

Minimal Position-Velocity Uncertainty Wave Packets in Relativistic and Non-relativistic Quantum Mechanics

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Abstract

We consider wave packets of free particles with a general energy-momentum dispersion relation $E(p)$. The spreading of the wave packet is determined by the velocity $v = \partial_p E$. The position-velocity uncertainty relation $\Delta x \Delta v \geq \frac{1}{2} |\langle \partial_p^2 E \rangle|$ is saturated by minimal uncertainty wave packets $\Phi(p) = A \exp(-\alpha E(p) + \beta p)$. In addition to the standard minimal Gaussian wave packets corresponding to the non-relativistic dispersion relation $E(p) = p^2/2m$, analytic calculations are presented for the spreading of wave packets with minimal position-velocity uncertainty product for the lattice dispersion relation $E(p) = -\cos(pa)/ma^2$ as well as for the relativistic dispersion relation $E(p) = \sqrt{p^2 + m^2}$. The boost properties of moving relativistic wave packets as well as the propagation of wave packets in an expanding Universe are also discussed.

1 Introduction

The spreading of Gaussian wave packets is a standard topic discussed in almost any textbook on quantum mechanics. While there still are interesting investigations and applications of wave packet dynamics [1, 2], it may seem unlikely that anything new can be said theoretically about a topic as elementary as this. Indeed, some of the results that will be presented below have been discussed before. Still, to the best of our knowledge, most of our results, although easy to derive, are new and seem to have escaped the attention of quantum physicists. The main goal of this paper is to present a general discussion of wave packet spreading, which generalizes the standard Gaussian wave packet describing a free non-relativistic particle to other minimal position-velocity uncertainty wave packets with a general energy-momentum dispersion relation $E(p)$.

As was first discussed by Ehrenfest [3], a free non-relativistic particle moving in one spatial dimension shows the following time-dependence of the position expectation value

$$\langle x \rangle(t) = \langle x \rangle(0) + \langle v \rangle t, \quad (1.1)$$

where $v = \partial_p E$ is the corresponding velocity. It is easy to show that this relation holds for any (relativistic or non-relativistic) dispersion relation $E(p)$. Similarly, one can show that the position uncertainty $\Delta x(t) = \sqrt{\langle x^2 \rangle(t) - \langle x \rangle(t)^2}$ varies as

$$\Delta x(t)^2 = \Delta x(0)^2 + [\langle vx + xv \rangle(0) - 2\langle v \rangle \langle x \rangle(0)] t + (\Delta v)^2 t^2. \quad (1.2)$$

This implies that the asymptotic speed of spreading for general wave packets is given by $\Delta v = \sqrt{\langle v^2 \rangle - \langle v \rangle^2}$. Furthermore, if position and velocity are correlated such that initially $\langle vx + xv \rangle(0) < 2\langle v \rangle \langle x \rangle(0)$, the wave packet is shrinking before it begins to spread.

It is interesting to investigate wave packets of minimal position-velocity uncertainty product $\Delta x \Delta v$. Putting $\hbar = 1$, one easily derives the generalized position-velocity uncertainty relation

$$\Delta x \Delta v \leq \frac{1}{2} |\langle \partial_p^2 E \rangle|, \quad (1.3)$$

which is valid for any dispersion relation, and which reduces to the standard Heisenberg uncertainty relation $\Delta x \Delta v \leq \frac{1}{2m}$ for a non-relativistic particle with energy $E(p) = p^2/2m$. For a particle hopping between neighboring sites on a lattice with spacing a , the dispersion relation is $E(p) = -\cos(pa)/ma^2$ and the uncertainty relation then takes the form

$$\Delta x \Delta v \leq \frac{a^2}{2} |\langle E \rangle|. \quad (1.4)$$

On the other hand, (also putting $c = 1$) for a relativistic particle with $E(p) = \sqrt{p^2 + m^2}$ the uncertainty relation reduces to

$$\Delta x \Delta v \leq \frac{m^2}{2} \langle E^{-3} \rangle. \quad (1.5)$$

For a general dispersion relation, minimal uncertainty wave packets $\Phi(p)$ saturate the inequality eq.(1.3) and obey the equation

$$(\partial_p + \alpha v - \beta) \Phi(p) = 0, \quad (1.6)$$

with

$$\alpha = \frac{1}{2(\Delta v)^2} |\langle \partial_p^2 E \rangle|, \quad \beta = \alpha \langle v \rangle - i \langle x \rangle. \quad (1.7)$$

In momentum space, they take the form

$$\Phi(p) = A \exp(-\alpha E(p) + \beta p). \quad (1.8)$$

In this paper, minimal uncertainty wave packets are constructed explicitly and their time-evolution is calculated analytically, both for the lattice and for the relativistic dispersion relation.

Several results of this paper belong to relativistic quantum mechanics. It should be noted that both the Dirac and the Klein-Gordon equation belong to quantum field theory and not to relativistic quantum mechanics. In particular, it is well-known that, due to the existence of negative energy solutions, a purely quantum mechanical single particle interpretation of these equations leads to problems such as the Klein paradox. When we discuss the relativistic quantum mechanics of a single free particle, we consider the Hamiltonian

$$H = \sqrt{p^2 + m^2}, \quad (1.9)$$

which only has positive energy solutions. While we know that the correct description of Nature at the most fundamental level accessible today is provided by the standard model of particle physics — which is a relativistic quantum field theory — there is nothing wrong with studying the Hamiltonian of eq.(1.9). In particular, while the corresponding square root would conflict with locality in quantum field theory, it is perfectly acceptable in the framework of single particle quantum mechanics. In fact, as will be discussed in appendix B, the above Hamiltonian correctly describes the single particle states of a free scalar quantum field theory. As borne out by the Reeh-Schlieder theorem [4], the localization of particles is a subtle issue in quantum field theory. The interpretation of the results obtained in relativistic quantum mechanics must take this into account. In particular, apparent violations of Einstein causality in relativistic quantum mechanics turn out to be unproblematical when viewed from this perspective [5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

In contrast to quantum field theory, relativistic quantum mechanics (as characterized above) seems not to be a well-studied subject. This is a pity, because relativistic quantum mechanics may help to bridge the large gap between non-relativistic quantum mechanics and relativistic quantum field theory that makes learning the latter rather non-trivial. Also for pedagogical reasons, it thus seems worthwhile to study relativistic quantum mechanics — even of just a single particle. Due to

Lorentz invariance, the possible interactions in relativistic quantum mechanics are highly constrained. In particular, if one attempts to include interactions only in the Hamiltonian and in the boost operator (but not in the momentum or angular momentum operators) of a multi-particle system, at the classical level Leutwyler has proved that only the free theory is consistent with Lorentz invariance [15]. The same was first shown for two particles by Currie, Jordan, and Sudarshan [16]. By including the interaction in the momentum (and not in the boost operator), interesting relativistic systems with a fixed number of interacting particles have been constructed and investigated in detail [17, 18, 19]. Here we concentrate on wave packet spreading of a single free relativistic particle. Although in this paper we do not focus on applications, the resulting expressions may be useful in studies of neutrino wave packets [20, 21]. For this reason, we also consider the spreading of wave packets in an expanding Universe.

The rest of the paper is organized as follows. In section 2 the time-dependence of position expectation values is considered for a general dispersion relation, the corresponding position-velocity uncertainty relation is derived, and the general minimal uncertainty wave packet is constructed. Section 3 investigates wave packet spreading in the non-relativistic case, both in the continuum and for a lattice dispersion relation. The spreading of minimal position-velocity uncertainty wave packets for a free relativistic particle is discussed in detail in section 4. Section 5 extends the discussion to wave packets propagating in an expanding Universe. Finally, section 6 contains our conclusions. Some non-trivial expectation values are evaluated in appendix A. The relation of relativistic quantum mechanics to quantum field theory and the related issues of particle localization are reviewed in appendix B.

2 Spreading of General Wave Packets

In this section we investigate general properties of spreading wave packets. We also derive a position-velocity uncertainty relation and we consider wave packets with a minimal position-velocity uncertainty product. While the results presented in this section are quite elementary, except for those in subsection 2.1, we have not been able to locate them in the physics literature. Although it seems likely that some of the material has been discussed elsewhere, it seems not to be well-known and may thus be worth studying in some detail.

2.1 Time-Evolution of Position Expectation Values

Let us consider a single free particle in one spatial dimension with momentum p and energy $E(p)$. The time-evolution of an initial wave function $\Psi(p)$ in momentum

space is then given by

$$\Psi(p, t) = \Psi(p) \exp(-iE(p)t). \quad (2.1)$$

The corresponding wave function in coordinate space takes the form

$$\begin{aligned} \Psi(x, t) &= \frac{1}{2\pi} \int dp \Psi(p, t) \exp(ipx) = \frac{1}{2\pi} \int dp \Psi(p) \exp(-iE(p)t + ipx) \\ &= \int dx' G(x - x', t) \Psi(x', 0), \end{aligned} \quad (2.2)$$

with the Green's function given by

$$G(x, t) = \frac{1}{2\pi} \int dp \exp(-iE(p)t + ipx). \quad (2.3)$$

Using $x = i\partial_p$ one then obtains

$$\begin{aligned} \langle x \rangle(t) &= \frac{1}{2\pi} \int dp \Psi(p, t)^* i\partial_p \Psi(p, t) = \frac{1}{2\pi} \int dp \Psi(p)^* (i\partial_p + \partial_p E t) \Psi(p) \\ &= \langle x \rangle(0) + \langle v \rangle t, \end{aligned} \quad (2.4)$$

where $v = \partial_p E$ is the particle's velocity. Similarly, one finds

$$\begin{aligned} \langle x^2 \rangle(t) &= \frac{1}{2\pi} \int dp \Psi(p, t)^* (i\partial_p)^2 \Psi(p, t) \\ &= \frac{1}{2\pi} \int dp \Psi(p)^* [(i\partial_p)^2 + (2\partial_p E i\partial_p + i\partial_p^2 E) t + (\partial_p E)^2 t^2] \Psi(p) \\ &= \langle x^2 \rangle(0) + \langle vx + xv \rangle(0)t + \langle v^2 \rangle t^2, \end{aligned} \quad (2.5)$$

where we have used

$$[x, v] = [i\partial_p, \partial_p E] = i\partial_p^2 E. \quad (2.6)$$

Combining eq.(2.4) with eq.(2.5) one then obtains

$$\Delta x(t)^2 = \Delta x(0)^2 + [\langle vx + xv \rangle(0) - 2\langle v \rangle \langle x \rangle(0)] t + (\Delta v)^2 t^2. \quad (2.7)$$

The sign of the connected position-velocity correlation $\langle vx + xv \rangle(0) - 2\langle v \rangle \langle x \rangle(0)$ determines whether the wave packet is initially spreading or shrinking. Asymptotically, for large times the packet is spreading with the velocity Δv , i.e. the velocity uncertainty determines the velocity of spreading. The time-dependence of moments of the position operator is well-known [1, 3] and has also been considered, for example, in [22].

2.2 General Position-Velocity Uncertainty Relation

As we have just seen, the spreading of general wave packets is controlled by the uncertainties Δx and Δv of position and velocity. This suggests to consider packets

with a minimal position-velocity uncertainty product. Before we construct such wave packets, let us derive a generalization of the non-relativistic Heisenberg uncertainty relation

$$\Delta x \Delta v = \frac{1}{m} \Delta x \Delta p \geq \frac{1}{2m} \quad (2.8)$$

to an arbitrary dispersion relation $E(p)$ with velocity $v = \partial_p E$. For this purpose, we define the operator

$$a = -ix + \alpha v - \beta = \partial_p + \alpha v - \beta, \quad (2.9)$$

(with $\alpha \in \mathbb{R}$ and $\beta = \beta_r + i\beta_i \in \mathbb{C}$ as arbitrary adjustable parameters) and we evaluate

$$\begin{aligned} \langle a^\dagger a \rangle &= \langle (ix + \alpha v - \beta^*) (-ix + \alpha v - \beta) \rangle \\ &= \langle x^2 + \alpha^2 v^2 + |\beta|^2 + i\alpha[x, v] - i(\beta - \beta^*)x - \alpha(\beta + \beta^*)v \rangle \\ &= \langle x^2 \rangle + \alpha^2 \langle v^2 \rangle + \beta_r^2 + \beta_i^2 - \alpha \langle \partial_p^2 E \rangle + 2\beta_i \langle x \rangle - 2\alpha\beta_r \langle v \rangle \geq 0. \end{aligned} \quad (2.10)$$

By construction $\langle a^\dagger a \rangle \geq 0$. In order to obtain the most stringent bound, we now vary the free parameters α , β_r , and β_i such that $\langle a^\dagger a \rangle$ is minimized. This implies the conditions

$$\begin{aligned} \frac{\partial \langle a^\dagger a \rangle}{\partial \alpha} &= 2\alpha \langle v^2 \rangle - \langle \partial_p^2 E \rangle - 2\beta_r \langle v \rangle = 0, \\ \frac{\partial \langle a^\dagger a \rangle}{\partial \beta_r} &= 2\beta_r - 2\alpha \langle v \rangle = 0, \\ \frac{\partial \langle a^\dagger a \rangle}{\partial \beta_i} &= 2\beta_i + 2\langle x \rangle = 0, \end{aligned} \quad (2.11)$$

which are satisfied when

$$\alpha = \frac{1}{2(\Delta v)^2} \langle \partial_p^2 E \rangle, \quad \beta_r = \alpha \langle v \rangle, \quad \beta_i = -\langle x \rangle. \quad (2.12)$$

Inserting these values in eq.(2.10) one obtains

$$(\Delta x)^2 - \frac{1}{4(\Delta v)^2} \langle \partial_p^2 E \rangle^2 \geq 0 \Rightarrow \Delta x \Delta v \geq \frac{1}{2} |\langle \partial_p^2 E \rangle|. \quad (2.13)$$

Indeed, in the non-relativistic case, $E(p) = p^2/2m$, this reduces to the standard Heisenberg uncertainty relation eq.(2.8). Obviously, this is just a special case of the general uncertainty relation $\Delta A \Delta B \leq \frac{1}{2} |\langle [A, B] \rangle|$.

2.3 Minimal Position-Velocity Uncertainty Wave Packets

By construction, it is clear that wave packets $\Phi(p)$ with a minimal position-velocity uncertainty product $\Delta x \Delta v$, which saturate the inequality eq.(2.13), must satisfy

$$a\Phi(p) = (\partial_p + \alpha v - \beta)\Phi(p) = 0. \quad (2.14)$$

This equation is easy to solve and one obtains

$$\Phi(p) = A \exp(-\alpha E(p) + \beta p). \quad (2.15)$$

Using eq.(2.12) and assuming that $\langle \partial_p^2 E \rangle > 0$, this can also be expressed as

$$\Phi(p) = A \exp\left(-\frac{\Delta x}{\Delta v} [E(p) - p\langle v \rangle] - ip\langle x \rangle\right). \quad (2.16)$$

In coordinate space, a minimal uncertainty wave packet takes the form

$$\begin{aligned} \Phi(x, t) &= \frac{1}{2\pi} \int dp \Phi(p) \exp[-iE(p)t + ipx] \\ &= \frac{A}{2\pi} \int dp \exp[-iE(p)(t - i\alpha) + ip(x - i\beta)] \\ &= A G(x - i\beta, t - i\alpha), \end{aligned} \quad (2.17)$$

and is thus given by analytic continuation of the Green's function $G(x, t)$ of eq.(2.3). It is straightforward to show that for a wave packet with minimal position-velocity uncertainty the initial position-velocity correlation vanishes, i.e. $\langle vx + xv \rangle(0) = 2\langle v \rangle \langle x \rangle(0)$. The general formula eq.(2.7) describing wave packet spreading then reduces to

$$\Delta x(t)^2 = \Delta x(0)^2 + (\Delta v)^2 t^2. \quad (2.18)$$

Since for a free particle $\langle v \rangle$ and $\langle v^2 \rangle$ are time-independent, a wave packet that initially has a minimal position-velocity uncertainty will obviously not maintain a minimal uncertainty product as time evolves. In fact, one obtains

$$\Delta x(t)\Delta v = \sqrt{\frac{1}{4} \langle \partial_p^2 E \rangle^2 + (\Delta v)^4 t^2}. \quad (2.19)$$

2.4 Generalization to Higher Dimensions

It is straightforward to extend the results of the previous subsections to higher dimensions. Let us consider a free particle with momentum \vec{p} and energy $E(\vec{p})$ moving in d dimensions. The time-evolution of an initial momentum space wave function $\Psi(\vec{p})$ is then given by

$$\Psi(\vec{p}, t) = \Psi(\vec{p}) \exp(-iE(\vec{p})t), \quad (2.20)$$

and the corresponding coordinate space wave function takes the form

$$\Psi(\vec{x}, t) = \frac{1}{(2\pi)^d} \int d^d p \Psi(\vec{p}) \exp(-iE(\vec{p})t + i\vec{p} \cdot \vec{x}) = \int d^d x' G(\vec{x} - \vec{x}', t) \Psi(\vec{x}', 0), \quad (2.21)$$

where the Green's function is given by

$$G(\vec{x}, t) = \frac{1}{(2\pi)^d} \int d^d p \exp(-iE(\vec{p})t + i\vec{p} \cdot \vec{x}). \quad (2.22)$$

In d dimensions we have

$$\vec{v} = \vec{\nabla}_p E, \quad [x_i, v_j] = [i\partial_{p_i}, \partial_{p_j} E] = i\partial_{p_i} \partial_{p_j} E, \quad (2.23)$$

where $\vec{\nabla}_p$ denotes the gradient in momentum space, and one then obtains

$$\begin{aligned} \langle \vec{x} \rangle(t) &= \langle \vec{x} \rangle(0) + \langle \vec{v} \rangle t, \\ \langle \vec{x}^2 \rangle(t) &= \langle \vec{x}^2 \rangle(0) + \langle \vec{v} \cdot \vec{x} + \vec{x} \cdot \vec{v} \rangle(0)t + \langle \vec{v}^2 \rangle t^2, \\ \Delta x(t)^2 &= \Delta x(0)^2 + [\langle \vec{v} \cdot \vec{x} + \vec{x} \cdot \vec{v} \rangle(0) - 2\langle \vec{v} \rangle \cdot \langle \vec{x} \rangle(0)]t + (\Delta v)^2 t^2, \end{aligned} \quad (2.24)$$

with $\Delta x = \sqrt{\langle \vec{x}^2 \rangle - \langle \vec{x} \rangle^2}$ and $\Delta v = \sqrt{\langle \vec{v}^2 \rangle - \langle \vec{v} \rangle^2}$.

In order to derive the generalized position-velocity uncertainty relation we define

$$\vec{a} = -i\vec{x} + \alpha\vec{v} - \vec{\beta} = \vec{\nabla}_p + \alpha\vec{v} - \vec{\beta}, \quad (2.25)$$

Minimizing $\langle \vec{a}^\dagger \cdot \vec{a} \rangle$ one obtains

$$\alpha = \frac{1}{2(\Delta v)^2} \langle \Delta_p E \rangle, \quad \vec{\beta} = \alpha \langle \vec{v} \rangle - i \langle \vec{x} \rangle, \quad (2.26)$$

which then implies

$$(\Delta x)^2 (\Delta v)^2 \geq \frac{1}{4} \langle \Delta_p E \rangle^2, \quad (2.27)$$

where Δ_p denotes the Laplace operator in momentum space. For the non-relativistic dispersion relation $E(\vec{p}) = \vec{p}^2/2m$ one obtains the d -dimensional Heisenberg uncertainty relation

$$\Delta x \Delta v \geq \frac{d}{2m}, \quad (2.28)$$

while for the relativistic dispersion relation $E(p) = \sqrt{\vec{p}^2 + m^2}$ one obtains

$$\Delta x \Delta v \geq \frac{dm^2}{2} \langle E^{-3} \rangle. \quad (2.29)$$

A minimal position-velocity uncertainty wave packet must obey

$$\vec{a}\Phi(\vec{p}) = \left(\vec{\nabla}_p + \alpha\vec{v} - \vec{\beta} \right) \Phi(\vec{p}) = 0, \quad (2.30)$$

which is solved by

$$\Phi(\vec{p}) = A \exp(-\alpha E(\vec{p}) + \vec{\beta} \cdot \vec{p}). \quad (2.31)$$

In coordinate space, a minimal uncertainty wave packet then takes the form

$$\begin{aligned} \Phi(\vec{x}, t) &= \frac{1}{(2\pi)^d} \int d^d p \Phi(\vec{p}) \exp[-iE(\vec{p})t + i\vec{p} \cdot \vec{x}] \\ &= \frac{A}{(2\pi)^d} \int d^d p \exp\left[-iE(\vec{p})(t - i\alpha) + i\vec{p} \cdot (\vec{x} - i\vec{\beta})\right] \\ &= A G(\vec{x} - i\vec{\beta}, t - i\alpha), \end{aligned} \quad (2.32)$$

3 Spreading of Non-relativistic Wave Packets in the Continuum and on a Lattice

In this section we consider free non-relativistic particles either in the continuum, i.e. with $E(p) = p^2/2m$, or on a lattice with spacing a and $E(p) = -\cos(pa)/ma^2$.

3.1 Spreading of Standard Gaussian Wave Packets

Although this subsection contains completely standard textbook material, we like to include it, in order to ease the transition to the relativistic case discussed in the next section. Let us consider a non-relativistic free particle with $E(p) = p^2/2m$. The minimal uncertainty wave packet then takes the standard Gaussian form

$$\Phi(p) = A \exp\left(-\alpha \frac{p^2}{2m} + \beta p\right). \quad (3.1)$$

For $\beta_i = 0$ the corresponding expectation values are given by

$$\begin{aligned} \langle x \rangle &= 0, & \langle x^2 \rangle &= (\Delta x)^2 = \frac{\alpha}{2m}, \\ \langle v \rangle &= \frac{\beta}{\alpha}, & \langle v^2 \rangle &= \frac{1}{2m\alpha} + \frac{\beta^2}{\alpha^2}, & (\Delta v)^2 &= \frac{1}{2m\alpha}. \end{aligned} \quad (3.2)$$

For $\beta_i \neq 0$ the wave packet is just shifted in space by $-\beta_i$. The wave function $\Phi(p)$ translates into the coordinate space form $\Phi(x, t) = AG(x - i\beta, t - i\alpha)$ with the Green's function given by

$$G(x, t) = \sqrt{\frac{m}{2\pi it}} \exp\left(\frac{imx^2}{2t}\right). \quad (3.3)$$

The spreading of two minimal position-velocity uncertainty wave packets is illustrated in figure 1.

As a preparation for the relativistic case to be discussed in section 4, let us consider the Galilean boost properties of the spreading Gaussian wave packet. The generators of the Galilean group are the Hamiltonian H , the momentum P , and the Galilean boost M , which are given by

$$H = \frac{p^2}{2m}, \quad P = p, \quad M = mx, \quad (3.4)$$

and which obey the commutation relations

$$[H, P] = 0, \quad [M, P] = im, \quad [M, H] = iP. \quad (3.5)$$

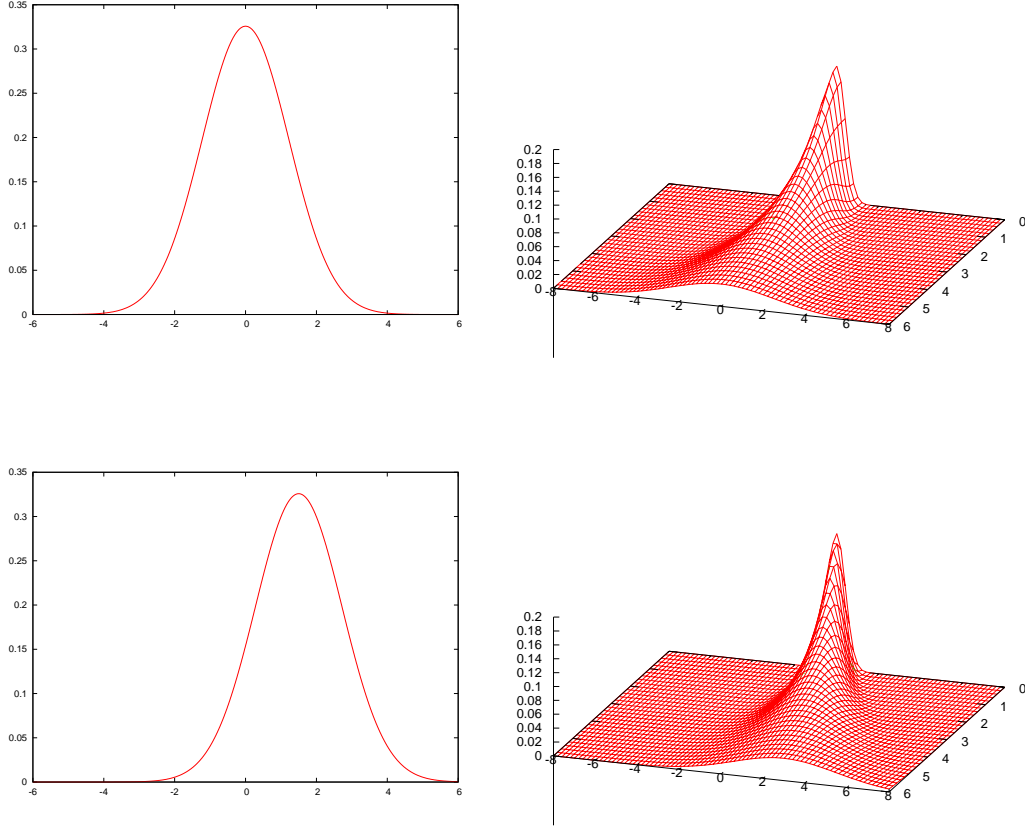


Figure 1: *Probability distributions of two non-relativistic minimal position-velocity uncertainty wave packets in momentum space (left) and spreading in coordinate space as a function of time (right) for $\alpha = 1$, $\beta = 0$ (top) and for $\alpha = 1$, $\beta = 1/2$ (bottom) with $m = 3$.*

The unitary transformation

$$U(u) = \exp(-iuM) = \exp(-iumx) = \exp(um\partial_p), \quad (3.6)$$

which implements a boost to an inertial frame moving with the velocity u , acts as a shift-operator on an arbitrary momentum space wave function, i.e.

$$\Psi_b(p') = U(u)\Psi(p') = \Psi(p), \quad p' = p - um. \quad (3.7)$$

In particular, for a minimal uncertainty wave packet we obtain

$$\begin{aligned} \Phi_b(p') &= U(u)\Phi(p') = \Phi(p) = A \exp\left(-\alpha \frac{(p' + um)^2}{2m} + \beta(p' + um)\right) \\ &= A' \exp\left(-\alpha' \frac{p'^2}{2m} + \beta' p'\right), \end{aligned} \quad (3.8)$$

with the parameters after the boost given by

$$\alpha' = \alpha, \quad \beta' = \beta - u\alpha \Rightarrow \Delta v' = \Delta v, \quad \langle v' \rangle = \langle v \rangle - u. \quad (3.9)$$

Both Δx and Δv remain unchanged after the boost, and hence a minimal position-velocity uncertainty wave packet has minimal uncertainty also from the point of view of a moving observer. As we will see later, this is different in the relativistic case.

3.2 Spreading of Wave Packets on a Lattice

Let us now consider a non-relativistic particle hopping between neighboring sites on a lattice with spacing a . The energy-momentum dispersion relation is then given by

$$E(p) = -\frac{\cos(pa)}{ma^2}. \quad (3.10)$$

In this case, $\partial_p^2 E = -a^2 E(p)$ such that the position velocity uncertainty relation takes the form

$$\Delta x \Delta v \leq \frac{a^2}{2} |\langle E \rangle|. \quad (3.11)$$

The general solution of the minimal position-velocity uncertainty wave packet given by $\Phi(p) = A \exp(-\alpha E(p) + \beta p)$ is not periodic over the Brillouin zone $]-\pi/a, \pi/a]$, and is thus not appropriate for the particle hopping on the lattice. Indeed, the periodicity requirement $\Phi(p + 2\pi/a) = \Phi(p)$ implies $\beta_r = 0$, and $\beta_i/a \in \mathbb{Z}$. Consequently, minimal position-velocity uncertainty wave packets on the lattice must obey $\langle v \rangle = 0$ as well as $\langle x \rangle/a \in \mathbb{Z}$, i.e. they do not move sideways and are centered at a lattice point. In contrast to the particle moving in the continuum, a moving wave packet on the lattice cannot have a minimal uncertainty product. This is due to the absence of Galilean symmetry, which is explicitly broken by the lattice. In fact, the lattice defines a preferred reference frame, which is the one in which minimal uncertainty wave packets (as well as the lattice itself) are at rest. The expectation values of various operators for these wave packets are worked out in appendix A and (for $\beta_i = 0$) one obtains

$$\begin{aligned} \langle x \rangle &= 0, & \langle x^2 \rangle &= (\Delta x)^2 = \frac{\alpha I_1}{2mI_0}, \\ \langle v \rangle &= 0, & \langle v^2 \rangle &= (\Delta v)^2 = \frac{I_1}{2m\alpha I_0}, \\ \langle E \rangle &= -\frac{I_1}{ma^2 I_0}, & \langle E^2 \rangle &= \frac{1}{m^2 a^4} \left(1 - \frac{ma^2 I_1}{2\alpha I_0} \right), \\ (\Delta E)^2 &= \frac{1}{m^2 a^4} \left(1 - \frac{ma^2 I_1}{2\alpha I_0} - \frac{I_1^2}{I_0^2} \right). \end{aligned} \quad (3.12)$$

Here I_0 and I_1 are modified Bessel functions of degree zero and one with

$$I_0 = I_0 \left(\frac{2\alpha}{ma^2} \right), \quad I_1 = I_1 \left(\frac{2\alpha}{ma^2} \right). \quad (3.13)$$

Despite the fact that they do not move sideways, it is still interesting to investigate the spreading of minimal uncertainty wave packets on the lattice. In this case, the Green's function takes the form

$$G(x, t) = \frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} dp \exp\left(\frac{i \cos(pa)t}{ma^2} + ipx\right) = \frac{1}{a} I_{x/a}\left(\frac{it}{ma^2}\right). \quad (3.14)$$

Here $I_{x/a}$ is a modified Bessel function of degree $n = x/a \in \mathbb{Z}$. It should be noted that the Green's function is restricted to the lattice sites, i.e. $x = na$ with $n \in \mathbb{Z}$. The spreading of a minimal uncertainty wave packet on the lattice, $\Phi(x, t) = AG(x, t - i\alpha)$, is illustrated in figure 2. Interestingly, the probability density shows an oscillatory behavior which is absent in the continuum. This effect arises for wave packets of large energy that are sensitive to the lattice spacing scale a .

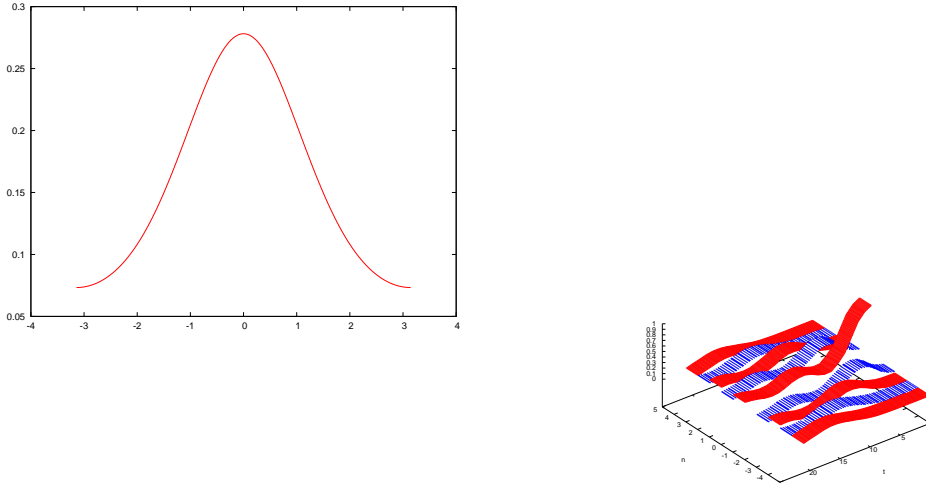


Figure 2: *Probability densities of a minimal position-velocity uncertainty wave packet on a lattice in momentum space (left) and spreading in coordinate space as a function of time (right) for $\alpha = 1$, $\beta = 0$ with $m = 3$ and $a = 1$.*

4 Spreading of Relativistic Wave Packets

As we have stressed in the introduction, we do not consider the Dirac or Klein-Gordon equations because those belong to quantum field theory and not to relativistic quantum mechanics with a finite number of degrees of freedom and a fixed number of particles. In this section we consider the spreading of minimal uncertainty wave packets for a single free relativistic particle. The spreading of other relativistic wave packets has been investigated in [23, 24]. As a preparation, we first study the Lorentz transformation properties of general wave functions.

4.1 Poincaré Algebra and Boost Properties of General Wave Functions

For a single free particle, it is trivial to satisfy the Poincaré algebra

$$[H, P] = 0, \quad [M, P] = iH, \quad [M, H] = iP, \quad (4.1)$$

by writing

$$H = \sqrt{p^2 + m^2}, \quad P = p, \quad M = \frac{1}{2} \left(x\sqrt{p^2 + m^2} + \sqrt{p^2 + m^2} x \right), \quad (4.2)$$

for the Hamiltonian, the momentum, and the boost operator, respectively. The unitary transformation that implements the boost to a frame moving with the velocity u is given by

$$U(u) = \exp(-iuM). \quad (4.3)$$

Under the corresponding Lorentz transformation, the momentum p of a particle turns into

$$p' = \gamma [p - uE(p)] = \gamma \left[p - u\sqrt{p^2 + m^2} \right], \quad \gamma = \frac{1}{\sqrt{1 - u^2}}. \quad (4.4)$$

Hence, the action of the boost operator on a plane wave is given by

$$U(u) \exp(ipx) = A(p) \exp(ip'x) = A(p) \exp(i\gamma [p - uE(p)]x). \quad (4.5)$$

The normalization condition

$$\begin{aligned} \int dx A(p_1)^* \exp(-ip_1'x) A(p_2) \exp(ip_2'x) &= \int dx \exp(-ip_1x) U(u)^\dagger U(u) \exp(ip_2x) \\ &= 2\pi \delta(p_1 - p_2), \end{aligned} \quad (4.6)$$

then implies

$$|A(p)|^2 = \gamma (1 - u\partial_p E) = \gamma (1 - uv). \quad (4.7)$$

When one applies the boost to an arbitrary wave packet

$$\Psi(x) = \frac{1}{2\pi} \int dp \Psi(p) \exp(ipx), \quad (4.8)$$

one hence obtains

$$\begin{aligned} \Psi_b(x) &= U(u)\Psi(x) = \frac{1}{2\pi} \int dp \Psi(p)A(p) \exp(ip'x) \\ &= \frac{1}{2\pi} \int dp' \gamma(1+uv') \Psi(p)A(p) \exp(ip'x). \end{aligned} \quad (4.9)$$

Here we have used

$$p = \gamma[p' + uE'(p')], \quad \frac{dp}{dp'} = \gamma(1 + u\partial_{p'}E') = \gamma(1 + uv'), \quad v' = \frac{v-u}{1-uv}, \quad (4.10)$$

with $E'(p') = \sqrt{p'^2 + m^2}$. Using eq.(4.7) as well as eq.(4.9), one may then identify

$$\Psi_b(p') = U(u)\Psi(p') = A(-p')\Psi(p), \quad (4.11)$$

with

$$|A(-p')|^2 = \gamma(1 + u\partial_{p'}E') = \gamma(1 + uv'). \quad (4.12)$$

In momentum space, the norm of the boosted wave function then takes the form

$$\begin{aligned} \frac{1}{2\pi} \int dp' |\Psi_b(p')|^2 &= \frac{1}{2\pi} \int dp' |A(-p')|^2 |\Psi(p)|^2 \\ &= \frac{1}{2\pi} \int dp' \frac{dp}{dp'} |\Psi(p)|^2 = \frac{1}{2\pi} \int dp |\Psi(p)|^2, \end{aligned} \quad (4.13)$$

which is thus indeed consistent.

Finally, in order to verify explicitly that $U(u) = \exp(-iuM)$, we expand for small u and obtain

$$\begin{aligned} U(u) - \mathbb{1} &= -iuM + \mathcal{O}(u^2) = -i\frac{u}{2} \left(x\sqrt{p^2 + m^2} + \sqrt{p^2 + m^2} x \right) + \mathcal{O}(u^2) \\ &= \frac{u}{2} (\partial_p E(p) + E(p)\partial_p) + \mathcal{O}(u^2) = \frac{uv}{2} + uE(p)\partial_p + \mathcal{O}(u^2). \end{aligned} \quad (4.14)$$

Acting with this operator on a momentum space wave function $\Psi(p')$ and keeping only the leading linear order in u we obtain

$$[U(u) - \mathbb{1}] \Psi(p') = \frac{uv}{2} \Psi(p) + uE(p)\partial_p \Psi(p) + \mathcal{O}(u^2). \quad (4.15)$$

Using eq.(4.11) and expanding for small u in the same manner one finds

$$\begin{aligned} [U(u) - \mathbb{1}] \Psi(p') &= A(-p')\Psi(p) - \Psi(p') \\ &= \left(1 + \frac{uv}{2} \right) \Psi(p) - \Psi(p) + uE(p)\partial_p \Psi(p) + \mathcal{O}(u^2) \\ &= \frac{uv}{2} \Psi(p) + uE(p)\partial_p \Psi(p) + \mathcal{O}(u^2), \end{aligned} \quad (4.16)$$

which is thus indeed consistent.

Using the boost properties of a general wave function, it is easy to show that

$$\begin{aligned}\langle E \rangle_b &= \gamma [\langle E \rangle - u \langle p \rangle], & \langle p \rangle_b &= \gamma [\langle p \rangle - u \langle E \rangle], & \langle v \rangle_b &= \langle v' \rangle = \left\langle \frac{v - u}{1 - uv} \right\rangle, \\ \langle x \rangle_b &= \gamma \left\langle x + \frac{u}{2} (v'x + xv') \right\rangle, & \langle x^2 \rangle_b &= \gamma^2 \left\langle \left[x + \frac{u}{2} (v'x + xv') \right]^2 \right\rangle.\end{aligned}\quad (4.17)$$

Here the subscript b refers to expectation values taken with the boosted wave function Ψ_b , while the expectation values without this subscript refer to the original wave function Ψ .

4.2 Relativistic Minimal Uncertainty Wave Packets

According to the general expression of eq.(2.15), the relativistic minimal position-velocity uncertainty wave packets take the form

$$\Phi(p) = A \exp(-\alpha E(p) + \beta p) = A \exp\left(-\alpha \sqrt{p^2 + m^2} + \beta p\right). \quad (4.18)$$

Similar wave packets have been discussed in the context of relativistic quantum walks [25]. Using eq.(4.11) it is easy to show that a boosted minimal uncertainty wave packet takes the form

$$\Phi_b(p') = U(u)\Phi(p') = A(-p')\Phi(p) = A(-p')A \exp(-\alpha' E(p') + \beta' p'), \quad (4.19)$$

with

$$\alpha' = \gamma(\alpha - u\beta), \quad \beta' = \gamma(\beta - u\alpha), \quad (4.20)$$

i.e. (α, β) transforms as a space-time vector. However, due to the factor $A(-p')$ (which is not constant), the boosted wave packet no longer has minimal position-velocity uncertainty. This is in contrast to the non-relativistic case, in which a Galilean boost does not increase the uncertainty product. In light of the relativistic position-velocity uncertainty relation

$$\Delta x \Delta v \geq \frac{m^2}{2} \langle E^{-3} \rangle, \quad (4.21)$$

this is not surprising because the uncertainty product does not transform covariantly. We hence conclude that the concept of minimal position-velocity uncertainty is frame-dependent.

It is possible to work out the expectation values of a variety of operators for relativistic wave packets with a minimal position-velocity uncertainty product. As

discussed in appendix A (for $\beta_i = 0$) one obtains

$$\begin{aligned}
\langle x \rangle &= 0, & \langle x^2 \rangle &= \alpha^2 - \beta^2 - \frac{2\alpha\sqrt{\alpha^2 - \beta^2}}{K_1} \int_{\alpha}^{\infty} d\alpha' K_0 \left(2m\sqrt{\alpha'^2 - \beta^2} \right), \\
\langle v \rangle &= \frac{\beta}{\alpha}, & \langle v^2 \rangle &= 1 - \frac{2\sqrt{\alpha^2 - \beta^2}}{\alpha K_1} \int_{\alpha}^{\infty} d\alpha' K_0 \left(2m\sqrt{\alpha'^2 - \beta^2} \right), \\
\langle p \rangle &= \frac{\beta}{\alpha^2 - \beta^2} \left(1 + m\sqrt{\alpha^2 - \beta^2} \frac{K_0}{K_1} \right), \\
\langle p^2 \rangle &= \frac{m^2\beta^2}{\alpha^2 - \beta^2} + \frac{\alpha^2 + 3\beta^2}{2(\alpha^2 - \beta^2)^2} \left(1 + m\sqrt{\alpha^2 - \beta^2} \frac{K_0}{K_1} \right), \\
\langle E \rangle &= \frac{\alpha}{\alpha^2 - \beta^2} \left(1 + m\sqrt{\alpha^2 - \beta^2} \frac{K_0}{K_1} \right) - \frac{1}{2\alpha}, \\
\langle E^2 \rangle &= \frac{m^2\alpha^2}{\alpha^2 - \beta^2} + \frac{\alpha^2 + 3\beta^2}{2(\alpha^2 - \beta^2)^2} \left(1 + m\sqrt{\alpha^2 - \beta^2} \frac{K_0}{K_1} \right).
\end{aligned} \tag{4.22}$$

Here K_0 and K_1 are modified Bessel functions of degree zero and one

$$K_0 = K_0 \left(2m\sqrt{\alpha^2 - \beta^2} \right), \quad K_1 = K_1 \left(2m\sqrt{\alpha^2 - \beta^2} \right). \tag{4.23}$$

4.3 Relativistic Wave Packet Spreading and Apparent Causality Violation

Let us consider the Green's function in the relativistic case

$$G(x, t) = \frac{1}{2\pi} \int dp \exp(-i\sqrt{p^2 + m^2} t + ipx). \tag{4.24}$$

Following methods presented in [26], we now write

$$\sqrt{p^2 + m^2} = m \cosh z, \quad p = m \sinh z, \quad \frac{dp}{dz} = m \cosh z, \tag{4.25}$$

as well as

$$\begin{aligned}
t = \sqrt{t^2 - x^2} \cosh \tau, & \quad x = \sqrt{t^2 - x^2} \sinh \tau, & \text{for } |x| < t, \\
t = \sqrt{x^2 - t^2} \sinh \tau, & \quad x = \sqrt{x^2 - t^2} \cosh \tau, & \text{for } |x| > t,
\end{aligned} \tag{4.26}$$

such that

$$\begin{aligned}
\sqrt{p^2 + m^2}t - px &= m\sqrt{t^2 - x^2} (\cosh z \cosh \tau - \sinh z \sinh \tau) \\
&= m\sqrt{t^2 - x^2} \cosh(z - \tau), & \text{for } |x| < t, \\
\sqrt{p^2 + m^2}t - px &= m\sqrt{x^2 - t^2} (\cosh z \sinh \tau - \sinh z \cosh \tau) \\
&= m\sqrt{x^2 - t^2} \sinh(z - \tau), & \text{for } |x| > t.
\end{aligned} \tag{4.27}$$

Inserting this in the expression for the Green's function we obtain

$$\begin{aligned}
G(x, t) &= \frac{i}{2\pi} \partial_t \int dp \frac{1}{\sqrt{p^2 + m^2}} \exp\left(-i\sqrt{p^2 + m^2} t + ipx\right) \\
&= \frac{i}{2\pi} \partial_t \int dz \exp\left(-im\sqrt{t^2 - x^2} \cosh(z - \tau)\right) \\
&= \frac{i}{\pi} \partial_t \int_0^\infty dz \left[\cos\left(m\sqrt{t^2 - x^2} \cosh z\right) \right. \\
&\quad \left. - i \sin\left(m\sqrt{t^2 - x^2} \cosh z\right) \right], \quad \text{for } |x| < t, \\
G(x, t) &= \frac{i}{2\pi} \partial_t \int dz \exp\left(-im\sqrt{x^2 - t^2} \sinh(z - \tau)\right) \\
&= \frac{i}{\pi} \partial_t \int_0^\infty dz \cos\left(m\sqrt{x^2 - t^2} \sinh z\right), \quad \text{for } |x| > t. \quad (4.28)
\end{aligned}$$

Finally, using

$$\begin{aligned}
\int_0^\infty dz \sin\left(m\sqrt{t^2 - x^2} \cosh z\right) &= \frac{\pi}{2} J_0\left(m\sqrt{t^2 - x^2}\right), \quad \text{for } |x| < t, \\
\int_0^\infty dz \cos\left(m\sqrt{t^2 - x^2} \cosh z\right) &= -\frac{\pi}{2} N_0\left(m\sqrt{t^2 - x^2}\right), \quad \text{for } |x| < t, \\
\int_0^\infty dz \cos\left(m\sqrt{x^2 - t^2} \sinh z\right) &= K_0\left(m\sqrt{x^2 - t^2}\right), \quad \text{for } |x| > t. \quad (4.29)
\end{aligned}$$

where J_0 , N_0 , and K_0 are Bessel functions of degree zero, one obtains

$$\begin{aligned}
G(x, t) &= \frac{1}{2} \partial_t \left[J_0\left(m\sqrt{t^2 - x^2}\right) - iN_0\left(m\sqrt{t^2 - x^2}\right) \right] \quad \text{for } |x| < t, \\
G(x, t) &= \frac{i}{\pi} \partial_t K_0\left(m\sqrt{x^2 - t^2}\right) \quad \text{for } |x| > t. \quad (4.30)
\end{aligned}$$

Using $\frac{i}{\pi} K_0(iz) = \frac{1}{2}[J_0(z) - iN_0(z)]$, one can write

$$G(x, t) = \frac{i}{\pi} \partial_t K_0\left(m\sqrt{x^2 - t^2}\right) = -\frac{imt}{\pi\sqrt{x^2 - t^2}} K_1\left(m\sqrt{x^2 - t^2}\right), \quad (4.31)$$

for all values of x and t . Here K_1 is a modified Bessel function of degree one.

Interestingly, the Green's function does not vanish at space-like distances $|x| > t$, which seems to violate causality [5]. As discussed in appendix B, the apparent violation of causality is resolved in the framework of quantum field theory, and is due to an inherent non-locality of single particle states [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. The probability density of a minimal position-velocity uncertainty wave packet given by $\Phi(x, t) = AG(x - i\beta, t - i\alpha)$ is illustrated in figure 3.

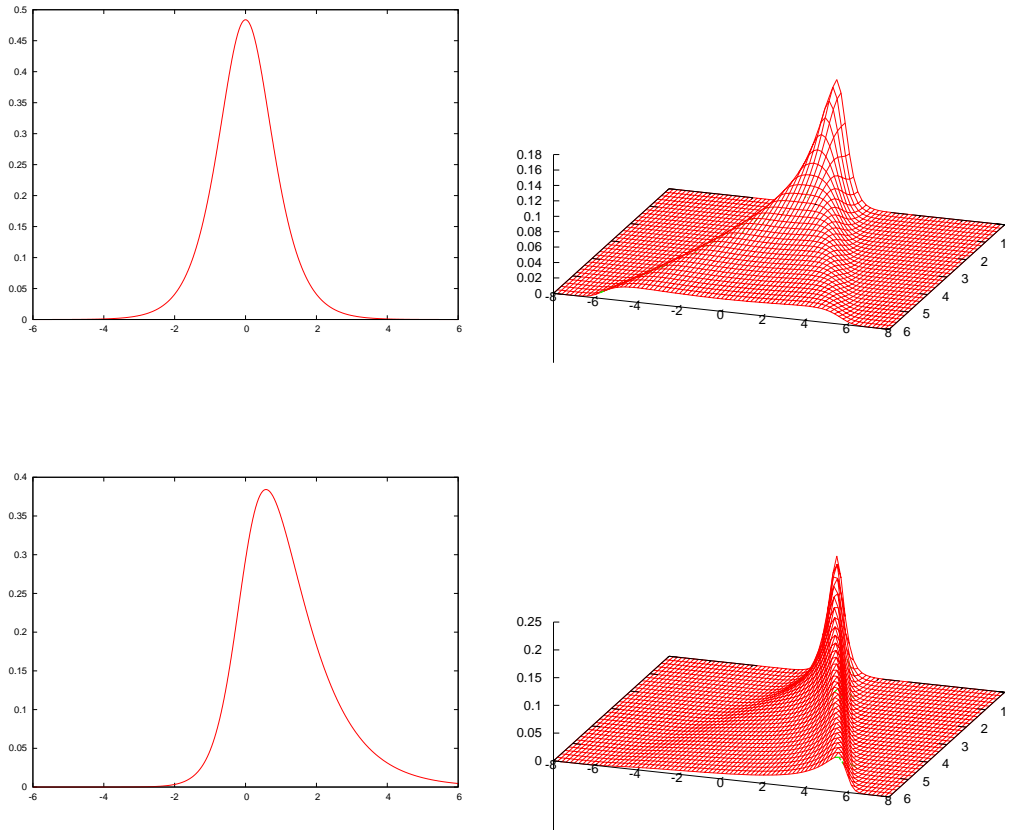


Figure 3: *Probability densities of two relativistic minimal position-velocity uncertainty wave packets in momentum space (left) and spreading in coordinate space as a function of time (right) for $\alpha = 1$, $\beta = 0$ (top) and for $\alpha = 1$, $\beta = 1/2$ (bottom) with $m = 1$.*

4.4 The Massless Limit

It is interesting to consider the massless limit $m \rightarrow 0$. A minimal position-velocity uncertainty wave packet then takes the form

$$\Phi(p) = A \exp(-\alpha|p| + \beta p). \quad (4.32)$$

It is straightforward to work out the expectation values of various operators and (for $\beta_i = 0$) one obtains

$$\begin{aligned}
\langle x \rangle &= 0, & \langle x^2 \rangle &= (\Delta x)^2 = \alpha^2 - \beta^2, \\
\langle v \rangle &= \frac{\beta}{\alpha}, & \langle v^2 \rangle &= 1, & (\Delta v)^2 &= 1 - \frac{\beta^2}{\alpha^2}, \\
\langle p \rangle &= \frac{\beta}{\alpha^2 - \beta^2}, & \langle p^2 \rangle &= \frac{\alpha^2 + 3\beta^2}{2(\alpha^2 - \beta^2)^2}, & (\Delta p)^2 &= \frac{\alpha^2 + \beta^2}{2(\alpha^2 - \beta^2)^2}, \\
\langle E \rangle &= \frac{\alpha^2 + \beta^2}{2\alpha(\alpha^2 - \beta^2)}, & \langle E^2 \rangle &= \frac{\alpha^2 + 3\beta^2}{2(\alpha^2 - \beta^2)^2}, & (\Delta E)^2 &= \frac{\alpha^4 + 4\alpha^2\beta^2 - \beta^4}{4\alpha^2(\alpha^2 - \beta^2)^2}.
\end{aligned} \tag{4.33}$$

It should be noted that, despite the fact that massless particles move with the speed of light, in general $|\langle v \rangle| \leq 1$, because a particle may move simultaneously both to the left and to the right with non-zero probability amplitude. Only for $|\beta| \rightarrow \alpha$ the particle is entirely left- or right-moving, $|\langle v \rangle| \rightarrow 1$, $\Delta v \rightarrow 0$, and the corresponding wave packet is not spreading.

In the massless limit, the Green's function takes the form

$$G(x, t) = \frac{1}{2\pi} \int dp \exp(-i|p|t + ipx) = \frac{i}{\pi} \frac{t}{x^2 - t^2}. \tag{4.34}$$

Accordingly, the wave function of a minimal uncertainty wave packet is given by $\Phi(x, t) = AG(x - i\beta, t - i\alpha)$. The time-dependence of two corresponding probability densities is shown in figure 4. In contrast to the non-relativistic case, the spreading of ultra-relativistic wave packets proceeds by wave packet splitting into two packets, one moving to the left and one moving to the right, each with the speed of light.

5 Propagation of Wave Packets in an Expanding Universe

In this section we consider the propagation of wave packets in an expanding Universe. For simplicity, we limit ourselves to one spatial dimension, but the generalization to higher dimensions is straightforward.

5.1 Free Falling Particle in an Expanding Universe

Let us consider an expanding 1-dimensional Universe with the Friedmann-Lemaître-Robertson-Walker-type metric

$$(ds)^2 = (dt)^2 - R(t)^2(d\rho)^2. \tag{5.1}$$

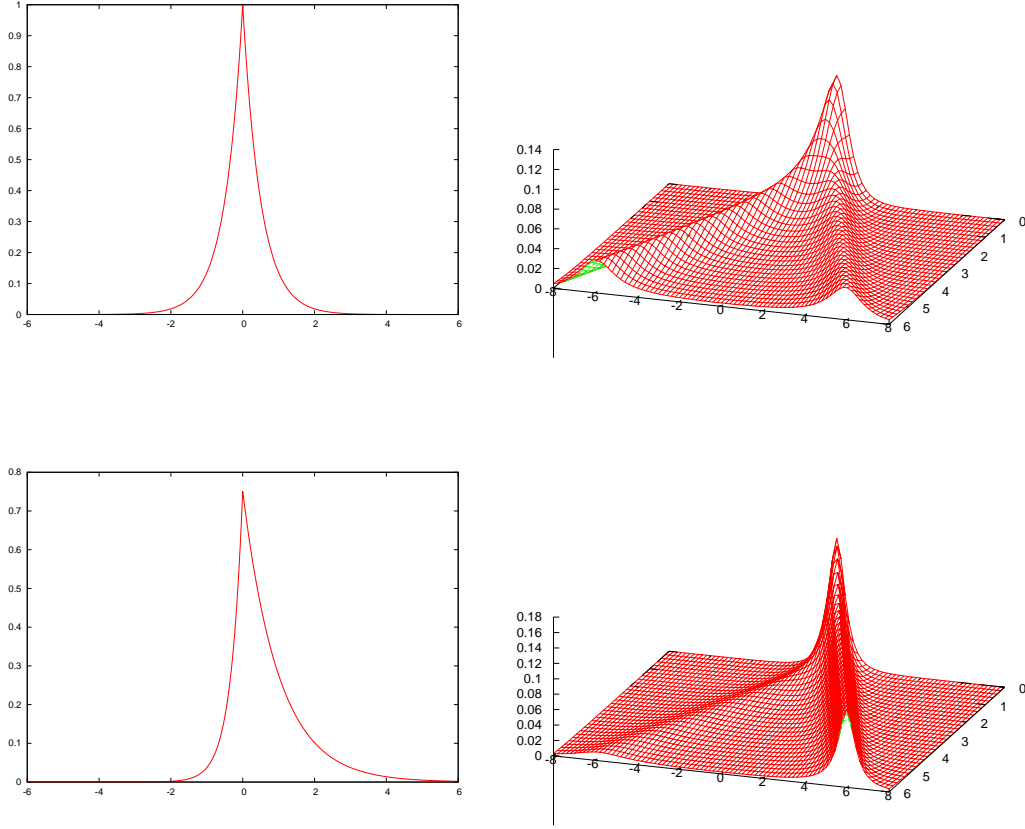


Figure 4: *Probability densities of two ultra-relativistic minimal position-velocity uncertainty wave packets in momentum space (left) and spreading in coordinate space as a function of time (right) for $\alpha = 1$, $\beta = 0$ (top) and for $\alpha = 1$, $\beta = 1/2$ (bottom) with $m = 0$.*

Here $R(t)$ is the scale parameter of the Universe whose time-dependence we consider as given. The position $x = R(t)\rho$ of a particle is described by the dimensionless coordinate ρ . The Lagrange function of a free falling particle then takes the form

$$L = -m \frac{ds}{dt} = -m \sqrt{1 - R(t)^2 \dot{\rho}^2}, \quad (5.2)$$

and the momentum canonically conjugate to the dimensionless coordinate ρ is given by

$$p_\rho = \frac{\partial L}{\partial \dot{\rho}} = \frac{m R(t)^2 \dot{\rho}}{\sqrt{1 - R(t)^2 \dot{\rho}^2}}, \quad (5.3)$$

while the dimensionful momentum conjugate to x is $p = p_\rho / R(t)$. The corresponding time-dependent classical Hamilton function hence takes the form

$$H(t) = p_\rho \dot{\rho} - L = \sqrt{p_\rho^2 / R(t)^2 + m^2}. \quad (5.4)$$

The classical equations of motion are thus given by

$$\dot{p}_\rho = -\frac{\partial H}{\partial \rho} = 0, \quad \dot{\rho} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho/R(t)^2}{\sqrt{p_\rho^2/R(t)^2 + m^2}}. \quad (5.5)$$

Integrating the two equations one obtains

$$\rho(t) = \rho(0) + \int_0^t dt' \frac{p_\rho/R(t')^2}{\sqrt{p_\rho^2/R(t')^2 + m^2}}. \quad (5.6)$$

The corresponding equation for the particle's position then takes the form

$$x(t) = R(t)\rho(t) = \frac{R(t)}{R(0)}x(0) + R(t) \int_0^t dt' \frac{v(t')}{R(t')}, \quad (5.7)$$

where we have identified the velocity as

$$v(t) = \frac{p_\rho/R(t)}{\sqrt{p_\rho^2/R(t)^2 + m^2}}. \quad (5.8)$$

In particular, as the Universe expands, the velocity $v(t)$ decreases, because the momentum $p = p_\rho/R(t)$ is red-shifted. The time-dependence of the velocity is given by

$$v(t) = \frac{R(0)}{R(t)} \frac{v(0)}{\sqrt{1 - v(0)^2 + v(0)^2 R(0)^2/R(t)^2}}. \quad (5.9)$$

Upon canonical quantization the Hamilton function turns into a Hamilton operator, and one postulates $[\rho, p_\rho] = i$, which is realized by $\rho = i\partial_{p_\rho}$. The Schrödinger equation then takes the form

$$i\partial_t \Psi(p_\rho, t) = H(t)\Psi(p_\rho, t), \quad (5.10)$$

which is solved by

$$\Psi(p_\rho, t) = \exp\left(-i \int dt \sqrt{p_\rho^2/R(t)^2 + m^2}\right) \Psi(p_\rho), \quad (5.11)$$

where $\Psi(p_\rho)$ is the momentum space wave function at $t = 0$.

5.2 Time-dependence of Expectation Values

Let us consider the time-dependence of the expectation value of the dimensionless coordinate

$$\begin{aligned}
\langle \rho \rangle(t) &= \frac{1}{2\pi} \int dp_\rho \Psi(p_\rho, t)^* i \partial_{p_\rho} \Psi(p_\rho, t) \\
&= \langle \rho \rangle(0) + \int_0^t dt' \frac{1}{2\pi} \int dp_\rho |\Psi(p_\rho)|^2 \frac{p_\rho/R(t')^2}{\sqrt{p_\rho^2/R(t')^2 + m^2}} \\
&= \langle \rho \rangle(0) + \int_0^t dt' \frac{\langle v(t') \rangle}{R(t')}.
\end{aligned} \tag{5.12}$$

Here the expectation value of the velocity $v(t')$ is evaluated for the initial wave function $\Psi(p_\rho)$. Correspondingly, one obtains

$$\langle x \rangle(t) = \frac{R(t)}{R(0)} \langle x \rangle(0) + R(t) \int_0^t dt' \frac{\langle v(t') \rangle}{R(t')}. \tag{5.13}$$

Similarly, one finds

$$\langle \rho^2 \rangle(t) = \langle \rho^2 \rangle(0) + \int_0^t dt' \frac{1}{R(t')} \langle v(t') \rho + \rho v(t') \rangle + \left\langle \left(\int_0^t dt' \frac{1}{R(t')} v(t') \right)^2 \right\rangle. \tag{5.14}$$

From this it is straightforward to obtain an expression for $\Delta x(t)$.

5.3 Propagation of Minimal Uncertainty Wave Packets in an Expanding Universe

For a relativistic minimal position-velocity uncertainty wave packet the initial wave function at $t = 0$ is given by

$$\Psi(p_\rho) = A \exp \left(-\alpha \sqrt{p_\rho^2/R(0)^2 + m^2} + \beta p_\rho/R(0) \right), \tag{5.15}$$

and the velocity expectation value takes the form

$$\begin{aligned}
\langle v(t) \rangle &= \frac{1}{2\pi} \int dp_\rho |\Psi(p_\rho)|^2 \frac{p_\rho/R(t)}{\sqrt{p_\rho^2/R(t)^2 + m^2}} \\
&= \frac{R(0)}{R(t)} \frac{|A|^2 R(0)}{2\pi} \int dp \exp \left(-2\alpha \sqrt{p^2 + m^2} + 2\beta p \right) \frac{p}{\sqrt{p^2 \frac{R(0)^2}{R(t)^2} + m^2}}.
\end{aligned} \tag{5.16}$$

We have not been able to simplify this integral any further. However, in the massless case it simplifies to

$$\langle v(t) \rangle = \frac{1}{2\pi} \int dp_\rho |\Psi(p_\rho)|^2 \text{sign}(p_\rho) = \langle v(0) \rangle = \frac{\beta}{\alpha}, \quad (5.17)$$

i.e. the average velocity is not red-shifted. One should keep in mind that $\langle v(0) \rangle$ receives contributions ± 1 , corresponding to the massless particle traveling to the left or to the right with the velocity of light. Similarly, in the non-relativistic limit

$$\begin{aligned} \langle v(t) \rangle &= \frac{1}{2\pi} \int dp_\rho |\Psi(p_\rho)|^2 \frac{p_\rho}{mR(t)} \\ &= \frac{R(0)}{R(t)} \frac{|A|^2 R(0)}{2\pi} \int dp \exp(-2\alpha(m + p^2/2m) + 2\beta p) \frac{p}{m}. \\ &= \frac{R(0)}{R(t)} \langle v(0) \rangle = \frac{R(0)}{R(t)} \frac{\beta}{\alpha}, \end{aligned} \quad (5.18)$$

i.e. the velocity is red-shifted in proportion to the scale parameter.

6 Conclusions

While most results obtained in this paper are rather simple, except for a few, we have not been able to find them in the physics literature. The standard textbook example of a spreading Gaussian wave packet is just the simplest case of a minimal position-velocity wave packet, which can be defined for an arbitrary relativistic or non-relativistic dispersion relation $E(p)$. Such wave packets saturate a generalized position-velocity uncertainty relation, and their time-evolution is described by analytic continuation of the corresponding Green's function. Detailed analytic solutions have been worked out for a non-relativistic particle in the continuum and on the lattice as well as for a relativistic particle, both in the massive and in the massless case.

Some of our results belong to relativistic quantum mechanics. Of course, since quantum field theory has been identified as the correct description of Nature at the most fundamental level that is accessible today, there is no urgent need for relativistic quantum mechanics. In particular, in view of Leutwyler's no-interaction theorem [15, 16], relativistic quantum mechanics seems to be limited to free theories, although we have already mentioned [17, 18, 19] to which the theorem does not apply. In fact, there may exist further interacting systems with a fixed number of particles in relativistic quantum mechanics, for example, with contact interactions. Even if no further systems of this kind should exist, we believe that the results presented here may be of some value. In particular, they may help bridging the gap between non-relativistic quantum mechanics and relativistic quantum field theory, which makes

learning the latter rather non-trivial. The explicit solutions of spreading relativistic wave packets illustrate in a simple setting what happens when both relativistic and quantum effects are present at the same time.

Although we have not elaborated on this, we can imagine that our results may have some use in neutrino physics. Indeed, the spreading of neutrino wave packets has been discussed in various places in the literature [20, 21], mostly using Gaussian wave packets. As we have discussed, Gaussian wave packets are natural to consider in non-relativistic quantum mechanics. In relativistic theories, on the other hand, the minimal position-velocity wave packets discussed above seem more natural, in particular, since closed analytic expressions have been obtained for a large variety of observables. Hence, in the relativistic case, there is no need to use Gaussian wave packets, which only yield approximate analytic results. Whether wave packet spreading of neutrinos (or other light particles) either in a static or in an expanding Universe is related to phenomenologically relevant questions remains an interesting topic for future investigations.

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A Evaluation of Various Expectation Values

In this appendix we work out the expectation values of various operators, for minimal position-velocity uncertainty wave packets both on the lattice and in the relativistic case.

A.1 Expectation Values for Wave Packets on the Lattice

In this subsection we calculate expectation values for the wave packet

$$\Phi(p) = A \exp\left(\frac{\alpha \cos(pa)}{ma^2}\right). \quad (\text{A.1})$$

The normalization condition then takes the form

$$\frac{1}{2\pi} \int dp |\Phi(p)|^2 = \frac{|A|^2}{2\pi} \int dp \exp\left(\frac{2\alpha \cos(pa)}{ma^2}\right) = \frac{|A|^2}{a} I_0 = 1 \Rightarrow |A|^{-2} = \frac{1}{a} I_0. \quad (\text{A.2})$$

For the energy one obtains

$$\begin{aligned} \langle E \rangle &= -\frac{1}{2\pi} \int dp |\Phi(p)|^2 \frac{\cos(pa)}{ma^2} \\ &= -\frac{|A|^2}{2\pi} \int dp \exp\left(\frac{2\alpha \cos(pa)}{ma^2}\right) \frac{\cos(pa)}{ma^2} \\ &= -\frac{|A|^2}{ma^3} I_1 = -\frac{I_1}{I_0 ma^2}, \end{aligned} \quad (\text{A.3})$$

where I_0 and I_1 are modified Bessel functions of degree zero and one

$$I_0 = I_0\left(\frac{2\alpha}{ma^2}\right), \quad I_1 = I_1\left(\frac{2\alpha}{ma^2}\right). \quad (\text{A.4})$$

For the energy squared we find

$$\begin{aligned} \langle E^2 \rangle &= \frac{1}{2\pi} \int dp |\Phi(p)|^2 \frac{\cos^2(pa)}{m^2 a^4} \\ &= \frac{|A|^2}{2\pi} \int dp \exp\left(\frac{2\alpha \cos(pa)}{ma^2}\right) \frac{\cos^2(pa)}{m^2 a^4} \\ &= \frac{|A|^2}{4a} \partial_\alpha^2 I_0 = \frac{\partial_\alpha^2 I_0}{4I_0} = \frac{1}{m^2 a^4} \left(1 - \frac{ma^2 I_1}{2\alpha I_0}\right). \end{aligned} \quad (\text{A.5})$$

The expectation value of the velocity squared takes the form

$$\langle v^2 \rangle = \left\langle \frac{\sin^2(pa)}{m^2 a^2} \right\rangle = \frac{1}{m^2 a^2} - a^2 \langle E^2 \rangle = \frac{I_1}{2m\alpha I_0}. \quad (\text{A.6})$$

Finally, we consider

$$\begin{aligned} \langle x^2 \rangle &= -\frac{1}{2\pi} \int dp \Phi(p)^* \partial_p^2 \Phi(p) = \left\langle \frac{\alpha \cos(pa)}{m} - \frac{\alpha^2 \sin^2(pa)}{m^2 a^2} \right\rangle \\ &= -\alpha a^2 \langle E \rangle - \alpha^2 \langle v^2 \rangle = \frac{\alpha I_1}{2m I_0}. \end{aligned} \quad (\text{A.7})$$

A.2 Expectation Values for Relativistic Wave Packets

In this subsection we calculate expectation values for the minimal position-velocity uncertainty wave packet

$$\Phi(p) = A \exp\left(-\alpha \sqrt{p^2 + m^2} + \beta p\right). \quad (\text{A.8})$$

Here we limit ourselves to $\beta \in \mathbb{R}$ which correspond to a wave packet centered at $x = 0$. Adding an imaginary part to β leads to a simple translation of the wave packet. The normalization condition takes the form

$$\frac{1}{2\pi} \int dp |\Phi(p)|^2 = \frac{|A|^2}{2\pi} \int dp \exp\left(-2\alpha\sqrt{p^2 + m^2} + 2\beta p\right) = 1. \quad (\text{A.9})$$

Introducing

$$\begin{aligned} \sqrt{p^2 + m^2} &= m \cosh z, & p &= m \sinh z, & \frac{dp}{dz} &= m \cosh z, \\ \alpha &= \sqrt{\alpha^2 - \beta^2} \cosh \lambda, & \beta &= \sqrt{\alpha^2 - \beta^2} \sinh \lambda, \end{aligned} \quad (\text{A.10})$$

one obtains

$$\alpha\sqrt{p^2 + m^2} - \beta p = \cosh z \cosh \lambda - \sinh z \sinh \lambda = \cosh(z - \lambda), \quad (\text{A.11})$$

and the normalization condition thus takes the form

$$\frac{|A|^2 m}{2\pi} \int dz \cosh z \exp\left(-2m\sqrt{\alpha^2 - \beta^2} \cosh(z - \lambda)\right) = 1. \quad (\text{A.12})$$

Let us first consider the integral

$$\begin{aligned} \frac{1}{2\pi} \int dz \exp(-2\alpha m \cosh z + 2\beta m \sinh z) &= \\ \frac{1}{2\pi} \int dz \exp\left(-2m\sqrt{\alpha^2 - \beta^2} \cosh z\right) &= \frac{1}{\pi} K_0. \end{aligned} \quad (\text{A.13})$$

The normalization condition can now be expressed as

$$-\frac{|A|^2}{2\pi} \partial_\alpha K_0 = \frac{|A|^2 m \alpha}{\pi \sqrt{\alpha^2 - \beta^2}} K_1 = 1 \Rightarrow |A|^{-2} = \frac{m \alpha}{\pi \sqrt{\alpha^2 - \beta^2}} K_1. \quad (\text{A.14})$$

Here K_0 and K_1 are modified Bessel functions of degree zero and one

$$K_0 = K_0\left(2m\sqrt{\alpha^2 - \beta^2}\right), \quad K_1 = K_1\left(2m\sqrt{\alpha^2 - \beta^2}\right). \quad (\text{A.15})$$

Next we consider the expectation value of the velocity

$$\begin{aligned} \langle v \rangle &= \frac{1}{2\pi} \int dp |\Phi(p)|^2 \frac{p}{\sqrt{p^2 + m^2}} \\ &= \frac{|A|^2 m}{2\pi} \int dz \sinh z \exp(-2\alpha m \cosh z + 2\beta m p \sinh z) \\ &= \frac{|A|^2}{2\pi} \partial_\beta K_0 = -\frac{\partial_\beta K_0}{\partial_\alpha K_0} = \frac{\beta}{\alpha}. \end{aligned} \quad (\text{A.16})$$

Similarly, we obtain

$$\begin{aligned}
\langle v^2 \rangle &= \frac{1}{2\pi} \int dp |\Phi(p)|^2 \frac{p^2}{p^2 + m^2} \\
&= 1 - \frac{|A|^2 m}{2\pi} \int dz \frac{1}{\cosh z} \exp(-2\alpha m \cosh z + 2\beta m p \sinh z) \\
&= 1 - \frac{2|A|^2 m}{\pi} \int_{\alpha}^{\infty} d\alpha' K_0 \left(2m\sqrt{\alpha'^2 - \beta^2} \right) \\
&= 1 - \frac{2\sqrt{\alpha^2 - \beta^2}}{\alpha K_1} \int_{\alpha}^{\infty} d\alpha' K_0 \left(2m\sqrt{\alpha'^2 - \beta^2} \right), \tag{A.17}
\end{aligned}$$

which he have not been able to simplify further. Next, we consider

$$\begin{aligned}
\langle p \rangle &= \frac{1}{2\pi} \int dp |\Phi(p)|^2 p \\
&= \frac{|A|^2 m^2}{2\pi} \int dz \cosh z \sinh z \exp(-2\alpha m \cosh z + 2\beta m p \sinh z) \\
&= -\frac{|A|^2}{4\pi} \partial_{\alpha} \partial_{\beta} K_0 = \frac{\partial_{\alpha} \partial_{\beta} K_0}{2\partial_{\alpha} K_0} = \frac{\beta}{\alpha^2 - \beta^2} \left(1 + m\sqrt{\alpha^2 - \beta^2} \frac{K_0}{K_1} \right). \tag{A.18}
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\langle p^2 \rangle &= \frac{1}{2\pi} \int dp |\Phi(p)|^2 p^2 \\
&= \frac{|A|^2 m^3}{2\pi} \int dz \cosh z \sinh^2 z \exp(-2\alpha m \cosh z + 2\beta m p \sinh z) \\
&= -\frac{|A|^2}{8\pi} \partial_{\alpha} \partial_{\beta}^2 K_0 = \frac{\partial_{\alpha} \partial_{\beta}^2 K_0}{4\partial_{\alpha} K_0} \\
&= \frac{m^2 \beta^2}{\alpha^2 - \beta^2} + \frac{\alpha^2 + 3\beta^2}{2(\alpha^2 - \beta^2)^2} \left(1 + m\sqrt{\alpha^2 - \beta^2} \frac{K_0}{K_1} \right). \tag{A.19}
\end{aligned}$$

Let us also consider the energy

$$\begin{aligned}
\langle E \rangle &= \frac{1}{2\pi} \int dp |\Phi(p)|^2 \sqrt{p^2 + m^2} \\
&= \frac{|A|^2 m^2}{2\pi} \int dz \cosh^2 z \exp(-2\alpha m \cosh z + 2\beta m p \sinh z) \\
&= \frac{|A|^2}{4\pi} \partial_{\alpha}^2 K_0 = -\frac{\partial_{\alpha}^2 K_0}{2\partial_{\alpha} K_0} = \frac{\alpha}{\alpha^2 - \beta^2} \left(1 + m\sqrt{\alpha^2 - \beta^2} \frac{K_0}{K_1} \right) - \frac{1}{2\alpha}, \tag{A.20}
\end{aligned}$$

as well as the energy squared

$$\langle E^2 \rangle = \langle p^2 \rangle + m^2 = \frac{m^2 \alpha^2}{\alpha^2 - \beta^2} + \frac{\alpha^2 + 3\beta^2}{2(\alpha^2 - \beta^2)^2} \left(1 + m\sqrt{\alpha^2 - \beta^2} \frac{K_0}{K_1} \right). \tag{A.21}$$

Finally, using the same methods one can show that

$$\frac{\alpha}{2} \left\langle \frac{m^2}{E^3} \right\rangle + \alpha^2 \left\langle \frac{m^2}{p^2 + m^2} \right\rangle = \alpha^2 - \beta^2, \quad (\text{A.22})$$

which then leads to

$$\langle x^2 \rangle = \frac{\alpha}{2} \left\langle \frac{m^2}{E^3} \right\rangle = \alpha^2 - \beta^2 - \frac{2\alpha\sqrt{\alpha^2 - \beta^2}}{K_1} \int_{\alpha}^{\infty} d\alpha' K_0 \left(2m\sqrt{\alpha'^2 - \beta^2} \right), \quad (\text{A.23})$$

B Relation between Relativistic Quantum Mechanics and Quantum Field Theory

In this appendix we review the relation between relativistic quantum mechanics and quantum field theory in the context of a simple free scalar field theory in $(1 + 1)$ dimensions.

B.1 Canonical Quantization

Let us consider a free field theory for a real-valued massive scalar field $\varphi(x, t) \in \mathbb{R}$ with the Lagrangian

$$\mathcal{L} = \frac{1}{2} [(\partial_t \varphi)^2 - (\partial_x \varphi)^2 - m^2 \varphi^2]. \quad (\text{B.1})$$

The momentum conjugate to the field φ is given by

$$\Pi(x) = \frac{\delta \mathcal{L}}{\delta \partial_t \varphi(x)} = \partial_t \varphi(x), \quad (\text{B.2})$$

and the classical Hamilton density thus takes the form

$$\mathcal{H} = \Pi \partial_t \varphi - \mathcal{L} = \frac{1}{2} [\Pi^2 + (\partial_x \varphi)^2 + m^2 \varphi^2]. \quad (\text{B.3})$$

Upon canonical quantization, the classical fields $\varphi(x)$ and $\Pi(x)$ are replaced by field operators with the commutation relations

$$[\varphi(x), \varphi(x')] = [\Pi(x), \Pi(x')] = 0, \quad [\varphi(x), \Pi(x')] = i\delta(x - x'). \quad (\text{B.4})$$

The classical Hamilton density then turns into the Hamilton operator

$$H = \int dx \mathcal{H} = \int dx \frac{1}{2} [\Pi^2 + (\partial_x \varphi)^2 + m^2 \varphi^2]. \quad (\text{B.5})$$

B.2 Particle Spectrum

In order to diagonalize the Hamiltonian, we go to momentum space by writing

$$\varphi(x) = \frac{1}{2\pi} \int dp \tilde{\varphi}(p) \exp(ipx), \quad \Pi(x) = \frac{1}{2\pi} \int dp \tilde{\Pi}(p) \exp(ipx). \quad (\text{B.6})$$

The field operators in momentum space satisfy

$$\tilde{\varphi}(p)^\dagger = \tilde{\varphi}(-p), \quad \tilde{\Pi}(p)^\dagger = \tilde{\Pi}(-p), \quad (\text{B.7})$$

and obey the commutation relations

$$[\tilde{\varphi}(p), \tilde{\varphi}(p')] = [\tilde{\Pi}(p), \tilde{\Pi}(p')] = 0, \quad [\tilde{\varphi}(p), \tilde{\Pi}(p')] = 2\pi i \delta(p + p'). \quad (\text{B.8})$$

The Hamilton operator is then given by

$$H = \frac{1}{2\pi} \int dp \frac{1}{2} \left[\tilde{\Pi}^\dagger \tilde{\Pi} + (p^2 + m^2) \tilde{\varphi}^\dagger \tilde{\varphi} \right]. \quad (\text{B.9})$$

Let us now introduce particle creation and annihilation operators

$$\begin{aligned} a(p) &= \frac{1}{\sqrt{2}} \left[(p^2 + m^2)^{1/4} \tilde{\varphi}(p) + i(p^2 + m^2)^{-1/4} \tilde{\Pi}(p) \right], \\ a(p)^\dagger &= \frac{1}{\sqrt{2}} \left[(p^2 + m^2)^{1/4} \tilde{\varphi}(p)^\dagger - i(p^2 + m^2)^{-1/4} \tilde{\Pi}(p)^\dagger \right], \end{aligned} \quad (\text{B.10})$$

which obey the commutation relations

$$[a(p), a(p')] = [a(p)^\dagger, a(p')^\dagger] = 0, \quad [a(p), a(p')^\dagger] = 2\pi \delta(p - p'), \quad (\text{B.11})$$

The Hamilton operator then takes the form

$$H = \frac{1}{2\pi} \int dp \sqrt{p^2 + m^2} \left[a(p)^\dagger a(p) + \pi \delta(0) \right], \quad (\text{B.12})$$

where the last term represents the divergent vacuum energy. The vacuum state $|0\rangle$ is characterized by $a(p)|0\rangle = 0$ for all values of p . The single particle states with momentum p and energy $E(p) = \sqrt{p^2 + m^2}$ are given by

$$|p\rangle = a(p)^\dagger |0\rangle. \quad (\text{B.13})$$

B.3 Localization of Particle States

According to the standard rules of quantum mechanics, one may construct a single particle position eigenstate

$$|x\rangle = \frac{1}{2\pi} \int dp \exp(-ipx) |p\rangle. \quad (\text{B.14})$$

Introducing

$$a(x)^\dagger = \frac{1}{2\pi} \int dp a(p)^\dagger \exp(-ipx), \quad (\text{B.15})$$

one then obtains

$$|x\rangle = a(x)^\dagger |0\rangle. \quad (\text{B.16})$$

According to our considerations in relativistic quantum mechanics, a wave packet composed of such states moves and spreads in a manner that seems to violate causality. Exactly the same behavior also arises in quantum field theory. Indeed, when we form single particle wave packets in the scalar field theory, they behave exactly as the ones in relativistic quantum mechanics that were considered in section 4.

In contrast to relativistic quantum mechanics, relativistic quantum field theory is based on the principle of locality. Hence, by construction, causality cannot be violated. Indeed, as we will now see, the apparent violation of causality observed in wave packet spreading is due to the fact that single particle states cannot be localized in a finite region. This is a consequence of the Reeh-Schlieder theorem [4]. The issues of particle localization have already been discussed by Newton and Wigner in 1949 [27], have been investigated further, for example, in [28], and continue to be a subject of controversial discussions [29, 30]. We notice that the operator

$$a(x)^\dagger = \frac{1}{2\pi} \int dp \frac{1}{\sqrt{2}} \left[(p^2 + m^2)^{1/4} \tilde{\varphi}(p)^\dagger - i(p^2 + m^2)^{-1/4} \tilde{\Pi}(p)^\dagger \right] \exp(-ipx), \quad (\text{B.17})$$

is not localized at x , but is instead non-local. In fact, the single particle position eigenstate $|x\rangle$ cannot be created from the vacuum by an application of the field operators $\varphi(x)$ and $\Pi(x)$ and their derivatives at the point x . Consequently, in quantum field theory single particles are inherently non-local objects. In non-relativistic quantum mechanics such subtleties do not arise and one interprets the state $|x\rangle$ as describing a single particle completely localized at the point x . When we do the same in relativistic quantum mechanics, we encounter an apparent violation of causality [5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. As the discussion of field theory shows, causality is not really violated, since the particle itself is a non-local object. This should be kept in mind when one interprets the results obtained in the framework of relativistic quantum mechanics. Consequently, the state $|x\rangle$ should not be viewed as describing a particle localized at the point x , but should be associated with the corresponding state in quantum field theory which is not localized.

References

- [1] B. M. Garraway and K.-A. Suominen, *Rep. Prog. Phys.* 58 (1995) 365.
- [2] M. Andrews, *Am. J. Phys.* 76 (2008) 1102.
- [3] P. Ehrenfest, *Z. Phys.* 45 (1927) 455.
- [4] H. Reeh and S. Schlieder, *Nuovo Cim.* 22 (1961) 1051.
- [5] G. N. Fleming, *Phys. Rev.* 139 (1965) 963.
- [6] G. C. Hegerfeldt, *Phys. Rev. D*10 (1974) 3320.
- [7] G. C. Hegerfeldt and S. N. M. Ruijsenaars, *Phys. Rev. D*22 (1980) 377.
- [8] G. C. Hegerfeldt, *Phys. Rev. Lett.* 54 (1985) 2395.
- [9] B. Rosenstein and M. Usher, *Phys. Rev. D*36 (1987) 2381.
- [10] S. N. Mosley and J. E. G. Farina, *J. Phys. A: Math. Gen.* 23 (1990) 3991.
- [11] G. C. Hegerfeldt, in “Extensions of Quantum Theory”, edited by A. Horzela and E. Kapuscik, published by Apeiron, Montreal (2001) 9.
- [12] N. Barat and J. C. Kimball, *Phys. Lett. A*308 (2003) 110.
- [13] F. Buscemi and G. Compagno, *Phys. Lett. A*334 (2005) 357.
- [14] F. Buscemi and G. Compagno, *J. Phys. B: At. Mol. Opt. Phys.* 39 (2006) 695.
- [15] H. Leutwyler, *Nuovo Cim.* 37 (1965) 543.
- [16] D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, *Rev. Mod. Phys.* 35 (1963) 350.
- [17] S. N. M. Ruijsenaars, *Ann. Phys.* 126 (1980) 399.
- [18] S. N. M. Ruijsenaars, *Commun. Math. Phys.* 110 (1987) 191.
- [19] S. N. M. Ruijsenaars, in “Integrable Structures of Exactly Solvable Two-Dimensional Models of Quantum Field Theory”, edited by S. Pakuliak and G. von Gehlen, published by Kluwer Academic Publishers, Netherlands (2001) 273.
- [20] C. Giunti, C. W. Kim, and U. W. Lee, *Phys. Rev. D*44 (1991) 3635.
- [21] C. Giunti, *Foundations of Physics Letters* 17 (2004) 103.
- [22] L. C. Baird, *Am. J. Phys.* 40 (1972) 327.

- [23] F. Bakke and H. Wergeland, *Physica* 69 (1973) 5.
- [24] C. Almeida and A. Jabs, *Am. J. Phys.* 52 (1984) 921.
- [25] F. W. Strauch, *Phys. Rev. A* 73 (2006) 069908.
- [26] G. Scharf, *Finite Quantum Electrodynamics*, Springer (1989).
- [27] T. D. Newton and E. P. Wigner, *Rev. Mod. Phys.* 21 (1949) 400.
- [28] S. N. M. Ruijsenaars, *Ann. Phys.* 137 (1981) 33.
- [29] G. N. Fleming, *Philosophy of Science* 67 (2000) 495.
- [30] H. Halverson, *Philosophy of Science* 68 (2001) 111.