From Decay to Complete Breaking: Pulling the Strings in $SU(2)$ Yang-Mills Theory

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We study $\{2Q+1\}$-strings connecting two static charges Q in $(2+1)$-d $SU(2)$ Yang-Mills theory. While the fundamental $\{2\}$-string between two charges $Q = \frac{1}{2}$ is unbreakable, the adjoint $\{3\}$-string connecting two charges $Q = 1$ can break. When a $\{4\}$-string is stretched beyond a critical length, it decays into a $\{2\}$-string by gluon pair creation. When a $\{5\}$-string is stretched, it first decays into a $\{3\}$-string, which eventually breaks completely. The energy of the screened charges at the ends of a string is well described by a phenomenological constituent gluon model.

Studies of the strings connecting two static color charges provide valuable insights into the physics of confinement in $SU(N)$ Yang-Mills theories. The properties of the string connecting a static quark-anti-quark pair with charges in the fundamental $\{N\}$ and $\{\overline{N}\}$ representations are described by a low-energy effective bosonic string theory. While the string tension σ determines the quark-anti-quark potential $V(r) = \sigma r$ at asymptotic distances, the massless modes corresponding to transverse fluctuations of the string give rise to the universal Lüscher term proportional to $1/r$ [1, 2], as well as to a diverging string thickness proportional to $\log r$ [3]. The effective string theory also makes detailed predictions for the excited states of the string [4]. Lattice gauge theory provides us with a powerful tool for investigating the string dynamics using Monte Carlo simulations. In this way, the linearly rising quark-anti-quark potential has been calculated at large distances [5]. By developing a highly efficient multi-level simulation technique [6], Lüscher and Weisz have studied the universal $1/r$-term in the quark-anti-quark potential at large distances [7].

In theories with dynamical fundamental charges, the confining string connecting two static color charges can break due to the creation of dynamical charge-anti-charge pairs which screen the external static sources. Numerical evidence for charge screening was obtained in a lattice gauge-Higgs model in which the dynamical fundamental charges are scalars [8]. Direct evidence for string breaking was first observed in the $SU(2)$ gauge-Higgs model [9, 10] and later also in the $\mathbb{Z}(2)$ gauge-Higgs model [11]. While the fundamental string is unbreakable in $SU(N)$ Yang-Mills theory, the string connecting static adjoint charges can break due to pair-creation of dynamical gluons. This effect has been investigated in [12, 13, 14, 15, 16, 17, 18]. Numerical evidence for string breaking in lattice QCD with dynamical quarks has been observed in [19]. The center symmetry of $SU(N)$ Yang-Mills theory is $\mathbb{Z}(N)$. Consequently, each $SU(N)$ representation and hence each external static charge can

be characterized by its N-ality $k = \{0, 1, ..., N - 1\}$. Strings connecting external charges with N-ality $k \neq 0$ are known as k-strings. These strings are unbreakable and have a k-dependent string tension, which may or may not be proportional to the Casimir operator of the corresponding representation [20, 21].

Since it is easiest to simulate numerically, in this letter we study the dynamics of strings in $(2+1)$-d $SU(2)$ Yang-Mills theory which has the center $\mathbb{Z}(2)$. Other theories in $(3 + 1)$ dimensions or with other gauge groups are expected to show similar behavior. Here we investigate the strings connecting two static charges Q in the $SU(2)$ representation $\{2Q + 1\}$, which we refer to as $\{2Q + 1\}$-strings, not to be confused with k-strings. The $\{2Q+1\}$-strings with integer Q have $k = 0$ and will eventually break at large distances, while the $\{2Q+1\}$-strings with half-integer Q have $k = 1$ and are unbreakable. At asymptotic distances all $\{2Q+1\}$-strings connecting half-integer charges have the same tension σ as the fundamental $\{2\}$-string. Since $SU(2)$ Yang-Mills theory has no dynamical fundamental charges, the static charges Q at the two ends of a $\{2Q+1\}$-string can only be screened by dynamical gluons. When a pair of gluons is created from the vacuum, the external sources are screened and thus reduced to $Q - 1$. As a consequence, the $\{2Q+1\}$-string decays to a $\{2Q-1\}$-string and abruptly reduces its tension accordingly [22, 23]. While some numerical evidence for string decay was presented in [24], using the multi-level simulation technique of [6], we are able, for the first time, to investigate string decay in detail.

We consider $(2 + 1)$-d $SU(2)$ Yang-Mills theory on a cubic lattice using the standard Wilson plaquette action for link variables in the fundamental representation. The external color charges Q at the two ends of a string are represented by Polyakov loops $\Phi_Q(x)$ in the $\{2Q + 1\}$ representation wrapping around the Euclidean time direction. The corresponding potential $V_Q(r)$ between the static sources is extracted from the Polyakov loop corre-

lator

$$\langle \Phi_Q(0)\Phi_Q(r)\rangle \sim \exp(-\beta V_Q(r)). \qquad (1)$$

In order to ensure a good projection on the ground state of the string, we have simulated at inverse temperatures as large as $\beta = 64$ in lattice units. The spatial lattice size was $L = 32$ and the bare gauge coupling was chosen as $4/g^2 = 6.0$ which puts the deconfinement phase transition at $\beta_c \approx 4$. While this is a moderate coupling, we are confident that our results remain unchanged, at least qualitatively, in the continuum limit. The values of the simulated Polyakov loop correlators range from 10^{-8} to 10^{-135}. Measuring such small signals would be completely impossible without the Lüscher-Weisz multi-level simulation technique. We have slightly refined this method by applying the segmentation of the lattice not only to slabs in time, but also to blocks in space. By carefully tuning the parameters of the multi-level algorithm, we have been able to extract the potentials $V_Q(r)$ for the {2}-, {3}-, {4}-, and {5}-strings. As shown in the top panel of figure 1, at distance $r \approx 8$, the {4}-string decays, thus reducing its tension to the one of the fundamental {2}-string. Similarly, the bottom panel shows that, at distance $r \approx 6$, the {5}-string decays and reduces its tension to the one of the adjoint {3}-string. Only at $r \approx 10$ the string breaks completely, at about the same distance as the adjoint {3}-string. Not unexpectedly, the tension of a string is the same, no matter whether it connects screened or unscreened external charges Q.

A fit of the fundamental potential to

$$V_{1/2}(r) = \sigma r - \frac{\pi}{24r} + 2M + \mathcal{O}(1/r^3), \qquad (2)$$

works very well and yields the asymptotic string tension $\sigma = 0.06397(3)$. In particular, the Monte Carlo data are in excellent agreement with the predicted coefficient $-\frac{\pi}{24}$ of the Lüscher term. The "mass" contribution of an external charge $Q = \frac{1}{2}$ to the total energy of the system is given by $M = 0.109(1)$. This "mass" itself is not physical because it contains ultra-violet divergent pieces. Since string decay occurs at moderate distances, its typical energy scale is not well separated from Λ_{QCD}. Consequently, unlike the string behavior at asymptotic distances, string decay can not be addressed in a fully systematic low-energy effective string theory. In particular, unlike the string tension σ of the unbreakable fundamental string, the tension σ_Q of an ultimately unstable {2Q + 1}-string (with $Q \geq 1$) is not defined unambiguously. Here we define σ_Q by a fit of the Monte Carlo data to a simple phenomenological model. In this model, we consider the {2Q + 1}-string as a multi-channel system. A channel containing a {2Q + 1}-string connecting two charges Q, which resulted from screening a larger charge $Q + n$ by n gluons, has the energy

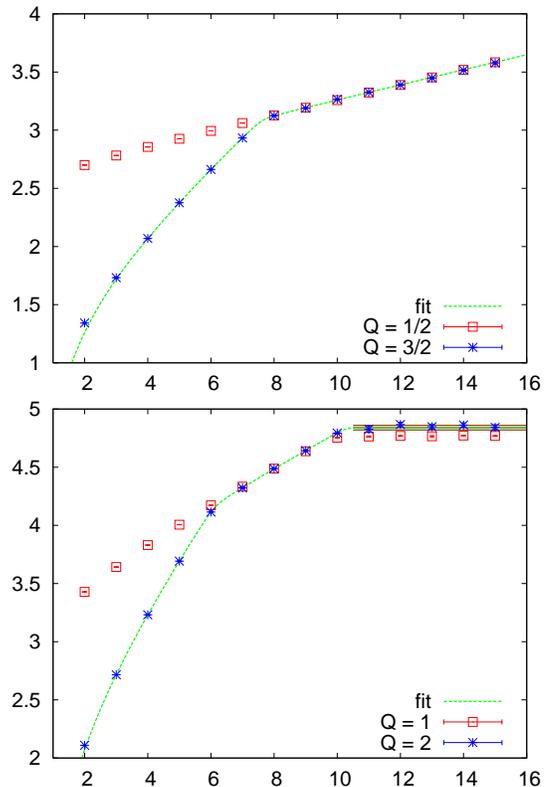$$E_{Q,n}(r) = \sigma_Q r - \frac{c_Q}{r} + 2M_{Q,n}. \qquad (3)$$

FIG. 1: *Top: Potential $V(r)$ of two static color charges with $Q = \frac{1}{2}$ (squares) and $Q = \frac{3}{2}$ (stars), shifted by a constant for a more convenient comparison of the slopes. Bottom: The same for $Q = 1$ (squares) and $Q = 2$ (stars). The lines are a fit of the multi-channel model to the Monte Carlo data. The horizontal band at $2M_{0,2} = 4.84(2)$ corresponds to twice the mass of a source of charge $Q = 2$ obtained from the measurement of a single Polyakov loop.*

Here c_Q is the coefficient of a sub-leading $1/r$ correction which does not necessarily assume the asymptotic Lüscher value $-\frac{\pi}{24}$. The "mass" $M_{Q,n}$ describes the contribution of an original charge $Q + n$ that has been screened to the value Q by n gluons. Just as the "mass" $M = M_{1/2,0}$, the "masses" $M_{Q,n}$ themselves are not physical, because they again contain ultra-violet divergent contributions. However, the mass differences $\Delta_{Q,n} = M_{Q-1,n+1} - M_{Q,n}$ are physical since the divergent pieces then cancel. The {3}- and {4}-string are described by the two-channel Hamiltonians H_1 and $H_{3/2}$, while the {5}-string is described by the three-channel Hamiltonian H_2 with

$$H_1(r) = \begin{pmatrix} E_{1,0}(r) & A \\ A & E_{0,1}(r) \end{pmatrix},$$

$$H_{3/2}(r) = \begin{pmatrix} E_{3/2,0}(r) & B \\ B & E_{1/2,1}(r) \end{pmatrix},$$

$$H_2(r) = \begin{pmatrix} E_{2,0}(r) & C & 0 \\ C & E_{1,1}(r) & A \\ 0 & A & E_{0,2}(r) \end{pmatrix}. \qquad (4)$$

Q	σ_Q	σ_Q/σ	$4Q(Q+1)/3$
1/2	0.06397(3)	1	1
1	0.144(1)	2.25(2)	8/3
3/2	0.241(5)	3.77(8)	5
2	0.385(5)	6.02(8)	8

TABLE I: *Fitted values of the string tensions σ_Q. The ratio σ_Q/σ with $\sigma = \sigma_{1/2}$ is compared with the value $4Q(Q+1)/3$ representing Casimir scaling.*

Q	$M_{Q,0}$	$M_{Q-1,1}$	$M_{Q-2,2}$	$\Delta_{Q,0}$	$\Delta_{Q-1,1}$
1/2	0.109(1)	—	—	—	—
1	0.37(3)	1.038(1)	—	0.67(3)	—
3/2	0.72(5)	1.32(5)	—	0.60(5)	—
2	1.04(3)	1.71(3)	2.42(1)	0.67(3)	0.71(3)

TABLE II: *Fitted values of the "mass" $M_{Q,n}$ of an original charge $Q+n$ that has been screened to the value Q by n gluons, together with the differences $\Delta_{Q,n} = M_{Q-1,n+1} - M_{Q,n}$.*

Here A, B, and C are decay amplitudes which we assume to be r-independent. The potential $V_Q(r)$ is the energy of the ground state of H_Q. Figure 2 compares the forces $F(r) = -dV(r)/dr$ in the {2}-, {3}-, {4}-, and {5}-string cases with the results of the multi-channel model. The tensions σ_Q listed in table 1 have been determined by a fit to the Monte Carlo data. The simple model works rather well. It is interesting to note that the ratios σ_Q/σ do not obey Casimir scaling, i.e. they are not equal to $4Q(Q+1)/3$. The "masses" $M_{Q,n}$ are listed in table 2. Remarkably, within the error bars, the mass differences $\Delta_{Q,0} = M_{Q-1,1} - M_{Q,0}$ all take the same value $M_G = 0.65(5)$, independent of Q. We interpret M_G as a constituent gluon mass which in units of the string tension takes the value $M_G/\sqrt{\sigma} = 2.6(2)$. It should be pointed out that, in contrast to the string tension, M_G is not unambiguously defined. It just results from the fit parameters of the phenomenological model. The value $\Delta_{1,1} = M_{0,2} - M_{1,1} = 0.71(3)$ indicates that the addition of a second constituent gluon costs an energy slightly larger than M_G. Interestingly, the mass of two constituent gluons $2M_G = 1.3(1)$ is close to the 0^+ glueball mass $M_{0^+} = 1.198(25)$ obtained in [25] at the same value of the bare coupling. M_G also sets the distance scale for string decay and string breaking. A leading order estimate for the distance at which the {4}-string decays into the {2}-string is $r \approx 2M_G/(\sigma_{3/2} - \sigma_{1/2}) = 7.3(6)$, while the distance at which the {3}- and the {5}-string ultimately break is estimated to be around $r \approx 2M_G/\sigma_1 = 9.0(7)$.

String decay can be viewed as a quantum analogue of the classical process of strand rupture in a cable consisting of a bundle of strands. When such a cable is
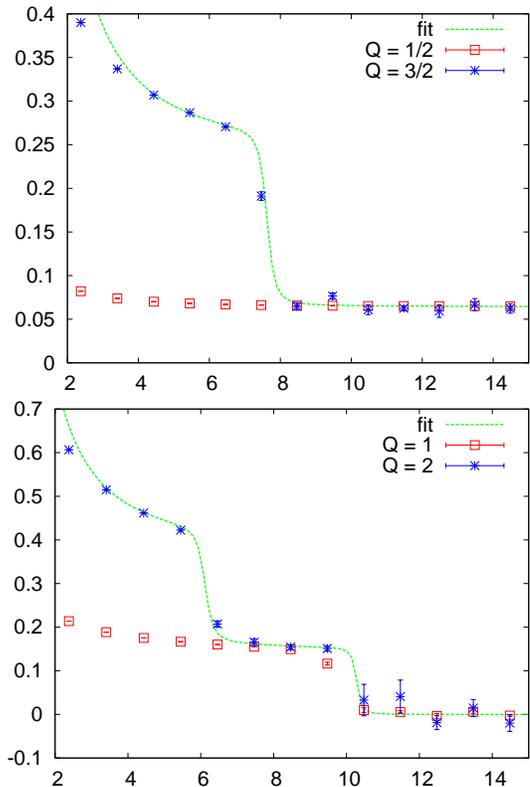

FIG. 2: *Top: Forces $F(r)$ that the {2}- and {4}-string exert on the external charges $Q = \frac{1}{2}$ (squares) and $Q = \frac{3}{2}$ (stars), respectively. Bottom: The same for the {3}- and {5}-string connecting external charges $Q = 1$ (squares) and $Q = 2$ (stars), respectively. The lines represent the fit of the multi-channel model to the Monte Carlo data.*

stretched further and further, individual strands eventually rupture, thereby abruptly reducing the tension of the cable. While strand rupture is well-known in the material science of centimeter thick steel cables with a tension of about 10^5 Newton, we have seen that a similar process occurs for the confining strings in non-Abelian gauge theories which have about the same tension but are 13 orders of magnitude thinner. Whether a strand picture may correctly describe the actual anatomy of decaying $\{2Q + 1\}$-strings is an interesting question that will require further investigations which go beyond the scope of the present letter.

It would be interesting to investigate string decay and string breaking for other $SU(N)$ gauge theories. In $SU(3)$ Yang-Mills theory the {3}-string connecting a quark in the {3} with an anti-quark in the {$\overline{3}$} representation is unbreakable, while the {8}-string connecting two adjoint sources can break by pair creation of gluons. When a {6}-string is stretched, the external source in the {6}-representation will eventually be screened to a {$\overline{3}$} by a gluon. The corresponding string decay should be analogous to the decay of the {4}-string in $SU(2)$ Yang-Mills theory discussed above. In analogy to the {5}-string in

$SU(2)$, the $\{10\}$-string in $SU(3)$ Yang-Mills theory is expected to decay to an adjoint $\{8\}$-string, before it breaks completely at larger distances. In QCD with dynamical quarks, strings can also decay by quark-anti-quark pair creation. Due to its $\mathbb{Z}(4)$ center symmetry, $SU(4)$ Yang-Mills theory has two distinct unbreakable strings, connecting external charges either in the $\{4\}$ and $\{\overline{4}\}$ or in the $\{6\}$-representation. For external sources with non-trivial N-ality, one then expects cascades of string decays down to the $\{4\}$-string for $k = 1, 3$ and down to the $\{6\}$-string for $k = 2$.

Studying gauge groups other than $SU(N)$ would also be interesting. For example, all $Sp(N)$ gauge theories have the same center $\mathbb{Z}(2)$. The first Lie group in this sequence is $Sp(1) = SU(2) = Spin(3)$, while the second is $Sp(2) = Spin(5)$, the universal covering group of $SO(5)$. In $Sp(2)$ Yang-Mills theory, only the fundamental $\{4\}$-string is absolutely stable. As usual, the adjoint $\{10\}$-string can break by pair creation of gluons. The representation $\{5\}$ is center-neutral. Since in $Sp(2)$

$$\{5\} \otimes \{10\} = \{5\} \oplus \{10\} \oplus \{35\}, \qquad (5)$$

a single gluon can screen a charge $\{5\}$ only to a $\{10\}$ or a $\{35\}$. We expect the unstable $\{5\}$-string to have a smaller tension than the adjoint $\{10\}$-string or the $\{35\}$-string. In that case, the $\{5\}$-string will break in one step by the creation of four gluons, without any intermediate string decay.

Finally, it would be interesting to investigate the importance of the center for the phenomenon of string decay. The exceptional group $G(2)$ is the simplest Lie group with a trivial center. Still, $G(2)$ Yang-Mills theory confines color (although without an asymptotic string tension) [26]. Furthermore, it has a first order deconfinement phase transition [27, 28]. In fact, as we have discussed in the context of $Sp(N)$ Yang-Mills theories, the order of the deconfinement phase transition is controlled by the size of the gauge group and not by the center [29]. In $G(2)$ Yang-Mills theory, even a charge in the fundamental $\{7\}$ representation can be screened by gluons in the adjoint $\{14\}$ representation. As a result, there are no unbreakable strings. Since in $G(2)$

$$\{7\} \otimes \{14\} = \{7\} \oplus \{27\} \oplus \{64\}, \qquad (6)$$

a single gluon can screen a charge $\{7\}$ only to a $\{27\}$ or a $\{64\}$. In $G(2)$ Yang-Mills theory, approximate Casimir scaling has been verified for unstable strings including the $\{27\}$- and the $\{64\}$-string [30]. As a consequence, the fundamental $\{7\}$-string is stable against decay due to the creation of a single pair of gluons. The same is true even for processes involving four gluons. Based on the group theory of $G(2)$, we expect the fundamental $\{7\}$-string to break due to the simultaneous creation of six gluons, without any intermediate string decay.

Using the Lüscher-Weisz multi-level algorithm, studying string decay in $SU(3)$, $SU(4)$, $Sp(2)$, $G(2)$, and other Yang-Mills theories is interesting and definitely feasible. One may also ask whether string decay can be studied analytically in supersymmetric theories.

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