

Quantum Link Models: A Discrete Approach to Gauge Theories *

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Abstract

We construct lattice gauge theories in which the elements of the link matrices are represented by non-commuting operators acting in a Hilbert space. These quantum link models are related to ordinary lattice gauge theories in the same way as quantum spin models are related to ordinary classical spin systems. Here $U(1)$ and $SU(2)$ quantum link models are constructed explicitly. As Hamiltonian theories quantum link models are nonrelativistic gauge theories with potential applications in condensed matter physics. When formulated with a fifth Euclidean dimension, universality arguments suggest that dimensional reduction to four dimensions occurs. Hence, quantum link models are also reformulations of ordinary quantum field theories and are applicable to particle physics, for example to QCD. The configuration space of quantum link models is discrete and hence their numerical treatment should be simpler than that of ordinary lattice gauge theories with a continuous configuration space.

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1 Introduction

Gauge theories provide the fundamental structure that is used to describe the interactions of elementary particles. For example, the dynamics of the Standard model is formulated as an $SU(3) \otimes SU(2) \otimes U(1)$ gauge theory. Quantum chromodynamics (QCD) — the $SU(3)$ gauge theory of quarks and gluons — is strongly coupled, and hence requires a nonperturbative formulation. At present the only formulation of this kind is provided by the lattice regularization, in which a gauge theory formally resembles a classical statistical mechanics system. The gauge degrees of freedom are then represented by parallel transporters, which are matrices taking values in the gauge group, and which are naturally associated with the links connecting neighboring lattice sites. Gauge symmetries also arise in condensed matter systems. For example, in superconductors the spontaneous break down of the $U(1)$ gauge symmetry of quantum electrodynamics (QED) plays a central role. Furthermore, effective gauge symmetries may be dynamically generated, even though they are not present at the fundamental level of the Standard model. For example, in nonrelativistic quantum Hall fluids an $SU(2)$ gauge symmetry results from the coupling of orbital and spin angular momenta [1]. In the context of condensed matter physics a physical crystal lattice often serves as a regulator. In these cases a nonrelativistic lattice gauge Hamiltonian can be used to describe the system. Lattice gauge symmetries may arise even at macroscopic scales, as for example in man-made arrays of Josephson junctions [2].

Both in elementary particle and in condensed matter physics gauge theories are usually formulated using path integrals. Then the gauge degrees of freedom are described by classical fields, and the theory is analogous to a system of classical statistical mechanics. Here we formulate lattice gauge theories such that the classical statistical mechanics problem is converted into one of quantum statistical mechanics. In other words, the classical Hamilton function (or classical Euclidean action in the context of field theory) is replaced by a Hamilton operator. As a consequence, the elements of the link matrices that are ordinary c-numbers in the standard formulation of lattice gauge theory now turn into non-commuting operators acting in a Hilbert space. In the context of spin models this kind of quantization is well established when one goes from a classical to a quantum spin system. For example, the classical $O(3)$ spin model with 3-component classical unit vectors on each lattice site is replaced by the quantum Heisenberg model, in which each spin is represented by a vector of Pauli matrices [3]. In the same sense, quantum link models are quantized versions of ordinary classical lattice gauge theories. Such models were first constructed by Horn [4] in 1981. In 1990 they were rediscovered under the name of gauge magnets and investigated in more detail by Orland and Rohrlich [5, 6].¹ Having rediscovered these models another time, we go beyond the earlier work by

¹We thank P. Orland for drawing our attention to refs.[4, 5, 6], which we were not aware of in the preprint version of this paper.

showing how quantum link models are related to ordinary quantum field theories via dimensional reduction, and how they may be used to improve numerical simulations of QCD.

The Hamilton operator of a quantum link model resembles the Hamilton function of the corresponding classical system. By construction such a system is nonrelativistic. At present we can only speculate about potential applications of quantum link models to real condensed matter systems. Since a variety of models becomes available, it is of theoretical interest to study their properties, and perhaps relate them to some experimental phenomenon, like, for example, high T_c superconductivity. In any case, quantum link models possess a rich mathematical structure, and they are as general as ordinary lattice gauge theories. Hence, one may expect that Nature has made use of them at some level.

Here we concentrate on the application of quantum link models to elementary particle physics. In this context, the Hamilton operator of the quantum link model replaces the Euclidean action of the corresponding ordinary lattice gauge theory. By construction the symmetries of the quantum link model are the same as the ones of the corresponding ordinary lattice gauge theory. Therefore, if one can take the continuum limit of a quantum link model, universality arguments suggest that it corresponds to the same continuum field theory as its classical counterpart. In this context the Hamilton operator of a quantum link model describes the evolution of the system in a fifth continuous Euclidean dimension that is distinct from Euclidean time, which is part of the lattice. The extent of the extra dimension resembles the inverse coupling constant of the corresponding 4-d lattice gauge theory. Hence, varying the extent of the extra dimension allows one to approach the continuum limit. In this limit dimensional reduction to four dimensions occurs. This suggests that quantum link models are also a reformulation of ordinary gauge theories, and hence are applicable to theories like QCD.

The configuration space of a quantum link model is discrete. Therefore numerical approaches to quantum link models should be simpler than in ordinary lattice gauge theories. In particular, it is easier to store large lattices, and it should require less computer time to generate configurations. Also it is conceivable that efficient numerical techniques, for example cluster algorithms, which do not work for ordinary lattice gauge theories, become available for quantum link models.

The paper is organized as follows. In section 2 we contrast classical with quantum spin systems, and we discuss in which sense the 2-d antiferromagnetic quantum Heisenberg model can be viewed as a discrete realization of the ordinary classical 2-d $O(3)$ model. In section 3 we construct the simplest quantum link model with a $U(1)$ gauge symmetry. Section 4 contains the construction of an $SU(2)$ quantum link model. In section 5 we formulate quantum link models with a fifth Euclidean dimension and discuss their dimensional reduction to ordinary 4-d quantum field theories. Finally, section 6 contains our conclusions.

2 Classical versus Quantum Spin Models

In this section we discuss quantum models in the well established context of spin systems, and we discuss how they are related to their classical counterparts. Later we will generalize these structures to gauge theories. Let us consider standard classical $O(N)$ symmetric spin systems on a d -dimensional cubic lattice with a classical N -component unit vector \vec{s}_x attached to each lattice point x . We discuss this problem in the language of lattice field theory with the classical action (or in the language of classical statistical mechanics the classical Hamilton function) of this model given by

$$S[\vec{s}] = - \sum_{x,\mu} \vec{s}_x \cdot \vec{s}_{x+\hat{\mu}}, \quad (2.1)$$

where $\hat{\mu}$ represents the unit vector in the μ -direction. One is interested in the partition function

$$Z = \int \mathcal{D}\vec{s} \exp\left(-\frac{1}{g}S[\vec{s}]\right), \quad (2.2)$$

which is a path integral over all classical spin field configurations $[\vec{s}]$. Here g is the coupling constant (or equivalently the temperature in classical statistical mechanics language). The $N = 2$ case is the XY -model, and $N = 3$ corresponds to the classical $O(3)$ model. The 2-d XY -model has a Kosterlitz-Thouless transition at a finite value of g , separating a massless spin-wave phase at small coupling from a massive strong coupling phase with a vortex condensate. One can take a continuum limit of this lattice model anywhere in the spin-wave phase. The resulting continuum field theory describes a free massless boson. The 2-d $O(3)$ -model, on the other hand, has only one phase with a nonperturbatively generated mass gap. The mass gap vanishes exponentially as g goes to zero. Hence, the corresponding continuum field theory is asymptotically free.

Heisenberg constructed a quantum version of the $O(3)$ -model by replacing the classical spins \vec{s}_x by quantum spin operators \vec{S}_x . The classical action (or Hamilton function) then turns into the Hamilton operator

$$H = J \sum_{x,\mu} \vec{S}_x \cdot \vec{S}_{x+\hat{\mu}}. \quad (2.3)$$

For $J < 0$ we have a ferromagnet, while $J > 0$ corresponds to an antiferromagnet. It is important to realize that the components of the spin vectors obey the standard commutation relations

$$[S_x^i, S_y^j] = i\delta_{xy}\epsilon_{ijk}S_x^k. \quad (2.4)$$

Of course, spin operators located at different lattice points commute with each other. Consequently, the above Hamilton operator commutes with the total spin

$$\vec{S} = \sum_x \vec{S}_x, \quad (2.5)$$

i.e. $[\vec{S}, H] = 0$, and hence, like the classical model, the quantum spin model has a global $O(3)$ symmetry. Now one is interested in the quantum partition function

$$Z = \text{Tr} \exp(-\beta H). \quad (2.6)$$

The trace is taken in the Hilbert space, which is a direct product of the Hilbert spaces of individual spins.

Quantum spin models can be realized with various representations of the spin. The corresponding theories may behave quite differently. For example, Haldane has conjectured that 1-d antiferromagnetic $O(3)$ quantum spin chains with integer spins have a mass gap, while those with half-integer spins are gapless [7]. In fact, the spin 1/2 antiferromagnetic Heisenberg chain has been solved by the Bethe ansatz, and indeed turns out to have no mass gap [8]. The same has been shown analytically for all half-integer spins [9]. On the other hand, there is strong numerical evidence for a mass gap in spin 1 and spin 2 systems [10]. In the classical limit of large spin S the mass gap vanishes as $m \propto \exp(-\pi S)$ thus approaching a continuum limit. It is interesting that 1-d antiferromagnetic quantum spin chains can be mapped to the 2-d classical $O(3)$ -model with a θ -vacuum term. Integer spins correspond to $\theta = 0$. Then the 2-d $O(3)$ -model indeed has a mass gap that vanishes as $m \propto \exp(-2\pi/g)$ in the low temperature limit. Hence, for large S one may identify the spin of the 1-d quantum model, $S = 2/g$, with the inverse coupling of the 2-d classical model. Half-integer spins correspond to $\theta = \pi$ and it turns out that the mass gap of the $O(3)$ -model then disappears [11]. In that case the 1-d quantum model corresponds to a 2-d conformal field theory — the $k = 1$ Wess-Zumino-Novikov-Witten model [12] — as was first argued by Affleck [13].

Also the 2-d antiferromagnetic spin 1/2 quantum Heisenberg model has very interesting properties. First of all, it describes the precursor insulators of high T_c superconductors — materials like La_2CuO_4 — whose ground states are Néel ordered with a spontaneously generated staggered magnetization. Indeed, there has been early numerical evidence that the ground state of the 2-d antiferromagnetic spin 1/2 quantum Heisenberg model shows long range order [14]. This has been confirmed by a very precise numerical study using a loop cluster algorithm [15]. Recently, the loop cluster algorithm has been reformulated to work in the Euclidean time continuum, allowing high-precision studies of the extreme low temperature limit [16].

Formulating the 2-d quantum model as a path integral in Euclidean time results in a 3-d $O(3)$ -symmetric classical model. At zero temperature of the quantum system we are in the infinite volume limit of the corresponding 3-d classical model. The Néel order of the ground state of the 2-d quantum system implies that the corresponding 3-d classical system is in the broken phase with massless Goldstone bosons — in this case two antiferromagnetic magnons (or spin-waves). One can use chiral perturbation theory to describe the dynamics of the Goldstone bosons at low

energies [17]. To lowest order the effective action then takes the form

$$S[\vec{s}] = \int_0^\beta dt \int d^2x \frac{\rho_s}{2} [\partial_\mu \vec{s} \cdot \partial_\mu \vec{s} + \frac{1}{c^2} \partial_t \vec{s} \cdot \partial_t \vec{s}]. \quad (2.7)$$

Here c and ρ_s are the spin-wave velocity and the spin stiffness. The 2-d quantum system at finite temperature corresponds to the 3-d classical model with finite Euclidean time extent β . For massless particles — in our case the Goldstone bosons — the finite temperature system appears dimensionally reduced to two dimensions, because the finite Euclidean time extent is then negligible compared to the correlation length. However, the Mermin-Wagner-Coleman theorem prevents the existence of interacting massless Goldstone bosons in two dimensions [18]. Indeed, the 2-d $O(3)$ model has a nonperturbatively generated mass gap. Hasenfratz and Niedermayer used a block spin renormalization group transformation to map the 3-d $O(3)$ -model with finite Euclidean time extent β to a 2-d lattice $O(3)$ -model [20]. They averaged the 3-d field over space-time volumes of size β in the Euclidean time direction and βc in the two space directions. Due to the large correlation length the field is essentially constant over these blocks. The averaged field naturally lives at the block centers, which form a 2-d lattice of spacing βc (which is different from the lattice spacing of the underlying quantum antiferromagnet). Hence the effective action of the averaged field defines a 2-d classical lattice $O(3)$ -model. Using chiral perturbation theory, Hasenfratz and Niedermayer expressed its coupling constant as

$$1/g = \beta \rho_s - \frac{3}{16\pi^2 \beta \rho_s} + \mathcal{O}(1/\beta^2 \rho_s^2). \quad (2.8)$$

Using the 3-loop β -function of the 2-d $O(3)$ -model together with its exact mass gap [19], they also extended an earlier result of Chakravarty, Halperin and Nelson [21] for the inverse correlation length of the quantum antiferromagnet to

$$m = \frac{16\pi\rho_s}{ec} \exp(-2\pi\beta\rho_s) [1 + \frac{1}{4\pi\beta\rho_s} + \mathcal{O}(1/\beta^2\rho_s^2)]. \quad (2.9)$$

Here e is the base of the natural logarithm. The above equation resembles the asymptotic scaling behavior of the 2-d classical $O(3)$ -model. Hence, one can view the 2-d antiferromagnetic quantum $O(3)$ -model in the zero temperature limit as a reformulation of the 2-d classical model. It is remarkable that this formulation is entirely discrete, even though the classical model is usually formulated with a continuous configuration space. Further, the quantum model can be treated with very efficient loop cluster algorithm techniques [22, 15]. Defining the path integral for discrete quantum systems does not require discretization of Euclidean time. This observation has recently led to a very efficient loop cluster algorithm operating directly in the Euclidean time continuum [16]. Of course, for the classical $O(3)$ -model the Wolff cluster algorithm is also available [23].

A quantum spin model with $O(2)$ symmetry has also been constructed and is known as the quantum XY model. Its Hamilton operator is given by

$$H = J \sum_{x,\mu} [S_x^1 S_{x+\hat{\mu}}^1 + S_x^2 S_{x+\hat{\mu}}^2], \quad (2.10)$$

in analogy with the corresponding classical model. In this case H only commutes with the third component of the total spin, i.e. $[S^3, H] = 0$. The phenomenon of dimensional reduction also occurs in the quantum XY -model. There is numerical evidence for a Kosterlitz-Thouless transition at finite temperature [24] just like in the classical XY -model. In the low temperature phase the theory describes free massless spin-waves. Hence, again the finite Euclidean time extent of the quantum model is negligible compared to the correlation length. In contrast to the $O(3)$ -model no mass gap is generated, and the Mermin-Wagner-Coleman theorem is evaded, because the massless particles do not interact in this case.

In the following we construct 4-d quantum link models with a gauge symmetry. When formulated as 5-d classical gauge theories with a finite extent in the fifth dimension, these models can be viewed as reformulations of ordinary 4-d gauge theories. Again, the configuration space of the quantum models is entirely discrete, and one may hope that cluster algorithms become available, even though they don't work in the standard formulation.

Finally, let us comment on quantum ferromagnets. These systems have a highly degenerate ground state (even in a finite volume) and a conserved order parameter. As a consequence, the dispersion relation of the corresponding Goldstone bosons is nonrelativistic, and the arguments from above do not apply.

3 The $U(1)$ Quantum Link Model

Let us discuss the simplest quantum link model — quantum pure $U(1)$ gauge theory. The corresponding classical model has a $U(1)$ parallel transporter

$$u_{x,\mu} = \exp(i\varphi_{x,\mu}) = \cos \varphi_{x,\mu} + i \sin \varphi_{x,\mu}, \quad (3.1)$$

associated with each link x, μ . The classical lattice action is given by

$$S[u] = -\frac{1}{2} \sum_{x,\mu>\nu} [u_{x,\mu} u_{x+\hat{\mu},\nu} u_{x+\hat{\nu},\mu}^\dagger u_{x,\nu}^\dagger + u_{x,\nu} u_{x+\hat{\nu},\mu} u_{x+\hat{\mu},\nu}^\dagger u_{x,\mu}^\dagger], \quad (3.2)$$

where a dagger denotes complex conjugation. By construction, the action is invariant under $U(1)$ gauge transformations

$$u'_{x,\mu} = \exp(i\alpha_x) u_{x,\mu} \exp(-i\alpha_{x+\hat{\mu}}). \quad (3.3)$$

One is interested in the partition function as a path integral over classical link configurations

$$Z = \int \mathcal{D}u \exp(-\frac{1}{g^2} S[u]). \quad (3.4)$$

Here g is the gauge coupling. Formally we can think about the system as one of classical statistical mechanics. Then the action $S[u]$ plays the role of the classical

Hamilton function, and g^2 plays the role of the temperature. The 4-d $U(1)$ lattice gauge theory has a phase transition separating a massless Coulomb phase at weak coupling from a massive confined phase with condensed monopoles at large g . The continuum limit of this model can be taken anywhere in the Coulomb phase, resulting in a theory of free massless photons. This is in close analogy to the 2-d classical XY -model.

Let us now construct the quantum counterpart of the standard $U(1)$ gauge theory. We want to replace the classical action by a quantum Hamilton operator

$$H = \frac{J}{2} \sum_{x,\mu>\nu} [U_{x,\mu}U_{x+\hat{\mu},\nu}U_{x+\hat{\nu},\mu}^\dagger U_{x,\nu}^\dagger + U_{x,\nu}U_{x+\hat{\nu},\mu}U_{x+\hat{\mu},\nu}^\dagger U_{x,\mu}^\dagger], \quad (3.5)$$

where $U_{x,\mu}$ now is an operator acting in a Hilbert space — not just a c-number. In quantum mechanics there is no Hermitean operator that replaces a classical angle $\varphi_{x,\mu}$. Instead, one should work with $\cos \varphi_{x,\mu}$ and $\sin \varphi_{x,\mu}$. Hence, we write

$$U_{x,\mu} = C_{x,\mu} + iS_{x,\mu}, \quad (3.6)$$

where $C_{x,\mu}$ and $S_{x,\mu}$ are Hermitean operators yet to be determined. Consequently, we write

$$U_{x,\mu}^\dagger = C_{x,\mu} - iS_{x,\mu}. \quad (3.7)$$

Here the dagger represents Hermitean conjugation in the Hilbert space. The gauge symmetry of the quantum link model requires that the above Hamilton operator commutes with the generators G_x of infinitesimal gauge transformations at each lattice site x . This is satisfied by construction if the quantum link operator transforms as

$$U'_{x,\mu} = \exp(-i\alpha_x G_x) U_{x,\mu} \exp(i\alpha_x G_x) = \exp(i\alpha_x) U_{x,\mu}, \quad (3.8)$$

under gauge transformations from the left, and as

$$U'_{x,\mu} = \exp(-i\alpha_{x+\hat{\mu}} G_{x+\hat{\mu}}) U_{x,\mu} \exp(i\alpha_{x+\hat{\mu}} G_{x+\hat{\mu}}) = U_{x,\mu} \exp(-i\alpha_{x+\hat{\mu}}), \quad (3.9)$$

under gauge transformations from the right. The unitary operator that represents a general gauge transformation then is $\prod_x \exp(i\alpha_x G_x)$ such that

$$U'_{x,\mu} = \prod_y \exp(-i\alpha_y G_y) U_{x,\mu} \prod_z \exp(i\alpha_z G_z) = \exp(i\alpha_x) U_{x,\mu} \exp(-i\alpha_{x+\hat{\mu}}). \quad (3.10)$$

The above structure implies the following commutation relations

$$\begin{aligned} [G_x, C_{y,\mu}] &= i(\delta_{x,y+\hat{\mu}} - \delta_{x,y}) S_{y,\mu}, \\ [G_x, S_{y,\mu}] &= i(\delta_{x,y} - \delta_{x,y+\hat{\mu}}) C_{y,\mu}, \end{aligned} \quad (3.11)$$

and hence

$$\begin{aligned} [G_x, U_{y,\mu}] &= (\delta_{x,y+\hat{\mu}} - \delta_{x,y}) U_{y,\mu}, \\ [G_x, U_{y,\mu}^\dagger] &= (\delta_{x,y} - \delta_{x,y+\hat{\mu}}) U_{y,\mu}^\dagger. \end{aligned} \quad (3.12)$$

It is straightforward to show that these relations are satisfied when one identifies

$$C_{x,\mu} = S_{x,\mu}^1, \quad S_{x,\mu} = S_{x,\mu}^2, \quad (3.13)$$

and

$$G_x = \sum_{\mu} (S_{x-\hat{\mu},\mu}^3 - S_{x,\mu}^3), \quad (3.14)$$

where $\vec{S}_{x,\mu}$ obeys angular momentum commutation relations, i.e.

$$[S_{x,\mu}^i, S_{y,\nu}^j] = i\delta_{x,y}\delta_{\mu\nu}\epsilon_{ijk}S_{x,\mu}^k. \quad (3.15)$$

We can now identify

$$\begin{aligned} U_{x,\mu} &= C_{x,\mu} + iS_{x,\mu} = S_{x,\mu}^1 + iS_{x,\mu}^2 = S_{x,\mu}^+, \\ U_{x,\mu}^\dagger &= C_{x,\mu} - iS_{x,\mu} = S_{x,\mu}^1 - iS_{x,\mu}^2 = S_{x,\mu}^-, \end{aligned} \quad (3.16)$$

i.e. a link variable is represented by a raising operator $S_{x,\mu}^+$, and its inverse by the lowering operator $S_{x,\mu}^-$. Like for quantum spin systems, the above commutation relations can be realized with any representation of $SU(2)$. In the simplest case one can use Pauli matrices on each link. Then the Hilbert space of the model is the direct product of 2-dimensional link Hilbert spaces.

By construction the above Hamilton operator is invariant under gauge transformations, i.e. $[G_x, H] = 0$ for all x . Also the generators of gauge transformations commute, i.e. $[G_x, G_y] = 0$. Hence, the eigenstates of H can be characterized by the eigenvalues of all G_x , i.e. there is a conserved quantity at each lattice site. In gauge theories Gauss' law restricts the physical Hilbert space to gauge invariant states $|\Psi\rangle$. In our formulation this means

$$G_x|\Psi\rangle = 0. \quad (3.17)$$

Let us contrast our construction with the Hamiltonian formulation of ordinary lattice gauge theories. There one often chooses an electric flux basis of the physical Hilbert space. In an ordinary $U(1)$ lattice gauge theory the electric flux is quantized in integer units. Gauss' law requires that the fluxes associated with links emanating from the same lattice point add up to zero. The electric part of the Hamiltonian is diagonal in the electric flux basis. The magnetic part associated with spatial plaquettes, on the other hand, induces a shift of the electric flux on all links of the plaquette. This is in close analogy to the $U(1)$ quantum link model. In particular, one can identify the eigenvalues of $S_{x,\mu}^3$ with electric fluxes associated with the links. Again, the Hamiltonian induces shifts in the electric fluxes around a plaquette. However, in this case the fluxes, being the eigenvalues of $S_{x,\mu}^3$, are restricted to a finite set (for example to $\pm 1/2$ when one chooses the fundamental representation of $SU(2)$). Thus, in contrast to ordinary lattice gauge theories the Hilbert space of a quantum link model is finite (on a finite lattice).

At this point we have constructed what we call the $U(1)$ quantum link model. As for ordinary lattice gauge models, solving the theory is a complicated problem. In two dimensions this has been discussed in ref.[6]. Before we discuss the dynamics let us construct an example of a nonabelian quantum link model.

4 The $SU(2)$ Quantum Link Model

Let us first recall standard $SU(2)$ lattice gauge theory. In that case there is an $SU(2)$ matrix

$$u_{x,\mu} = u_{x,\mu}^0 + i\vec{u}_{x,\mu} \cdot \vec{\sigma}, \quad (4.1)$$

associated with each link. Here $\vec{\sigma}$ is a vector of Pauli matrices. Further, $u_{x,\mu}^0$ and $\vec{u}_{x,\mu}$ are real and satisfy the constraint $u_{x,\mu}^0 u_{x,\mu}^0 + \vec{u}_{x,\mu} \cdot \vec{u}_{x,\mu} = 1$. The action is given by

$$S[u] = - \sum_{x,\mu>\nu} \text{Tr}[u_{x,\mu} u_{x+\hat{\mu},\nu} u_{x+\hat{\nu},\mu}^\dagger u_{x,\nu}^\dagger + u_{x,\nu} u_{x+\hat{\nu},\mu} u_{x+\hat{\mu},\nu}^\dagger u_{x,\mu}^\dagger], \quad (4.2)$$

where the dagger denotes Hermitean conjugation. The action is invariant under $SU(2)$ gauge transformations

$$u'_{x,\mu} = \exp(i\vec{\alpha}_x \cdot \vec{\sigma}) u_{x,\mu} \exp(-i\vec{\alpha}_{x+\hat{\mu}} \cdot \vec{\sigma}). \quad (4.3)$$

Again, the path integral is given by

$$Z = \int \mathcal{D}u \exp(-\frac{1}{g^2} S[u]), \quad (4.4)$$

where g is the nonabelian gauge coupling. In contrast to $U(1)$ gauge theory, $SU(2)$ gauge theory in four dimensions has only one phase, in which the gluons are confined. This is analogous to the 2-d $O(3)$ -model.

As in the $U(1)$ case we can construct a quantum version of the $SU(2)$ model by replacing the classical action by a Hamilton operator

$$H = J \sum_{x,\mu>\nu} \text{Tr}[U_{x,\mu} U_{x+\hat{\mu},\nu} U_{x+\hat{\nu},\mu}^\dagger U_{x,\nu}^\dagger + U_{x,\nu} U_{x+\hat{\nu},\mu} U_{x+\hat{\mu},\nu}^\dagger U_{x,\mu}^\dagger]. \quad (4.5)$$

Here the elements of the 2×2 link matrices $U_{x,\mu}$ are operators acting in a Hilbert space. Naturally, the dagger now represents Hermitean conjugation in both the Hilbert space and the 2×2 matrix space. In analogy to the classical expression of eq.(4.1) we write

$$U_{x,\mu} = U_{x,\mu}^0 + i\vec{U}_{x,\mu} \cdot \vec{\sigma}, \quad (4.6)$$

where $U_{x,\mu}^0$ and $\vec{U}_{x,\mu}$ are Hermitean operators. We also have

$$U_{x,\mu}^\dagger = U_{x,\mu}^0 - i\vec{U}_{x,\mu} \cdot \vec{\sigma}. \quad (4.7)$$

In analogy with the $U(1)$ case, gauge covariance requires

$$U'_{x,\mu} = \prod_y \exp(-i\vec{\alpha}_y \cdot \vec{G}_y) U_{x,\mu} \prod_z \exp(i\vec{\alpha}_z \cdot \vec{G}_z) = \exp(i\vec{\alpha}_x \cdot \vec{\sigma}) U_{x,\mu} \exp(-i\vec{\alpha}_{x+\hat{\mu}} \cdot \vec{\sigma}). \quad (4.8)$$

This implies the following commutation relations

$$\begin{aligned} [\vec{G}_x, U_{y,\mu}] &= \delta_{x,y+\hat{\mu}} U_{y,\mu} \vec{\sigma} - \delta_{x,y} \vec{\sigma} U_{y,\mu}, \\ [\vec{G}_x, U_{y,\mu}^\dagger] &= \delta_{x,y} U_{y,\mu}^\dagger \vec{\sigma} - \delta_{x,y+\hat{\mu}} \vec{\sigma} U_{y,\mu}^\dagger \end{aligned} \quad (4.9)$$

One way to find representations that satisfy these relations is to define

$$\vec{G}_x = \sum_\mu (\vec{R}_{x-\hat{\mu},\mu} + \vec{L}_{x,\mu}). \quad (4.10)$$

Here $\vec{R}_{x,\mu}$ and $\vec{L}_{x,\mu}$ are generators of right and left gauge transformations of the link variable $U_{x,\mu}$. As such, they obey the following commutation relations

$$\begin{aligned} [R_{x,\mu}^i, R_{y,\nu}^j] &= 2i\delta_{x,y}\delta_{\mu\nu}\epsilon_{ijk}R_{x,\mu}^k, \\ [L_{x,\mu}^i, L_{y,\nu}^j] &= 2i\delta_{x,y}\delta_{\mu\nu}\epsilon_{ijk}L_{x,\mu}^k, \\ [R_{x,\mu}^i, L_{y,\nu}^j] &= 0, \end{aligned} \quad (4.11)$$

i.e. $\vec{R}_{x,\mu}$ and $\vec{L}_{x,\mu}$ generate an $SU(2)_R \otimes SU(2)_L$ algebra on each link. The commutation relations of eq.(4.9) imply

$$\begin{aligned} [\vec{R}_{x,\mu}, U_{y,\nu}] &= \delta_{x,y}\delta_{\mu\nu}U_{x,\mu}\vec{\sigma}, \\ [\vec{L}_{x,\mu}, U_{y,\nu}] &= -\delta_{x,y}\delta_{\mu\nu}\vec{\sigma}U_{x,\mu}. \end{aligned} \quad (4.12)$$

For each link the above relations can be realized by using the generators of an $SO(5)$ algebra, with the $SU(2)_L \otimes SU(2)_R$ algebra embedded in it. In the spinorial representation, for example, the ten generators of $SO(5)$ can be written as

$$\begin{aligned} \vec{R} &= \begin{pmatrix} \vec{\tau} & 0 \\ 0 & 0 \end{pmatrix}, \quad \vec{L} = \begin{pmatrix} 0 & 0 \\ 0 & \vec{\tau} \end{pmatrix}, \\ U^0 &= \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \vec{U} = \begin{pmatrix} 0 & -i\vec{\tau} \\ i\vec{\tau} & 0 \end{pmatrix}. \end{aligned} \quad (4.13)$$

Here $\mathbf{1}$ is a 2×2 unit matrix, and $\vec{\tau}$ is a vector of Pauli matrices (not to be confused with $\vec{\sigma}$, which acts in a different space). Note that (U^0, \vec{U}) resembles a four-vector of Euclidean Dirac matrices. The commutators of the components of the link matrices then take the form

$$\begin{aligned} [U_{x,\mu}^0, U_{y,\nu}^0] &= 0, \\ [U_{x,\mu}^0, U_{y,\nu}^i] &= 2i\delta_{x,y}\delta_{\mu\nu}(R_{x,\mu}^i - L_{x,\mu}^i), \\ [U_{x,\mu}^i, U_{y,\nu}^j] &= 2i\delta_{x,y}\delta_{\mu\nu}\epsilon_{ijk}(R_{x,\mu}^k + L_{x,\mu}^k). \end{aligned} \quad (4.14)$$

The above commutation relations can be realized with any representation of $SO(5)$. The simplest choice is the spinorial representation that was used above. Then the Hilbert space of the model is the direct product of 4-dimensional link Hilbert spaces. By construction we have $[\vec{G}_x, H] = 0$. To impose the Gauss law in the $SU(2)$ case one requires

$$\vec{G}_x|\Psi\rangle = 0, \quad (4.15)$$

for all physical states $|\Psi\rangle$.

5 Reduction from five to four Dimensions

As we have seen, a 2-d $O(3)$ quantum spin model can be viewed as a discrete reformulation of a 2-d classical $O(3)$ spin model, because dimensional reduction from three to two dimensions occurs. We also know that the 2-d quantum $O(3)$ -model at finite temperature corresponds to a 3-d classical $O(3)$ -model with finite extent β in the Euclidean time direction. Due to the Néel order of the underlying 2-d quantum antiferromagnet the theory is in the broken phase with massless Goldstone bosons. In the leading order of chiral perturbation theory the effective action of the 3-d model takes the form

$$S[\vec{s}] = \int_0^\beta dt \int d^2x \frac{\rho_s}{2} [\partial_\mu \vec{s} \cdot \partial_\mu \vec{s} + \frac{1}{c^2} \partial_t \vec{s} \cdot \partial_t \vec{s}]. \quad (5.1)$$

In fact, the extent of the third dimension — the inverse temperature β of the quantum antiferromagnet — resembles the inverse coupling of the induced 2-d classical model, i.e. at large β

$$1/g = \beta \rho_s. \quad (5.2)$$

Thus, the continuum limit of the asymptotically free 2-d classical $O(3)$ -model at $g \rightarrow 0$ corresponds to the zero temperature limit of the 2-d quantum antiferromagnet, which also corresponds to the infinite volume limit ($\beta \rightarrow \infty$) of the 3-d classical model. When β is finite, the theory is effectively two dimensional and the Mermin-Wagner-Coleman theorem implies that the Goldstone bosons then pick up a mass

$$m \propto \exp(-2\pi\beta\rho_s). \quad (5.3)$$

This is consistent with the asymptotic freedom of the 2-d classical $O(3)$ -model. The correlation length $1/m$ is exponentially large compared to the Euclidean time extent β , and hence, in contrast to naive expectations, dimensional reduction occurs at large β .

The above observations imply an interesting relation between $O(3)$ -models in two and three dimensions. In the infinite volume ($\beta = \infty$) the 3-d model describes the physics of massless Goldstone bosons. The corresponding fixed point of the renormalization group resembles a conformal field theory. Once we make β finite

(which implies $g > 0$) we explore a relevant direction in the vicinity of this fixed point. Approaching the fixed point along the relevant direction yields the 2-d $O(3)$ -model. In the following we argue that nonabelian gauge theories in four and five dimensions are related in a similar way.

Nonabelian lattice gauge theories in five dimensions have a confinement phase at strong coupling, which is separated from a massless weak coupling phase [25]. When a gauge theory is dimensionally reduced, usually the Polyakov loop in the extra dimension appears as an adjoint scalar field. Here we want to obtain pure 4-d Yang-Mills theory (without charged scalars) after dimensional reduction. This can be achieved if one does not impose Gauss' law for the states propagating in the fifth dimension, because the Polyakov loop is a Lagrange multiplier field that enforces the Gauss law. Formally, this can be realized simply by putting the fifth component of the gauge potential to zero, i.e.

$$A_5 = 0. \quad (5.4)$$

In the infinite volume limit of the 5-d theory this restriction has no effect on the dynamics. With finite extent in the fifth direction, however, it deviates from the standard formulation of gauge theories. The leading terms in the effective action of the 5-d gauge theory take the form

$$S[A_\mu] = \int_0^\beta dx_5 \int d^4x \frac{1}{2e^2} [\text{Tr } F_{\mu\nu} F_{\mu\nu} + \frac{1}{c^2} \text{Tr } \partial_5 A_\mu \partial_5 A_\mu]. \quad (5.5)$$

Here e is the dimensionful gauge coupling, which is analogous to ρ_s from eq.(2.7) for quantum antiferromagnets, and c is the velocity of light of the 5-d theory. Note that μ runs over 4-d indices only. At finite β the above theory has only a 4-d gauge invariance, because we have fixed $A_5 = 0$, i.e. we have not imposed Gauss' law. On the other hand, for $\beta = \infty$ a full 5-d gauge symmetry is recovered, although the above action then still is in $A_5 = 0$ gauge. Since we are interested in dimensional reduction, a 4-d gauge symmetry is sufficient for our purposes. In analogy to eq.(5.2) for large β the gauge coupling of the induced 4-d theory is given by

$$1/g^2 = \beta/e^2. \quad (5.6)$$

For spin models the Mermin-Wagner-Coleman theorem implies that in the 3-d theory with finite extent in the third direction the Goldstone bosons acquire mass nonperturbatively. In gauge theories, on the other hand, an analogous theorem, stating that massless gauge bosons cannot exist in four dimensions unless they do not interact with each other, has not yet been proven. In fact, proving such a theorem would mean proving confinement. However, we can turn the argument around and use the confinement hypothesis to argue that the dimensionally reduced 5-d theory indeed has a nonperturbatively generated mass gap. Using the β -function of $SU(2)$ gauge theory, in analogy with the $O(3)$ -model, we expect

$$m \propto \exp\left(-\frac{12\pi^2\beta}{11e^2}\right). \quad (5.7)$$

Thus starting from a 5-d nonabelian gauge theory in the massless phase one can obtain the corresponding 4-d nonabelian gauge theory by making the extent β of the fifth dimension finite. In fact, β plays the role of the inverse gauge coupling of the 4-d theory, and hence, due to asymptotic freedom, we are interested in the large β limit. As before, in contrast to naive expectations, dimensional reduction occurs when the extent of the fifth dimension becomes large. Again, we want to emphasize that it was important not to impose Gauss' law, i.e. to put $A_5 = 0$.

Let us also discuss dimensional reduction for Abelian theories. The 2-d quantum XY -model has a Kosterlitz-Thouless transition with a massless phase at low temperatures. The corresponding 3-d classical model with a finite extent in the third direction has an $O(2)$ symmetry with free massless particles. In particular, no mass gap is generated in this case. This is not in conflict with the Mermin-Wagner-Coleman theorem because the particles do not interact. Again, dimensional reduction occurs — now already at finite β — and the resulting 2-d theory is the classical XY -model. This is analogous to what happens in Abelian gauge theories between five and four dimensions. A 5-d Abelian gauge theory with finite extent in the fifth direction and with $A_5 = 0$ has massless photons. After dimensional reduction we end up in the Coulomb phase of 4-d Abelian gauge theory. On the lattice compact $U(1)$ gauge theory has a phase transition that separates the Coulomb phase at weak coupling from a confined phase with condensed monopoles. In the Coulomb phase the monopoles have a mass of the order of the cut-off. Hence, if one takes the continuum limit anywhere in the Coulomb phase one obtains a free theory of photons.

Why have we formulated 4-d gauge theory in a 5-d context? First of all, the reformulation may shed some light on the structure of fixed points in four and five dimensional gauge theories. In the context of standard approaches to lattice gauge theory one would probably prefer to work directly in four dimensions. However, for quantum link models the situation is different. We have defined these models as Hamiltonian theories. Of course, one can also define a Hamiltonian for ordinary gauge theories. In that case the spectrum of the Hamilton operator defined on the 3-d space reflects Poincaré invariance of the corresponding 4-d action. On the other hand, by construction quantum link models are nonrelativistic gauge theories with no symmetry between space and time. Hence, a quantum link model with a Hamilton operator defined on a 3-d spatial lattice can in general not describe a system of elementary particles, simply because its spectrum does not reflect Poincaré invariance. However, in analogy to 2-d quantum spin systems, we expect that in the special case of quantum link models defined on a 4-d lattice, there are massless modes with a relativistic dispersion relation characterized by the velocity of light c of the corresponding 5-d theory. The massless modes correspond to the deconfined gauge bosons in the weak coupling phase of a 5-d gauge theory. Of course, we cannot interpret the fifth Euclidean direction as time, and hence the spectrum of the 4-d Hamilton operator of the quantum link model is not the physical spectrum.

However, when we make the extent β of the extra dimension finite, we can make use of the above scenario of dimensional reduction, in which β plays the role of the inverse coupling constant of the resulting 4-d gauge theory. In that case one must not impose Gauss' law ($\vec{G}_x|\Psi\rangle = 0$), i.e. gauge variant states also propagate in the fifth direction. We are then interested in the quantum statistical partition function

$$Z = \text{Tr} \exp(-\beta H). \quad (5.8)$$

In contrast to the standard formulation of gauge theories we have not included a projection operator on gauge invariant states. From this point of view the Hamilton operator of the quantum link model is defined on a 4-d space-time lattice, and describes the evolution of the system in the fifth unphysical direction. In particular, all the information about the physical spectrum of the 4-d theory is contained in correlation functions in the Euclidean time direction, which is part of the 4-d lattice. In the continuum limit $g \rightarrow 0$, which we approach by increasing the extent β of the fifth dimension, we are probing the low lying states in the spectrum of the 4-d Hamilton operator of the quantum link model. The space-time correlations in these unphysical states of the 4-d Hamiltonian contain the information about the physical spectrum.

The partition function of eq.(5.8) can be written as a 5-d path integral of discrete variables — in the $SU(2)$ case the eigenstates of the diagonal generators of $SO(5)$ on each link. In many respects this path integral resembles that of quantum spin systems, which can be simulated by very efficient loop cluster algorithms. Due to the discrete nature of the Hilbert space, one can even work directly in the continuum for the extra Euclidean direction [16]. It is plausible that cluster algorithms can also be constructed for quantum link models, which would allow high precision simulations in gauge theories.

6 Conclusions

Quantum link models are another class of lattice gauge theories with applications in particle and possibly also in condensed matter physics. In the context of particle physics quantum link models formulated with a fifth Euclidean dimension of finite extent resemble ordinary 4-d gauge theories. From the point of view of numerical simulations it may seem easier to work directly in four dimensions using the standard formulation of lattice gauge theories. However, it could be advantageous to work in five dimensions, because the existence of cluster algorithms seems plausible for quantum link models. Due to the discrete nature of quantum link models, a discretization of the fifth direction is not even necessary. The path integral can be defined directly in the continuum, and can perhaps be simulated with an algorithm analogous to the one for quantum spins [16].

Although in this paper we have presented explicit constructions only for pure $U(1)$ and $SU(2)$ quantum link models, it is straightforward to construct models for other gauge groups, and with couplings to charged matter fields. In fact, we have also constructed $U(N)$ quantum link models, quantum Higgs models, as well as quantum $CP(N)$ -models [26]. The inclusion of quarks is a nontrivial issue, which is presently under investigation.

At present, we can only speculate about applications of quantum link models to condensed matter physics. However, due to their general structure, we believe that they will be at least as useful as ordinary gauge theories. As we have seen, there are close analogies between quantum spin systems in two dimensions and quantum link models in four dimensions. Perhaps there are similar analogies between 1-d quantum spin chains and 3-d quantum link models. In fact, we find it plausible that Haldane's conjecture has a gauge analog. Perhaps a 3-d $SU(2)$ quantum link model with the spinorial representation of $SO(5)$ on each link corresponds to a 4-d $SU(2)$ lattice gauge theory with a nontrivial θ -vacuum angle, while in the vector representation the corresponding vacuum angle might vanish. If so, one could learn about the effect of θ by solving the 3-d quantum link model, which may be possible if cluster algorithms become available. One can also imagine to extend Haldane's conjecture to 1-d quantum $CP(N)$ -models in a similar way. Furthermore, quantum link models allow us to gauge the standard quantum models of condensed matter physics, for example the Heisenberg model or the Hubbard model. Such models equipped with a lattice gauge symmetry may eventually be useful to describe phenomena like, for example, high T_c superconductivity or the quantum Hall effect.

In conclusion, there is a whole class of models in various dimensions and with various symmetries that are of theoretical, and in some cases even of phenomenological interest. A lot of work needs to be done before it will be clear how useful quantum link models are in particle and condensed matter physics.

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