice sites. We derived a Euclidean path integral for the partition function of this theory, where the degrees of freedom are no longer local fields, but the elements of two big matrices, each living in the coset space  $U(MV)/[U(MV/2) \times U(MV/2)]$ . We showed how to calculate correlation functions of this theory in a strong coupling expansion.

We also formulated the quantum link model as a  $\Phi$ -model, and found that it has a local  $U(M)$  symmetry as well. Hence, there is a duality transformation, which interchanges the roles played by colour and rishon flavour in the theory. The resulting  $\Phi$ -variables live on the lattice sites, while the gauge transformations live on the lattice links. This is just the opposite of usual gauge theories. We showed that the new  $\Phi$ -model confines both colour and rishon flavour in the strong coupling limit. Unfortunately, we found that in order to take the 't Hooft large  $N$  limit of this model, we had to let  $M \rightarrow \infty$  at the same time. However, we hope that interesting properties about gauge theories may be learned from  $\Phi$ -models at finite  $M$  and  $N$ . An important question is whether such models are asymptotically free in the weak coupling limit, and we will address this question in future research.

The  $\Phi$ -model idea can be extended to include colour neutral bosonic fields to describe the quark degrees of freedom. This means that the theory can be completely bosonised, leading to a model that does not suffer from sign problems, which usually arise in simulations of fermionic systems on the computer.

# Appendix A

## Some Facts About Compact Semi-Simple Lie Groups

The importance of symmetries in physics can hardly be underestimated. They are the primary tool for constructing and understanding physical models. While the theory of finite groups is rich in results and has many applications to physics, continuous groups, or Lie groups as they are usually called, have found their way into the standard tool-box of any particle physicist. In this thesis, we use many results for continuous groups. In order to make the discussion reasonably self-contained, in this appendix we put the most important of these results into context. The mathematical theory of Lie groups is explained in [47]. Nice introductions to Lie groups, Lie algebras and their representations, written primarily for physicists, are given in [48] and [49].

Finite-dimensional continuous groups are groups whose elements are parametrised by a finite number of continuous variables. Such a group is a finite-dimensional differentiable manifold. If this manifold is compact, then the group is a compact Lie group. For example, the group  $U(1)$  is the one-dimensional group of complex phase rotations. Its elements can be represented by complex phases,

$$g_Q(\alpha) = \exp(i\alpha Q), \quad \text{for } \alpha \in [0, 2\pi), \quad Q \in \mathbb{Z}. \quad (\text{A.1})$$

Clearly, the group manifold is isomorphic to the unit circle,  $S^1$ , which is compact. We



are most interested in the unitary and special unitary groups, which are all compact. The Lorentz group on the other hand is not. Only compact groups can be represented by finite-dimensional unitary matrices. A representation of a Lie group,  $G$ , is a homomorphism

$$\begin{aligned} T : G &\rightarrow \mathbb{C}^{n \times n} \\ g &\mapsto T(g) \end{aligned} \tag{A.2}$$

from the group to the space of  $n \times n$  complex matrices. The size  $n$  of the representation matrices is, in general, different for different representations of the same group. The statement that  $T$  is a homomorphism simply means that for two group elements  $g_1$  and  $g_2$ , being mapped to matrices  $T(g_1)$  and  $T(g_2)$  respectively, the following property holds,

$$T(g_1 g_2) = T(g_1) T(g_2). \tag{A.3}$$

Thus, the group multiplication table is preserved under homomorphisms. Two representations  $T$  and  $\tilde{T}$  are equivalent if there is a unitary transformation  $U$  which relates them,

$$T(g) = U \tilde{T}(g) U^\dagger, \quad \text{for all } g \in G. \tag{A.4}$$

Furthermore, if  $T : g \mapsto T(g)$  is a representation of the group  $G$ , then  $T^* : g \mapsto T^*(g)$  is also a representation of  $G$ , which maps each group element  $g$  to the complex conjugate of the matrix  $T(g)$ . This is called the conjugate representation of  $T$ . If a representation is equivalent to its conjugate, then this representation is called real.

## A.1 The Group Algebra

A Lie algebra  $\mathfrak{G}$  over  $\mathbb{R}$  is a real vector space  $\mathfrak{G}$  which also has a multiplication operation  $\mathfrak{G} \times \mathfrak{G} \rightarrow \mathfrak{G}$ . This is a bilinear operation called the Lie bracket and

denoted by  $[Y_1, Y_2]$ . It satisfies the properties

$$\begin{aligned} [Y_1, Y_2] &= -[Y_2, Y_1] && \text{(anti-commutativity)} \\ [[Y_1, Y_2], Y_3] + [[Y_2, Y_3], Y_1] + [[Y_3, Y_1], Y_2] &= 0 && \text{(Jacobi identity)}. \end{aligned} \quad (\text{A.5})$$

We are mostly concerned with matrix algebras. In this case, if  $A$  and  $B$  are two elements of the Lie algebra, then their Lie bracket is simply  $[A, B] = AB - BA$ . A Lie algebra is called Abelian, if all of its Lie brackets vanish. A Lie algebra is completely determined once the Lie brackets of its basis elements are known. Denoting a basis for the Lie algebra  $\mathfrak{G}$  by  $\{X^a\}$ , we have

$$[X^a, X^b] = f^{abc} X^c. \quad (\text{A.6})$$

The constants  $f^{abc}$  are called structure constants. Of course, they depend on the choice of basis.

The importance of Lie algebras is due to the fact that to every Lie group one associates a certain finite-dimensional Lie algebra. The properties of the Lie group are intimately related to the properties of its Lie algebra. In particular, connected, simply-connected Lie groups are uniquely determined (up to isomorphisms) by their Lie algebras. The connection between the two is established by the exponential mapping, which is a generalisation of matrix exponentiation. In particular, we can write group elements which are sufficiently close to the identity as

$$g \approx \mathbf{1} + \alpha^a X^a + \mathcal{O}(\alpha^2). \quad (\text{A.7})$$

Here, the  $\alpha^a$  are presumed small.

A subalgebra of the Lie algebra  $\mathfrak{G}$  is a subspace of  $\mathfrak{G}$ , which is closed under the Lie bracket. Such a subalgebra is called invariant, if for every  $Y^a$  in the subalgebra, and every  $X^b \in \mathfrak{G}$ , there exists a  $Y^c$  also in the subalgebra, such that

$$[Y^a, X^b] = Y^c. \quad (\text{A.8})$$

A Lie algebra is called simple, if it is not Abelian, and if it does not have a proper invariant Lie subalgebra. A Lie group is called simple, if its real Lie algebra is simple. A Lie group is called semi-simple, if it can be decomposed into the direct product of simple groups. It is the compact semi-simple Lie algebras, which are most useful for applications to physics, and from now on all results quoted are true for all compact semi-simple Lie groups and algebras.

A subalgebra is called Abelian if all Lie brackets of elements of the subalgebra vanish. The maximal Abelian subalgebra of a Lie algebra  $\mathfrak{G}$  is called the Cartan subalgebra of  $\mathfrak{G}$ , denoted by  $\mathfrak{H}$ . It corresponds to the maximal Abelian semi-simple subgroup  $H$  of  $G$ , called the Cartan subgroup. For any compact semi-simple Lie algebra, we can always choose a specific basis called the Cartan basis. We start with a particular basis for the Cartan subalgebra,  $\{H_j\}$  say, and extend it to a basis of the full algebra  $\mathfrak{G}$  in such a way that the basis elements satisfy the following commutation relations,

$$\begin{aligned} [H_j, H_k] &= 0, & [H_j, E_\alpha] &= \alpha_j E_\alpha, & [E_\alpha, E_{-\alpha}] &= \alpha_j H_j, \\ [E_\alpha, E_\beta] &= N_{\alpha\beta} E_{\alpha+\beta}, & & \text{if } \alpha + \beta \in R, \\ [E_\alpha, E_\beta] &= 0, & & \text{if } \alpha + \beta \notin R \text{ and } \alpha + \beta \neq 0. \end{aligned} \tag{A.9}$$

The dimension of the Cartan subalgebra is called the rank,  $r$ , of the Lie algebra. In eq.(A.9),  $R$  is the root system of the Lie algebra  $\mathfrak{G}$ . This is a set of vectors in  $\mathbb{R}^n$ , denoted by  $\{\alpha = (\alpha_1, \dots, \alpha_r)\}$ , that label those basis elements which are not elements of the Cartan subalgebra. The  $N_{\alpha\beta}$  are simple normalisation constants.

## A.2 Representations

As mentioned above, a representation of a group  $G$  is a homomorphism from  $G$  to  $\mathbb{C}^{n \times n}$ , i.e. a map that preserves the group multiplication table. We can also talk about representations of Lie algebras  $\mathfrak{G}$ . Such a representation is a map from  $\mathfrak{G}$  to  $\mathbb{C}^{n \times n}$ , which preserves the Lie bracket. The matrices of a given representation

act as operators in the vector space  $\mathbb{C}^n$ , which is called the carrier space of the representation. For a given representation it is possible, that the carrier space has a number of proper invariant subspaces, i.e. different subspaces, which do not get mixed under the action of all representation matrices on the carrier space. In this case, we can find a similarity transformation which transforms all representation matrices into matrices that all exhibit the same block-diagonal structure. If such invariant subspaces exist, we say that the representation is reducible. If no invariant subspaces exist, the representation is called irreducible.

While the root system associated with the Cartan basis of a particular Lie algebra  $\mathfrak{G}$  specifies the algebra, there is a similar approach towards classifying all irreducible unitary representations of a simple compact Lie algebra  $\mathfrak{G}$ . Any such representation can be characterized by its highest weight vector,  $\lambda = (\lambda_1, \dots, \lambda_r)$ . This is a vector in  $\mathbb{R}^r$ , where  $r$  is of course the rank of the algebra. So to denote a particular representation characterized by  $\lambda$ , we may write  $T(g) = T^\lambda(g)$ . In the carrier space  $\mathcal{H}^\lambda$ , a corresponding highest weight vector  $|\lambda\rangle$  exists, which satisfies

$$T^\lambda(E_\alpha) |\lambda\rangle = 0, \text{ for } \alpha \in R, \quad T^\lambda(H_j) |\lambda\rangle = \lambda_j |\lambda\rangle. \quad (\text{A.10})$$

Here,  $T^\lambda(E_\alpha)$  and  $T^\lambda(H_j)$  are operators in  $\mathcal{H}^\lambda$  representing elements of the Lie algebra  $\mathfrak{G}$ . Furthermore, we can find a basis  $\{|\mu\rangle\}$  for the carrier space  $\mathcal{H}^\lambda$ , whose elements are eigenvectors of the matrices representing the  $H_j$  operators,

$$T^\lambda(H_j) |\mu\rangle = \mu_j |\mu\rangle. \quad (\text{A.11})$$

These basis vectors are called weight vectors. They generate a system of vectors, that determine the representation in a way analogous to the root system used to specify Lie algebras. The action of the other representation matrices of Lie algebra generators

on weight vectors is given by

$$\begin{aligned} T^\lambda(E_\alpha) |\mu\rangle &= N_\mu^\alpha |\mu + \alpha\rangle, \quad \text{if } \mu + \alpha \in \mathcal{W}, \\ T^\lambda(E_\alpha) |\mu\rangle &= 0, \quad \text{if } \mu + \alpha \notin \mathcal{W}. \end{aligned} \tag{A.12}$$

Here,  $\mathcal{W}$  is the system of weight vectors of the representation, and the  $\alpha$  are the same root vectors mentioned in eq.(A.9). The  $N_\mu^\alpha$  are normalisation constants.

Another way of classifying representations is by the value of the corresponding Casimir operators. In general, for an algebra of rank  $r$ , there is a set of at most  $r$  independent operators, which commute with all elements of the algebra. These operators, called Casimir operators, are themselves not elements of the algebra. Although general formulae exist for calculating general Casimir operators (see [50] for an overview of the subject), this can be a very tedious task, and in practice the most important one is the quadratic Casimir operator,

$$C_2 = \sum_a X^a X^a \tag{A.13}$$

where  $\{X^a\}$  is a basis for the Lie algebra. A result of utmost importance is Shur's lemma, which states that any operator which commutes with all matrices of an irreducible representation is proportional to the identity operator. So Casimir operators are proportional to the identity, and the constant of proportionality is what determines the representation. Representations of the group  $SU(2)$  for example are completely determined just by the value of the quadratic Casimir operator, which in this case is takes on values  $S(S + 1)$ , where  $S$  is just the spin of the representation. In other words, for  $SU(2)$ , representations are labeled by the spin  $S$ .

### A.3 $SU(N)$

Let us now consider the group  $SU(N)$ . The properties of this Lie group, and its algebra can be studied by investigating the properties of a concrete set of matrices, which

define the group structure of  $SU(N)$ , and are therefore called the defining representation. The group elements are  $N \times N$  complex unitary matrices with determinant equal to one;  $U^\dagger U = \mathbf{1}$  and  $\det U = 1$ . Such matrices are obtained by exponentiating anti-hermitian, traceless  $N \times N$  complex matrices, which defines the space of the corresponding Lie algebra.

To construct a basis for the Lie algebra, consider the  $N \times N$  matrices  $\{S^{ij} | i, j = 1, \dots, N\}$  whose elements are given by

$$[S^{ij}]^{\alpha\beta} = \delta^{i\alpha} \delta^{j\beta}. \quad (\text{A.14})$$

Thus, the matrix  $S^{ij}$  has a 1 in position  $(ij)$ , and zeros elsewhere. The commutation relations are

$$[S^{ij}, S^{k\ell}] = \delta^{kj} S^{i\ell} - \delta^{\ell i} S^{kj}. \quad (\text{A.15})$$

From these, we can construct a basis, first for the Cartan subalgebra,

$$H_i = S^{ii} - S^{NN}, \text{ for } i = 1, \dots, N - 1. \quad (\text{A.16})$$

There are  $N - 1$  independent such matrices, so  $SU(N)$  has rank  $N - 1$ . We extend these basis vectors to a basis of the full Lie algebra by including the matrices

$$\begin{aligned} E_R^{ij} &= S^{ij} - S^{ji}, \quad i < j, \\ E_I^{ij} &= i(S^{ij} + S^{ji}), \quad i < j. \end{aligned} \quad (\text{A.17})$$

There are  $(N - 1)N/2$  matrices of type  $E_R^{ij}$  and as many matrices of type  $E_I^{ij}$ . Hence, adding the number of basis elements of the  $SU(N)$  algebra, we find that there are  $2(N - 1)N/2 + N - 1 = N^2 - 1$  generators altogether.

There are other popular bases for  $SU(N)$  Lie algebras. In particular, it is common to choose a basis  $\{\lambda^a\}$ ,  $a = 1, \dots, N^2 - 1$  so that in a representation  $T^\mu$ ,

$$\text{Tr} (T^\mu(\lambda^a) T^\mu(\lambda^b)) = C_\mu \delta^{ab}. \quad (\text{A.18})$$

The constant of proportionality  $C_\mu$  depends on the representation. The structure constants are then given by

$$f^{abc} = \frac{1}{C_\mu} \text{Tr} ([T^\mu(\lambda^a), T^\mu(\lambda^b)] T^\mu(\lambda^c)). \quad (\text{A.19})$$

Hence, the  $f^{abc}$  are totally antisymmetric.

There is a very nice way of enumerating the irreducible representations of  $SU(N)$  and obtaining the weights of the generators, using Young graphs and Young tableaux. In this paragraph, we distinguish between Young graphs and tableaux for clarity, but throughout other parts of this thesis, any graph will be called a Young tableaux. The method is based on properties of the permutation groups, but here we only summarize those results that are important in the present context of representations of  $SU(N)$ . A Young graph is a graph consisting of a certain number of boxes, aligned into rows and columns, where the rows are flush aligned on the left of the graph. Furthermore, there are never more boxes in a given row than in the one right above it. An example of a Young graph is given in figure A-1(a). A Young graph with  $n$  boxes corresponds to an irreducible representation of  $SU(N)$  whose carrier space is described by rank  $n$  tensors. Let us label the boxes of the Young tableau by  $\ell$ , so that  $\ell$  runs from 1 to  $n$ , the number of boxes. The label  $\ell$  increases from left to right along the first row, then from left to right along the second row and so on. If the  $\ell$ th and  $\ell'$ th index of the rank  $n$  tensor correspond to boxes in the same row of the Young graph, then the tensors subtending the representation are symmetric under exchange of these two indices. If they correspond to boxes in the same column, then the tensors are anti-symmetric in these two indices. We can determine the dimension of such a representation by counting the number of permissible Young tableaux that can be obtained from a particular Young graph. A Young tableaux is obtained from a graph, by inserting the numbers 1 to  $N$  into the boxes of the graph, as shown in figure A-1(b). For such a tableau to be permissible, the numbers in a row must be non-decreasing when read from left to right and they must be strictly increasing when read from top to bottom along a column. For example, figure A-1(b) is a permissible tableau for

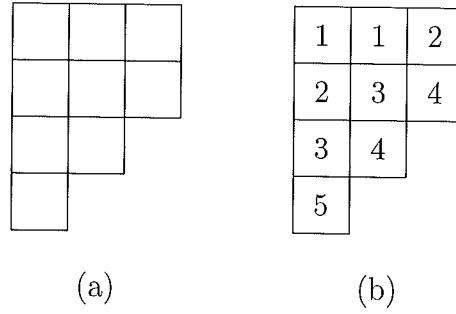


Figure A-1: Young graphs and permissible Young tableaux

$SU(5)$ . So each permissible tableau corresponds to a particular basis element of the representation, or weight vector as we called it before. The corresponding weights can be obtained as follows. Denote by  $i_\ell$  the number in the  $\ell$ th box of the Young tableau. Then the weight of the permissible tableau is given in terms of the fundamental weights  $\{\mu^j | j = 1, \dots, N - 1\}$  as

$$m_{i_1, \dots, i_n} = \sum_{\ell=1}^n \mu^{i_\ell}. \quad (\text{A.20})$$

Remember, that the  $\mu^j$ , are  $(N - 1)$ -component weight vectors, and so the sum is a sum of vectors.



# Appendix B

## Coherent States

### B.1 Generalised Coherent States

Following references [20] and [35], let us review the definition and properties of systems of coherent states. These results can then be used to derive coherent state path integrals for the quantum spin and quantum link models of this thesis. We start with an arbitrary Lie group  $G$  and a unitary irreducible representation of the group,  $T(g)$ , which acts in the Hilbert space  $\mathcal{H}$ . Let  $|\psi_0\rangle$  be a fixed vector in  $\mathcal{H}$ . We now consider the properties of the set of vectors obtained by acting on  $|\psi_0\rangle$  with all representation matrices. We denote the elements of this set by  $\{|\psi_g\rangle\}$ , where  $|\psi_g\rangle = T(g)|\psi_0\rangle$ ,  $g \in G$ .

Two such vectors correspond to the same physical state if they differ only by a phase, i.e. if  $|\psi_{g_1}\rangle = e^{i\alpha} |\psi_{g_2}\rangle$ , and  $|e^{i\alpha}| = 1$ . We can make use of the homomorphism property of  $T(g)$  and write this condition as

$$T(g_2^{-1}g_1) |\psi_0\rangle = e^{i\alpha} |\psi_0\rangle. \quad (\text{B.1})$$

Thus, the action of the group element  $g_2^{-1}g_1$  on  $|\psi_0\rangle$  does not change this state. Consider then the set of all such group elements,  $h$ , that do not change the initial state  $|\psi_0\rangle$ ,

$$T(h) |\psi_0\rangle = e^{i\alpha(h)} |\psi_0\rangle, \quad (\text{B.2})$$

From the elements of this set, we can form a subgroup of  $G$ , call it  $H$ . When  $H$  is maximal, it is called the *isotropy group* for the state  $|\psi_0\rangle$ . The whole group  $G$  can thus be divided up into left coset classes with respect to this new group  $H$ . Given a particular coset class,  $x$  say, we see that for all  $g \in x$ , the corresponding vectors  $|\psi_g\rangle$  only differ by a complex phase and therefore determine the same state. If we now choose a representative  $g(x)$  from each equivalence class  $x$ , we obtain a set of states  $\{|\psi_{g(x)}\rangle\}$ , where  $x \in X = G/H$ . Let us denote this set by  $\{|x\rangle\}$ . Using these concepts, we can make the following

**Definition 1** *The system of states  $\{|\psi_g\rangle\}$ ,  $|\psi_g\rangle = T(g)|\psi_0\rangle$ , where  $g \in G$ , is called a coherent state system  $\{T, |\psi_0\rangle\}$ .*

*Let  $H$  be the isotropy subgroup for the state  $|\psi_0\rangle$ . Then a coherent state is determined by a point  $x \in G/H$ , such that if  $x(g) = x$  for some group element  $g$ , then  $|\psi_g\rangle = e^{i\alpha} |x\rangle$ . In this notation,  $|\psi_0\rangle = |0\rangle$ .*

Having defined a system of coherent states, we explore some of its properties and, most importantly, we derive a completeness relation that can be used to construct a path integral.

We start by examining the subgroup  $H$ , which plays an important role. Consider two elements of  $H$ ,  $h_1$  and  $h_2$ . Combining the fact that  $T(h_j)|\psi_0\rangle = \exp(i\alpha(h_j))|\psi_0\rangle$ ,  $j = 1, 2$ , and making use of the homomorphism property of  $T(g)$ , we find that

$$\exp(i\alpha(h_2)) \exp(i\alpha(h_1)) = \exp(i\alpha(h_2h_1)). \quad (\text{B.3})$$

Hence,  $\exp(i\alpha(h))$  is the one-dimensional unitary representation (character) of the subgroup  $H$ .

More generally, if we let the operator  $T(g)$  act on the state  $|\psi_0\rangle = |0\rangle$ , we can write the result as

$$T(g)|0\rangle = e^{i\alpha(g)} |x(g)\rangle. \quad (\text{B.4})$$

The function  $\alpha(g)$  is now defined for any  $g \in G$ , while for  $g \in H$  it coincides with the

function  $\alpha(h)$  considered above. Replacing  $g$  in (B.4) with  $gh$  gives

$$\begin{aligned} T(gh)|0\rangle &= T(g)T(h)|0\rangle = e^{i\alpha(gh)}|x(gh)\rangle = e^{i\alpha(g)}e^{i\alpha(h)}|x(g)\rangle, \\ &\Leftrightarrow \alpha(gh) = \alpha(g) + \alpha(h). \end{aligned} \quad (\text{B.5})$$

The second line follows, because  $g$  and  $gh$  are in the same left coset class with respect to  $H$ , and so  $x(gh) = x(g)$ . Notice that in (B.5) it is important that  $h \in H$ . For arbitrary group elements  $g_1$  and  $g_2$  it is of course not true that  $\alpha(g_1g_2) = \alpha(g_1)\alpha(g_2)$ . If we let  $T(g_1)$  act on an arbitrary coherent state, we find that,

$$\begin{aligned} T(g_1)|x\rangle &= e^{-i\alpha(g)}T(g_1)T(g)|0\rangle = e^{-i\alpha(g)}e^{i\alpha(g_1g)}|g_1x\rangle = e^{i\beta(g_1,g)}|x_1\rangle, \\ &\text{where } \beta(g_1,g) = \alpha(g_1g) - \alpha(g). \end{aligned} \quad (\text{B.6})$$

Here,  $x = x(g)$  and  $g_1x = x_1 \in X$ . For (B.6) to be a well defined relation on the coset space  $X$ , it must not depend on the choice of  $g$ , but only on the equivalence class  $x(g)$ . In view of (B.5), this is indeed correct, and we can write

$$\beta(g_1,g) = \tilde{\beta}(g_1,x). \quad (\text{B.7})$$

Let us now look at the scalar product of two coherent states. With  $|x_1\rangle = |x(g_1)\rangle$  and  $|x_2\rangle = |x(g_2)\rangle$ , it is given by

$$\begin{aligned} \langle x_1|x_2\rangle &= \langle 0|T(g_1)^\dagger e^{i\alpha(g_1)}e^{-i\alpha(g_2)}T(g_2)|0\rangle \\ &= e^{i(\alpha(g_1)-\alpha(g_2))} \langle 0|T(g_1^{-1}g_2)|0\rangle, \end{aligned} \quad (\text{B.8})$$

and is independent of the choice of equivalence class members  $g_1$  and  $g_2$ , because of (B.5). Since the representation  $T(g)$  is unitary, the scalar product between two

coherent states has the following properties,

$$\begin{aligned}
|\langle x_1|x_2\rangle| &< 1 \text{ for } x_1 \neq x_2, \\
\langle x_1|x_2\rangle^* &= \langle x_2|x_1\rangle, \\
\langle gx_1|gx_2\rangle &= \exp\left[i\left(\tilde{\beta}(g, x_1) - \tilde{\beta}(g, x_2)\right)\right] \langle x_1|x_2\rangle.
\end{aligned} \tag{B.9}$$

**Completeness and Expansion in Coherent States Systems** The completeness property of coherent states systems is a direct consequence of the irreducibility of the representation  $T(g)$ , which allows us to apply Schur's Lemma to a certain operator, which we are about to introduce.

Suppose that for the group  $G$ , there exists a measure  $d\mu(g)$ , that is invariant under left and right shifts. For the unitary group  $U(N)$  for example, this is just the Haar measure (see appendix C). This will induce an invariant measure,  $dx$  say, in the homogeneous space  $X = G/H$ . Assuming that convergence conditions are satisfied, consider the operator

$$\hat{B} = \int dx |x\rangle \langle x| \tag{B.10}$$

where  $|x\rangle \langle x|$  is the projection operator onto the state  $|x\rangle$ . Performing a similarity transformation on the operator  $\hat{B}$  using arbitrary representation matrices, we find that  $\hat{B}$  is invariant,

$$\begin{aligned}
T(g)\hat{B}[T(g)]^{-1} &= \int dx T(g) |x\rangle \langle x| [T(g)]^{-1} \\
&= \int dx \exp\left(i\tilde{\beta}(g, x)\right) |gx\rangle \langle gx| \exp\left(-i\tilde{\beta}(g, x)\right) \\
&= \int dx |gx\rangle \langle gx| \\
&= \hat{B}.
\end{aligned} \tag{B.11}$$

Thus,  $\hat{B}$  commutes with all the operators  $T(g)$ . Since  $T(g)$  is an irreducible representation, we can invoke Schur's lemma to conclude that it must be equal to the identity

operator times a numerical constant,

$$\widehat{B} = d\widehat{I}. \quad (\text{B.12})$$

To determine the constant  $d$ , we compute the expectation value of  $\widehat{B}$  for a state  $|y\rangle$ . Note that we have normalised states to satisfy  $\langle y|y\rangle = 1$ ,

$$\langle y|\widehat{B}|y\rangle = \int |\langle y|x\rangle|^2 dx = \int |\langle 0|x\rangle|^2 dx = d. \quad (\text{B.13})$$

Of course, the convergence of the integral in (B.13) is a necessary condition for the operator  $\widehat{B}$  to exist. If the integral does converge, then the system of coherent states is called *square integrable*. Coherent states systems are square integrable in a number of cases, including all representations of compact semisimple Lie groups.

The proportionality constant  $d$  can be included in the measure on  $X$ , i.e.  $d\mu(x) = \frac{1}{d}dx$ , and hence, we get the following identity (*Resolution of Unity*):

$$\int d\mu(x) |x\rangle \langle x| = \widehat{I}. \quad (\text{B.14})$$

This allows us to expand any state in a superposition of coherent states,

$$|\psi\rangle = \int d\mu(x) |x\rangle \langle x|\psi\rangle = \int d\mu(x) c(x) |x\rangle, \quad (\text{B.15})$$

where  $c(x) = \langle x|\psi\rangle$ . In terms of the components  $c(x)$ , the square of the norm of a state  $|\psi\rangle$  is simply,

$$\langle \psi|\psi\rangle = \int d\mu(x) |c(x)|^2. \quad (\text{B.16})$$

Furthermore, the components  $c(x)$  satisfy the following relation,

$$c(x) = \langle x|\psi\rangle = \int d\mu(y) \langle x|y\rangle \langle y|\psi\rangle = \int d\mu(y) \langle x|y\rangle c(y). \quad (\text{B.17})$$

Choosing  $|\psi\rangle = |z\rangle$  in this equation, we see that the kernel  $K(x, y) = \langle x|y\rangle$  is

reproducing,

$$K(x, z) = \int d\mu(y) K(x, y) K(y, z). \quad (\text{B.18})$$

However,  $K(x, y)$  is not in general equal to a Dirac delta function, and so

$$|x\rangle = \int d\mu(y) \langle y|x\rangle |y\rangle, \quad (\text{B.19})$$

which shows that the coherent states are not all linearly independent. Rather, they form an overcomplete set of states.

## B.2 Coherent States for $SU(2N)$

Using the concepts introduced in the previous section, we construct a coherent state system for the types of representations of  $SU(2N)$  that we are interested in. These are representations with rectangular Young tableaux, as described in chapter 3. We must first pick a vector in the vector space  $\mathcal{H}$  that the representation  $T(g)$  acts in. For a vector that is a linear combination of weight vectors, the corresponding isotropy subgroup is, in general, a subgroup of the Cartan subgroup, and possibly a proper one. It is therefore convenient, to choose a weight vector  $|\mu\rangle$  as the initial element of the coherent state system. More generally, when the initial vector is in the same class as the weight vector  $|\mu\rangle$ , the isotropy subgroup  $H_\mu$  is isomorphic to the Cartan subgroup  $H$ , and coherent states are characterised by the elements of  $X = G/H$ . We will see that for our choice of initial vector, the corresponding isotropy subgroup, i.e. the subgroup of  $SU(2N)$  transformations which leave the state defined by this vector unchanged, is  $SU(N)_L \times SU(N)_R \times U(1)_{L=R}$ . This is a consequence of the chosen type of representation with rectangular Young tableaux of  $N$  rows. Coherent states are then characterized by the elements of  $X = SU(2N)/[SU(N)_L \times SU(N)_R \times U(1)_{L=R}]$ .

We now proceed to constructing a concrete set of coherent states for representations of  $SU(2N)$  with rectangular Young tableaux of the kind previously described. This discussion follows the steps outlined in [20]. Our basis of generators is given by (3.9). We can choose the Cartan sub-algebra to be spanned by the following set of

generators,

$$\widehat{H}_i = \widehat{S}^{ii} = \sum_{\alpha} \widehat{c}^{i\alpha\dagger} \widehat{c}^{i\alpha} - \frac{M}{2}, \quad (\text{B.20})$$

for  $i = 1, \dots, 2N$ . Recall that in (3.9), there is one constraint imposed on this set of generators, consistent with the fact that  $SU(2N)$  has rank  $2N - 1$ . The remaining operators  $\widehat{S}^{ij}$  with  $i \neq j$  are the “raising” and “lowering” operators which complete the canonical Cartan basis for the Lie algebra.

For the given representation we obtain the highest-weight vector as follows,

$$|\psi_0\rangle = \mathcal{C} [\varepsilon^{ab\dots} c^{a\alpha\dagger} c^{b\alpha\dagger} \dots] [\varepsilon^{cd\dots} c^{c\beta\dagger} c^{d\beta\dagger} \dots] \dots |0\rangle, \quad (\text{B.21})$$

where there are  $N$  creation operators in each square bracket, and there are  $M$  square bracketed terms all together. The indices  $a, b, \dots$  run through all values 1 to  $N$ , while  $\alpha, \beta, \dots$  run from 1 to  $M$ . We are symmetrizing the column indices of the Young tableau, and antisymmetrizing the row indices. The normalization constant  $\mathcal{C}$  is chosen so that  $\langle \psi_0 | \psi_0 \rangle = 1$ . The weight of this state is given by

$$\widehat{S}^{ii} |\psi_0\rangle = \begin{cases} \frac{M}{2} |\psi_0\rangle, & \text{for } 1 \leq i \leq N; \\ -\frac{M}{2} |\psi_0\rangle, & \text{for } N + 1 \leq i \leq 2N. \end{cases} \quad (\text{B.22})$$

We can obtain a coherent state system by applying all possible group transformations modulo the isotropy subgroup of  $|\psi_0\rangle$  to our chosen initial vector. In terms of the chosen basis for the Lie algebra we obtain a group element of  $SU(2N)$  by exponentiating an anti-Hermitian combination of generators, and so we have,

$$|q\rangle = \exp \left( -q^{ij} \widehat{S}^{ji} + q^{ij*} \widehat{S}^{ij} \right) |\psi_0\rangle, \quad (\text{B.23})$$

where the index  $j$  runs through the values 1 to  $N$ , and  $i$  runs from  $N + 1$  to  $2N$ . It is easily checked that if  $i$  and  $j$  were to fall in the same range of values, i.e. either both are between 1 and  $N$  or both are between  $N + 1$  and  $2N$ , then  $|q\rangle$  would just be equal (or in the case  $i = j$  proportional) to  $|\psi_0\rangle$ . Hence, the isotropy subgroup of the vector  $|\psi_0\rangle$  is  $SU(N)_L \times SU(N)_R \times U(1)_{L=R}$  as mentioned above. The  $q^{ij}$  are

$N^2$  independent complex numbers, which can be arranged into a  $GL(N, \mathbb{C})$  matrix. Under  $SU(2N)$  rotations, the generators  $\widehat{S}^{ij}$  transform in the adjoint representation,

$$\begin{aligned} & \exp\left(q^{k\ell}\widehat{S}^{\ell k} - q^{k\ell*}\widehat{S}^{k\ell}\right)\widehat{S}^{ij}\exp\left(-q^{mn}\widehat{S}^{nm} + q^{mn*}\widehat{S}^{mn}\right) \\ &= \left[\exp\begin{pmatrix} 0 & q^\dagger \\ -q & 0 \end{pmatrix}\right]^{ik}\widehat{S}^{kl}\left[\exp\begin{pmatrix} 0 & -q^\dagger \\ q & 0 \end{pmatrix}\right]^{\ell j}. \end{aligned} \quad (\text{B.24})$$

The states  $|q\rangle$  are normalized to one, and from (B.24) they clearly satisfy the following important identity

$$\langle q|\widehat{S}^{ij}|q\rangle = \frac{M}{2}Q^{ij}, \quad (\text{B.25})$$

where  $Q$  is given by

$$Q = \exp\left[\begin{pmatrix} 0 & q^\dagger \\ -q & 0 \end{pmatrix}\right]\begin{pmatrix} \mathbb{1}_N & 0 \\ 0 & -\mathbb{1}_N \end{pmatrix}\exp\left[\begin{pmatrix} 0 & -q^\dagger \\ q & 0 \end{pmatrix}\right]. \quad (\text{B.26})$$

In particular, if we write the matrix  $Q$  as

$$Q = \begin{pmatrix} L & V \\ V^\dagger & R \end{pmatrix}, \quad (\text{B.27})$$

we see that with  $\widehat{U}$  defined as in (3.6),

$$\langle q|\widehat{U}^{ij}|q\rangle = \frac{M}{2}V^{ij}. \quad (\text{B.28})$$

From the discussion of how to set up a coherent state path integral in the case of the Heisenberg model, we know that a Berry phase term of the form

$$S_B = \int_0^\beta d\tau \left( \frac{\langle q(\tau + \varepsilon)|q(\tau)\rangle - 1}{\varepsilon} \right). \quad (\text{B.29})$$

is generated at each site/link as part of the action. To manipulate this term, notice



that for any operator  $\widehat{M}$ , we have [51],

$$\frac{d}{dx} e^{\widehat{M}} = \int_0^1 dv e^{\widehat{M}(1-v)} \frac{d\widehat{M}}{dx} e^{\widehat{M}v}. \quad (\text{B.30})$$

Using the expression for  $|q\rangle$  found in (B.23), we can write the Berry phase term as,

$$\begin{aligned} S_B &= - \int_0^\beta d\tau \langle q(\tau) | \frac{d}{d\tau} | q(\tau) \rangle \\ &= - \int_0^\beta \langle \psi_0 | \exp(q^{ij} \widehat{S}^{ji} - q^{ij*} \widehat{S}^{ij}) \frac{d}{d\tau} \exp(-q^{ij} \widehat{S}^{ji} + q^{ij*} \widehat{S}^{ij}) | \psi_0 \rangle \\ &= - \int_0^\beta d\tau \int_0^1 dv \langle \psi_0 | \exp[-v(-q^{ij} \widehat{S}^{ji} + q^{ij*} \widehat{S}^{ij})] \\ &\quad \times \left( -\frac{\partial q^{ij}}{\partial \tau} \widehat{S}^{ji} + \frac{\partial q^{ij*}}{\partial \tau} \widehat{S}^{ij} \right) \exp[v(-q^{ij} \widehat{S}^{ji} + q^{ij*} \widehat{S}^{ij})] | \psi_0 \rangle. \end{aligned} \quad (\text{B.31})$$

If we now write

$$\langle vq(\tau) | = \widehat{S}^{ij} | vq(\tau) \rangle \equiv \frac{M}{2} Q^{ij}(\tau, v), \quad (\text{B.32})$$

we can simplify the above to

$$\begin{aligned} S_B &= -\frac{n}{2} \int_0^\beta d\tau \int_0^1 dv \left( -\frac{\partial q^{ij}}{\partial \tau} Q^{ji}(\tau, v) + \frac{\partial q^{ij*}}{\partial \tau} Q^{ij}(\tau, v) \right) \\ &= -\frac{n}{2} \int_0^\beta d\tau \int_0^1 dv \text{Tr} \left[ \begin{pmatrix} 0 & \frac{\partial q^\dagger}{\partial \tau} \\ -\frac{\partial q}{\partial \tau} & 0 \end{pmatrix} Q(\tau, v) \right]. \end{aligned} \quad (\text{B.33})$$

As a function of  $v$  the matrix  $Q(\tau, v)$  now satisfies

$$Q(\tau, 0) = \begin{pmatrix} \mathbf{1}_N & 0 \\ 0 & -\mathbf{1}_N \end{pmatrix}, \quad (\text{B.34})$$

and  $Q(\tau, 1) \equiv Q(\tau)$ . Integrating (B.33) by parts gives

$$S_B = \frac{M}{2} \int_0^\beta d\tau \int_0^1 dv \text{Tr} \left[ \begin{pmatrix} 0 & q^\dagger(\tau) \\ -q(\tau) & 0 \end{pmatrix} \frac{\partial}{\partial \tau} Q(\tau, v) \right]. \quad (\text{B.35})$$

It is not hard to see that

$$\begin{pmatrix} 0 & q^\dagger(\tau) \\ -q(\tau) & 0 \end{pmatrix} = -\frac{1}{2}Q(\tau, v)\frac{\partial Q(\tau, v)}{\partial v}, \quad (\text{B.36})$$

and this leads to the final result for the Berry phase,

$$S_B = -\frac{M}{4} \int_0^\beta d\tau \int_0^1 dv \left[ \text{Tr} \left( Q(\tau, v) \frac{\partial Q(\tau, v)}{\partial v} \frac{\partial Q(\tau, v)}{\partial \tau} \right) \right]. \quad (\text{B.37})$$

In the above derivation we used a specific dependence of  $Q(\tau, v)$  on the variable  $v$ , which satisfies the boundary conditions

$$\begin{aligned} Q(\tau, 0) &= Q(\tau', 0), \quad \text{for all } \tau, \tau'; \\ Q(\tau, 1) &= Q(\tau); \quad Q(0, v) = Q(\beta, v). \end{aligned} \quad (\text{B.38})$$

Thus, the field  $Q(\tau, v)$  lives in a rectangle  $0 \leq \tau \leq \beta$  and  $0 \leq v \leq 1$ . From the periodic boundary conditions in the  $\tau$  direction, we can interpret  $Q(\tau)$  as defining a closed curve, parametrized by  $\tau$ , and with the parameter  $v$  filling in the space enclosed by the curve to form a disc in the Grassmann manifold  $G(N, 2N) = SU(2N)/[SU(N)_L \times SU(N)_R \times U(1)_{L=R}]$ . In [20] it was shown that  $S_B$  is independent of the particular surface that has this boundary, up to integer multiples of  $2\pi Mi$ . This result was derived as a direct consequence of the fact that the second homotopy group of this Grassmann manifold is just  $\Pi_2(G(N, 2N)) = \mathbb{Z}$ , the group of integers.

Consider now the  $GL(N, \mathbb{C})$  matrix  $q$  of (B.23). As shown in section B.3 of this appendix, we can decompose such a matrix into the product of a left-coset Hermitian matrix  $B$  and a unitary matrix  $U^\dagger$ ,  $q = BU^\dagger$ . Upon substituting this decomposition into (B.26), we obtain

$$Q = \begin{pmatrix} U \cos(2B)U^\dagger & -U \sin(2B) \\ -\sin(2B)U^\dagger & -\cos(2B) \end{pmatrix}. \quad (\text{B.39})$$

In order to use this result to simplify the Berry phase term in the action, we

represent  $S_B$  as

$$S_B = \frac{M}{8} \int d^2\xi \varepsilon_{pq} \text{Tr} [Q \partial_p Q \partial_q Q], \quad (\text{B.40})$$

where  $p, q$  take the values 1, 2, and  $\xi_1 = \tau$ , and  $\xi_2 = v$ , and the integral is over a rectangle in  $(\tau, v)$  space. We parametrize  $Q(t, v)$  in the following way,

$$Q(\tau, v) = \exp \left[ \begin{pmatrix} 0 & q^\dagger(\tau, v) \\ -q(\tau, v) & 0 \end{pmatrix} \right] \begin{pmatrix} \mathbb{1}_N & 0 \\ 0 & -\mathbb{1}_N \end{pmatrix} \exp \left[ \begin{pmatrix} 0 & -q^\dagger(\tau, v) \\ q(\tau, v) & 0 \end{pmatrix} \right], \quad (\text{B.41})$$

where  $q(\tau, v)$  is a smooth function on the rectangle, such that the boundary conditions (B.38) are satisfied. We can then decompose the matrix  $q(\tau, v) = B(\tau, v)U^\dagger(\tau, v)$  as before and have

$$Q = \begin{pmatrix} U(\tau, v) \cos(2B(\tau, v))U^\dagger(\tau, v) & -U(\tau, v) \sin(2B(\tau, v)) \\ -\sin(2B(\tau, v))U^\dagger(\tau, v) & -\cos(2B(\tau, v)) \end{pmatrix}. \quad (\text{B.42})$$

Substituting this expression into the integrand of (B.40), we find that after some algebra it reduces to

$$\varepsilon_{pq} \text{Tr} [Q \partial_p Q \partial_q Q] = -4\varepsilon_{pq} \text{Tr} [\partial_q (\cos(2B(\tau, v))U^\dagger(\tau, v)\partial_p U(\tau, v))]. \quad (\text{B.43})$$

Hence, the Berry phase term simplifies to

$$\begin{aligned} S_B &= -\frac{M}{2} \int_0^\beta d\tau \int_0^1 dv (\partial_v \{ \text{Tr} [\cos(2B)U^\dagger \partial_\tau U] \} - \partial_\tau \{ \text{Tr} [\cos(2B)U^\dagger \partial_v U] \}) \\ &= -\frac{M}{2} \int_0^\beta d\tau \text{Tr} [\cos(2B)U^\dagger \partial_\tau U], \end{aligned} \quad (\text{B.44})$$

where we have used the boundary conditions on  $B(\tau, v)$  and  $U(\tau, v)$  to obtain the last line.

### B.3 Coset Decomposition of a $GL(N, \mathbb{C})$ matrix

For the sake of completeness, we show in this section how to decompose a non-singular  $GL(N, \mathbb{C})$  matrix  $q$  into the product of a Hermitian matrix  $B$  and a unitary matrix  $U$ . First, let  $A = qq^\dagger$ , which is Hermitian positive semidefinite. So it can be diagonalized by a unitary transformation,

$$VAV^\dagger = A_D = \text{diag}(a_1, a_2, \dots, a_N). \quad (\text{B.45})$$

We can then define the square root of  $A_D$  as

$$\sqrt{A_D} = \text{diag}(\sqrt{a_1}, \sqrt{a_2}, \dots, \sqrt{a_N}). \quad (\text{B.46})$$

Now let

$$B = V^\dagger A_D^{1/2} V, \quad U = B^{-1} q = V^\dagger A_D^{-1/2} V q. \quad (\text{B.47})$$

It is easily seen that such a  $U$  is unitary. We also want to determine the transformation properties of the  $B$  and  $U$  matrices if  $q$  transforms under  $U(N)_L \times U(N)_R$  transformations as

$$q \rightarrow q' = LqR^\dagger, \quad (\text{B.48})$$

where  $LL^\dagger = RR^\dagger = \mathbb{1}$ . Then  $q' = B'U'$ , and

$$A' = q'q'^\dagger = LqR^\dagger Rq^\dagger L^\dagger = LAL^\dagger. \quad (\text{B.49})$$

We also have,

$$A'_D = V' A' V'^\dagger = V' L A L^\dagger V'^\dagger = A_D. \quad (\text{B.50})$$

Thus,  $V'L = DV$  or  $V' = DVL^\dagger$ , where  $D$  is a non-degenerate diagonal matrix. So we find the transformation properties

$$B' = V'^\dagger \sqrt{A'_D} V' = LV^\dagger D^\dagger \sqrt{A_D} DV L^\dagger = LBL^\dagger. \quad (\text{B.51})$$

and

$$U' = B'^{-1}q' = LB^{-1}L^\dagger LqR^\dagger = LUR^\dagger. \quad (\text{B.52})$$

Furthermore, observe that

$$qq^\dagger = A = V^\dagger A_D V = (V^\dagger \sqrt{A_D} V)(V^\dagger \sqrt{A_D} V) = B^2, \quad q^\dagger q = U^\dagger B^2 U. \quad (\text{B.53})$$

# Appendix C

## Integrals over $U(dM)$ and $U(2dM)/[U(dM) \times U(dM)]$ manifolds

Recall that the  $\Phi$  matrices are products of the form  $\Phi = U\Lambda U^\dagger$ , where  $U$  is a unitary matrix and  $\Lambda = \text{diag}(\mathbb{1}_{dM}, -\mathbb{1}_{dM})$ . Hence, the measure of integration over  $\Phi$  is induced by the Haar measure of integration over  $U(2dM)$ . Therefore, let us first consider integration over elements of unitary matrices. The most important properties of such integrals are that

$$\int dU = 1, \tag{C.1}$$

and that the Haar measure is shift-invariant under left and right multiplication by unitary matrices,  $U \rightarrow LUR^\dagger$ . So for example, if we write  $I_i^j \equiv \int dU U_i^j$ , then we have

$$L_\ell^i I_i^j R_j^k = \int dU L_\ell^i U_i^j R_j^k = \int dV V_\ell^k = I_\ell^k. \tag{C.2}$$

Here we have let  $V = LUR^\dagger$ , and used the fact that the measure is shift-invariant. So  $I = LIR^\dagger$  for any unitary  $L$  and  $R$ , and hence,  $I = 0$ . Similarly it can be shown that any integral of the form

$$\int dU F(U, U^\dagger), \tag{C.3}$$

where  $F(U, U^\dagger)$  is some monomial in  $U$  and  $U^\dagger$ , must vanish, unless the result is invariant under left and right shifts of the integration variable. The only way this

can be achieved is if there are an equal number of  $U$  and  $U^\dagger$  in the integrand. In this case, we can construct tensors which are invariant under left and right shifts. As an example, consider

$$\begin{aligned}
I_{im}^{jn} &\equiv \int dU U_i^j U_m^\dagger{}^n \\
&= \int dV (L_i^a V_a^b R_b^\dagger{}^j) (R_m^c V_c^d L_d^\dagger{}^n) \\
&= L_i^a L_d^\dagger{}^n R_b^\dagger{}^j R_m^c I_{ac}^{bd}.
\end{aligned} \tag{C.4}$$

Hence, we have that  $I_{im}^{jn} \propto \delta_i^n \delta_m^j$ , and the constant of proportionality is found to be  $1/(2dM)$  by setting  $m = j$  and summing. Graphically, we can represent this result as

$$\int dU U_i^j U_m^\dagger{}^n = \frac{1}{2dM} \begin{pmatrix} n & j \\ i \updownarrow & \updownarrow m \end{pmatrix}. \tag{C.5}$$

This result can be applied to determine the integral over one element of a matrix in the coset space  $U(2dM)/[U(dM) \times U(dM)]$ . In fact, it immediately follows that the integration over one  $\Phi$  variable must vanish, because  $\Lambda$  is traceless,

$$\begin{aligned}
\int d\Phi \Phi_\mu{}^\nu &= \int dU U_\mu{}^\lambda \Lambda_\lambda{}^\delta U_\delta^\dagger{}^\nu \\
&= \frac{1}{2dM} \delta_\mu{}^\nu \delta_\delta{}^\lambda \Lambda_\lambda{}^\delta = \frac{1}{2dM} \delta_\mu{}^\nu \text{Tr} \Lambda = 0.
\end{aligned} \tag{C.6}$$

In general, integration over an odd number of  $\Phi$  variables always gives zero. This is a consequence of the fact that permutations are unitary transformations, and so

$$\Phi' = (UP)\Lambda(P^\dagger U^\dagger) = -U\Lambda U^\dagger = -\Phi, \quad \text{where } P = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{C.7}$$

Thus, for every  $\Phi$  there is a  $-\Phi$ , and the value of an integral over an odd number of them must therefore vanish.

In (C.5) we introduced a graphical notation for the Kronecker deltas that result from integration over elements of  $U(2dM)$  matrices. The general rule is that lower

(upper) indices of a  $U$  always need to be joined to upper (lower) indices of a  $U^\dagger$ , to give a Kronecker delta. For the case of four matrices in the integrand, we have

$$\int dU U_i^j U_k^\dagger U_m^\ell U_o^\dagger U^p = C_1 \left( \begin{array}{cccc} \ell & p & j & n \\ i \downarrow & \downarrow & \cdot & \downarrow \\ m & k & o & \end{array} + \begin{array}{cccc} \ell & p & j & n \\ i \times & \times & \cdot & \times \\ m & k & o & \end{array} \right) \\ + C_2 \left( \begin{array}{cccc} \ell & p & j & n \\ i \downarrow & \downarrow & \cdot & \times \\ m & k & o & \end{array} + \begin{array}{cccc} \ell & p & j & n \\ i \times & \times & \cdot & \downarrow \\ m & k & o & \end{array} \right). \quad (\text{C.8})$$

If we now contract  $j$  with  $k$ , and  $n$  with  $o$  on both sides of the equation, we find that  $C_1 = 1/((2dM)^2 - 1)$  and  $C_2 = 1/[2dM(1 - (2dM)^2)]$ . We can use this result to evaluate the integral over two  $\Phi$  fields as follows,

$$\int d\Phi \Phi_\mu^\nu \Phi_\rho^\sigma = \left( \int dU U_\mu^\lambda U_\delta^\dagger U_\rho^\alpha U_\beta^\dagger \right) \Lambda_\alpha^\beta \Lambda_\lambda^\delta \\ = \frac{1}{(2dM)^2 - 1} (2dM \delta_\mu^\sigma \delta_\rho^\nu - \delta_\mu^\nu \delta_\rho^\sigma). \quad (\text{C.9})$$

Here we have made use of the fact that  $\Lambda$  is traceless and its square is equal to the identity.

For the strong coupling expansion of section 6.4 we also need to evaluate integrals involving products of four  $\Phi$  variables, i.e. eight entries of unitary matrices,

$$\int d\Phi \Phi_{\mu_1}^{\nu_1} \Phi_{\mu_2}^{\nu_2} \Phi_{\mu_3}^{\nu_3} \Phi_{\mu_4}^{\nu_4} = \left( \int dU U_{\mu_1}^{\alpha_1} U_{\beta_1}^{\dagger \nu_1} U_{\mu_2}^{\alpha_2} U_{\beta_2}^{\dagger \nu_2} U_{\mu_3}^{\alpha_3} U_{\beta_3}^{\dagger \nu_3} U_{\mu_4}^{\alpha_4} U_{\beta_4}^{\dagger \nu_4} \right) \Lambda_{\alpha_1}^{\beta_1} \Lambda_{\alpha_2}^{\beta_2} \Lambda_{\alpha_3}^{\beta_3} \Lambda_{\alpha_4}^{\beta_4} \\ = \left[ \begin{array}{cccc} \nu_1 & \nu_2 & \nu_3 & \nu_4 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ & & & & & & & \\ & & & & & & & \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{array} \right] \Lambda_{\alpha_1}^{\beta_1} \Lambda_{\alpha_2}^{\beta_2} \Lambda_{\alpha_3}^{\beta_3} \Lambda_{\alpha_4}^{\beta_4}. \quad (\text{C.10})$$

The term in square brackets indicates that the integral is proportional to all possible pairings of  $\mu$ s with  $\nu$ s and of  $\alpha$ s with  $\beta$ s to form Kronecker deltas. It is clear from this expression, that the final answer for the integral is a tensor in  $\mu$  and  $\nu$  indices, and each term in this tensor is proportional to a product of Kronecker deltas pairing



$\mu$ s and  $\nu$ s. Using symmetry considerations, we can make an ansatz for the value of this integral,

$$\begin{aligned}
I_{\mu_1\mu_2\mu_3\mu_4}^{\nu_1\nu_2\nu_3\nu_4} = & C_1 \cdot \left( \begin{array}{cccc} \nu_1 & \nu_2 & \nu_3 & \nu_4 \\ \uparrow & \uparrow & \uparrow & \uparrow \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{array} \right) + C_2 \cdot \left( \begin{array}{cccc} \nu_1 & \nu_2 & \nu_3 & \nu_4 \\ \uparrow & \uparrow & \swarrow & \searrow \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{array} + 5 \text{ permutations} \right) \\
& + C_3 \cdot \left( \begin{array}{cccc} \nu_1 & \nu_2 & \nu_3 & \nu_4 \\ \uparrow & \swarrow & \searrow & \swarrow \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{array} + 7 \text{ perms.} \right) + C_4 \cdot \left( \begin{array}{cccc} \nu_1 & \nu_2 & \nu_3 & \nu_4 \\ \swarrow & \swarrow & \swarrow & \swarrow \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{array} + 2 \text{ perms.} \right) \\
& + C_5 \cdot \left( \begin{array}{cccc} \nu_1 & \nu_2 & \nu_3 & \nu_4 \\ \swarrow & \swarrow & \swarrow & \swarrow \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{array} + 5 \text{ perms.} \right) \quad (\text{C.11})
\end{aligned}$$

To evaluate the constants  $C_1, C_2, C_3, C_4$  and  $C_5$ , we can contract over some of the indices and compare both sides of the resulting equation. For example, if we set  $\mu_1 = \nu_1$  and sum over  $\nu_1$ , the value of the integral vanishes, because the  $\Phi$  matrices are traceless. So,

$$\begin{aligned}
0 = & (2dMC_1 + 3C_2) \left( \begin{array}{ccc} \nu_2 & \nu_3 & \nu_4 \\ \uparrow & \uparrow & \uparrow \\ \mu_2 & \mu_3 & \mu_4 \end{array} \right) + (2dMC_2 + 2C_3 + C_4) \left( \begin{array}{ccc} \nu_2 & \nu_3 & \nu_4 \\ \uparrow & \swarrow & \searrow \\ \mu_2 & \mu_3 & \mu_4 \end{array} + 2 \text{ perms.} \right) \\
& + (2dMC_3 + 3C_5) \left( \begin{array}{ccc} \nu_2 & \nu_3 & \nu_4 \\ \swarrow & \swarrow & \swarrow \\ \mu_2 & \mu_3 & \mu_4 \end{array} + 1 \text{ perm.} \right) \quad (\text{C.12})
\end{aligned}$$

Since all terms of products of Kronecker deltas are linearly independent tensors, each coefficient must vanish independently. Hence, we obtain the equations

$$C_1 = -3C_2/(2dM), \quad C_4 = -2dMC_2 - 2C_3, \quad C_5 = -2dMC_3/3. \quad (\text{C.13})$$

Next, we contract  $\mu_1$  with  $\nu_2$  and  $\mu_2$  with  $\nu_1$ . The resulting integral is one of only

two  $\Phi$  variables, and its value is given in (C.9). So we have

$$\begin{aligned}
& -\frac{2dM}{(2dM)^2-1} \begin{pmatrix} \nu_3 & \nu_4 \\ \uparrow & \uparrow \\ \mu_3 & \mu_4 \end{pmatrix} + \frac{(2dM)^2}{(2dM)^2-1} \begin{pmatrix} \nu_3 & \nu_4 \\ \swarrow & \searrow \\ \mu_3 & \mu_4 \end{pmatrix} \\
& = (2dMC_1 + 4C_2 + (2dM)^2C_2 + 4(2dM)C_3 + 2C_5) \begin{pmatrix} \nu_3 & \nu_4 \\ \uparrow & \uparrow \\ \mu_3 & \mu_4 \end{pmatrix} \\
& + (2dMC_2 + 4C_3 + ((2dM)^2 + 2)C_4 + 4(2dM)C_5) \begin{pmatrix} \nu_3 & \nu_4 \\ \swarrow & \searrow \\ \mu_3 & \mu_4 \end{pmatrix} \quad (C.14)
\end{aligned}$$

The resulting two equations are in fact not both linearly independent from those in (C.13). Combining the information in (C.13) and (C.14), we obtain the additional constraint

$$((2dM)^2 + 1)C_2 + \frac{10}{3}2dMC_3 = -\frac{2dM}{(2dM)^2 - 1}. \quad (C.15)$$

Finally, we contract all indices, namely  $\mu_1$  with  $\nu_4$ ,  $\mu_2$  with  $\nu_1$ ,  $\mu_3$  with  $\nu_2$  and  $\mu_4$  with  $\nu_3$ . Since the integral over the  $\Phi$  variables is normalised, the value of this integral is simply the trace of the unit matrix times 1. Hence,

$$\begin{aligned}
2dM &= 2dMC_1 + 6(2dM)^2C_2 + 4(2dM)((2dM)^2 + 1)C_3 \\
&+ 2dM(2(2dM)^2 + 1)C_4 + (2dM)^2((2dM)^2 + 5)C_5. \quad (C.16)
\end{aligned}$$

Solving all of these linear equations, we obtain

$$\begin{aligned}
C_1 &= C_3 = \frac{3}{((2dM)^2 - 1)((2dM)^2 - 9)} \\
C_2 &= C_5 = -\frac{2dM}{((2dM)^2 - 1)((2dM)^2 - 9)} \\
C_4 &= \frac{(2dM)^2 - 6}{((2dM)^2 - 1)((2dM)^2 - 9)}. \quad (C.17)
\end{aligned}$$

In order to generalise this procedure to integrals of more  $\Phi$ -variables it is worth

thinking about the structure of the answer. Each combination of Kronecker  $\delta$ s can be associated with an element of the permutation group in a straightforward way. For example,

$$\delta_{\mu_1}^{\nu_1} \delta_{\mu_2}^{\nu_4} \delta_{\mu_3}^{\nu_2} \delta_{\mu_4}^{\nu_3} \longleftrightarrow \begin{pmatrix} \nu_1 & \nu_2 & \nu_3 & \nu_4 \\ \uparrow & \swarrow & \nearrow & \searrow \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} \longleftrightarrow (3\ 4\ 2)(1), \quad (\text{C.18})$$

using two standard notations for denoting permutations in the symmetric group  $S_4$ . The right-most notation is the cycle notation, which should be read from right to left as “1 goes to 1 and 2 goes to 4 goes to 3 goes to 2.” Every permutation can be written as a product of independent cycles, i.e. such that each number appears in exactly one cycle. Hence, the total number of terms in an integral of  $2k$   $\Phi$ -variables is equal to the total number of permutations of  $2k$  objects, which is  $(2k)!$ . Earlier, we used symmetry arguments to reduce the number of independent coefficients. Namely, all terms which are just permutations of the same Kronecker  $\delta$  structure should appear with the same coefficient in the integral. Equivalently, all terms that belong to the same conjugacy class of  $S_{2k}$  appear with the same coefficient. Thus, the number of independent coefficients is equal to the number of conjugacy classes of  $S_{2k}$ . There are as many conjugacy classes in  $S_{2k}$  as there are Young graphs that can be constructed from  $2k$  boxes. Alternatively, we know that all the permutations of  $S_{2k}$  with the same cycle structure belong to the same equivalence class, and all the permutations in a class have the same cycle structure [48]. So to find out how many equivalent terms in the integral have the same coefficient, we need to determine the appropriate conjugacy class and count how many elements are in that class. Let a particular class correspond to a cycle structure  $(\ell) = (\ell_1^{i_1}, \ell_2^{i_2}, \dots)$ , where  $\ell_j$  is the length of a cycle, and  $i_j$  is the number the cycle of length  $\ell_j$  appears. Then  $\sum_j \ell_j i_j = 2k$ . The number of elements in that conjugacy class is given by [48],

$$h_{(\ell)} = \frac{(2k)!}{\prod_j \ell_j^{i_j} i_j!}. \quad (\text{C.19})$$

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